Polar coordinates

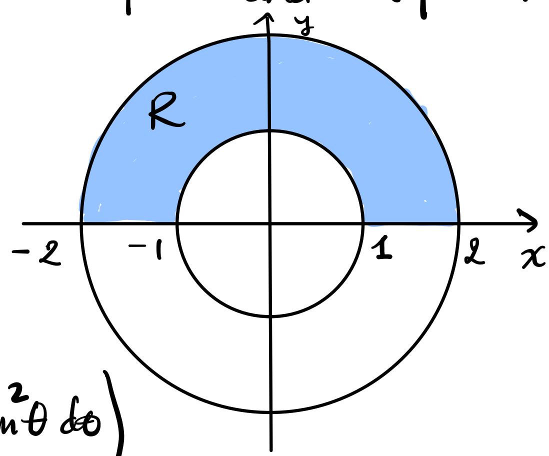
$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, y\, r\sin\theta)\,.\,r\,dr\,d\theta$$

$R \to$ (x,y)-plane

$S \uparrow$ (r,θ)-plane

$r \uparrow$ Jacobian.

Ex 1. $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper-half plane bonded by $x^2+y^2=1$ and $x^2+y^2=4$



In polar coordinate

$$1 \le r \le 2, \quad 0 \le \theta \le \pi$$

$$\int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,.\,r\,dr\,d\theta$$

$$= \left(\int_1^2 3r^2\,dr\right)\left(\int_0^\pi \cos\theta\,d\theta\right) + \left(\int_1^2 4r^3\,dr\right)\left(\int_0^\pi \sin^2\theta\,d\theta\right)$$

$$\left(r^3\Big|_1^2\right).\underbrace{\left(\sin\theta\Big|_0^\pi\right)}_{0} + \left(r^4\Big|_1^2\right)\left(\int_0^\pi \frac{1-\cos 2\theta}{2}\,d\theta\right)$$

$$15.\left(\frac{\pi}{2} - \frac{1}{2}\int_0^\pi \cos 2\theta\,d\theta\right)$$

$$15.\left(\frac{\pi}{2} - \underbrace{\frac{\sin 2\theta}{2}\Big|_0^\pi}_{0}\right) = \boxed{\frac{15\pi}{2}}$$

Note : $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

Ex2. Find the volume of the solid region bonded by $z=0$ & $z=1-x^2-y^2$

volume under the surface

R = intersection

$$\iint_R (1-x^2-y^2)\,dA$$

$z=0 \Rightarrow 1-x^2-y^2=0$

R is the circle $x^2+y^2 \le 1$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\int_0^{2\pi}\int_0^{1} (1-r^2)\, r\, dr\, d\theta = 2\pi \cdot \left(\int_0^1 (r - r^3)\, dr \right)$$

(r: Jacobian)

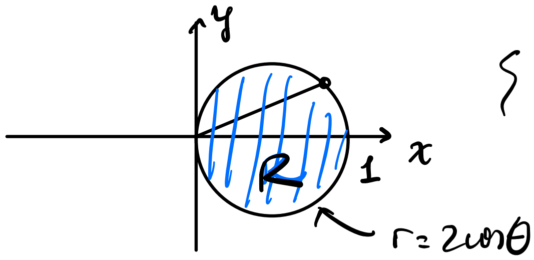
$$= 2\pi \cdot \left(\frac{r^2}{2}\Big|_0^1 - \frac{r^4}{4}\Big|_0^1 \right)$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

Ex 3. Find the volume of the solid region
$$\begin{cases} \text{under } z = x^2 + y^2 \\ \text{above } z = 0 \\ \text{inside } x^2 + y^2 = 2x \end{cases}$$
(inside $x^2+y^2=2x$: the base R)

Note: $x^2 + y^2 = 2x$

$\Rightarrow (x-1)^2 + y^2 = 1$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow r = 2\cos\theta$$

Hence in polar coordinate

$$\left\{ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta \right\}$$

$$\text{Volume} = \iint_R (x^2+y^2)\, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_0^{2\cos\theta} d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} 16\cos^4\theta\, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

Note: $\cos^4\theta = (\cos^2\theta)^2 = \left(\dfrac{\cos 2\theta + 1}{2} \right)^2$

$$= \left(\frac{\cos^2 2\theta + 2\cos 2\theta + 1}{4}\right)$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta + \frac{1}{8}$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

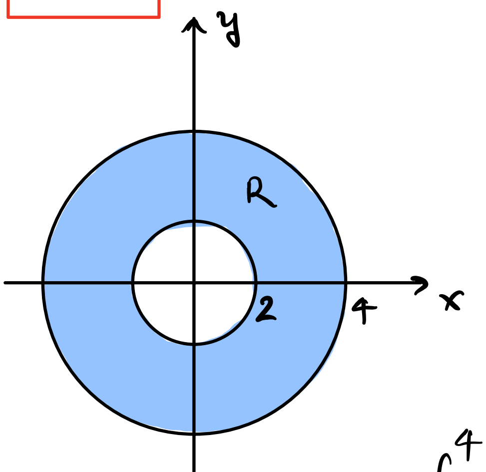
$$4\int_{-\pi/2}^{\pi/2}\left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right)d\theta$$

$$4\left(\frac{3}{8}\cdot\pi + \underbrace{\frac{1}{2}\,\frac{\sin 2\theta}{2}\Big|_{-\pi/2}^{\pi/2}}_{0} + \underbrace{\frac{1}{8}\cdot\frac{\sin 4\theta}{4}\Big|_{-\pi/2}^{\pi/2}}_{0}\right)$$

$$4\left(\frac{3\pi}{8} \quad 0 \quad 0\right)$$

$$= \boxed{\cancel{2\pi}} = \boxed{\frac{3\pi}{2}} \approx 4.71$$

Ex 4. Find the volume of the solid region inside the sphere $x^2+y^2+z^2=16$ outside of the cylinder $x^2+y^2=4$



Symmetry $\frac{\text{upper}}{\text{lower}}$

$$2\iint_R \sqrt{16-x^2-y^2}\, dA \qquad 2 \le r \le 4$$

$$2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\cdot r\, dr\, d\theta$$

$$= 4\pi\int_2^4 r\sqrt{16-r^2}\, dr$$

Let $u = 16 - r^2$

$du = -2r\,dr$

r	2	4
u	12	0

$$= 4\pi\int^{12}\sqrt{u}\cdot\frac{du}{2} = 2\pi\cdot\frac{u^{3/2}}{3/2}\Big|^{12} = 2\cdot\frac{2\pi}{3}\left(12^{3/2}\right)$$

0 2 3/2 0 3

Ex 5. Find the area of the 2D region bounded by $\begin{cases} \text{inside } r = 1+\cos\theta \\ \text{outside } r = 3\cos\theta \end{cases}$

$$3\cos\theta \le r \le 1+\cos\theta$$

$$\Rightarrow \quad 2\cos\theta \le 1 \Rightarrow \cos\theta \le \frac{1}{2} \Rightarrow -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$$

$$\int_{-\pi/3}^{\pi/3} \int_{3\cos\theta}^{1+\cos\theta} 1 \cdot r \, dr\, d\theta = \int_{-\pi/3}^{\pi/3} \frac{r^2}{2}\Big|_{3\cos\theta}^{1+\cos\theta} d\theta$$

(r ↓ Jacobian)

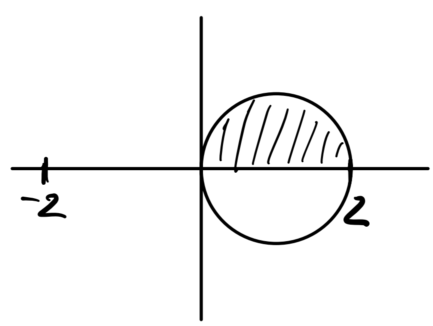
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((1+\cos\theta)^2 - 9\cos^2\theta\right) d\theta$$

$$\cos^2\theta - 7\cos\theta + 1 = \frac{1+\cos 2\theta}{2} - 7\cos\theta + 1$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(\frac{3}{2} + \frac{1}{2}\cos 2\theta - 7\cos\theta\right) d\theta$$

$$= \frac{1}{2}\left(3 \cdot \frac{\pi}{3} + \frac{1}{2}\,\frac{\sin 2\theta}{2}\Big|_{-\pi/3}^{\pi/3} - 7 \cdot \sin\theta \Big|_{-\pi/3}^{\pi/3}\right) \quad - - -$$

Ex. 6 Compute $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$



Use $x = r\cos\theta$
$y = r\sin\theta$

$$0 \le y \le \sqrt{2x-x^2}$$

$$y^2 \le 2x - x^2$$

$$\Rightarrow \quad y^2 + (x-1)^2 \le 1, \text{ and } y \ge 0$$

$$x^2 + y^2 \le 2x$$

then $0 \le \theta \le \pi/2$

$0 \le r \le 2\cos\theta$

$r^2 \le 2r\cos\theta \Rightarrow r \le 2\cos\theta$

Thus $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r \cdot \boxed{r}\, dr\, d\theta$

($\boxed{r}$ → Jacobian)

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta = \frac{1}{3}\int_0^{\pi/2} 8\cos^3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \left(\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\right) d\theta$$

$$= \frac{8}{3}\left(\frac{1}{4}\cdot\frac{\sin 3\theta}{3}\Big|_0^{\pi/2} + \frac{3}{4}\sin\theta\Big|_0^{\pi/2}\right)$$

$$= \frac{8}{3}\left(\frac{1}{12}\cdot(-1) + \frac{3}{4}\right) = \frac{8}{3}\cdot\frac{8}{12} = \boxed{\frac{16}{9}}$$

Note: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$