x	a	b
y	$\underbrace{g^{-1}(a)}_{c}$	$\underbrace{g^{-1}(b)}_{d}$

Change of variables

Motivated from the 1D - formula

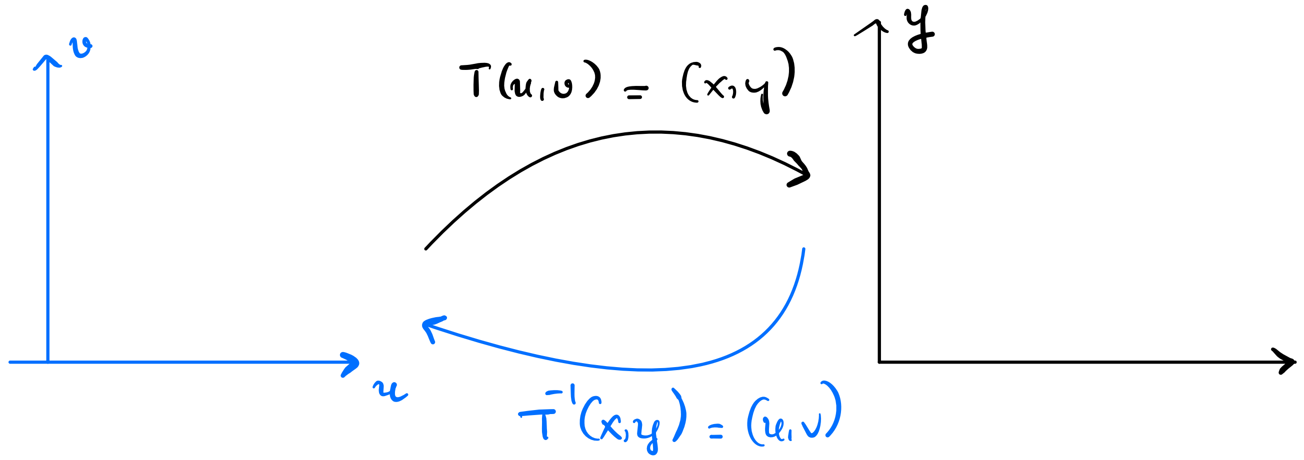
$$\int_b^a f(x)\,dx$$

if $x = g(y)$ then $dx = g'(y)\,dy$

hence

$$\int_a^b f(x)\,dx = \int_c^d f(g(y))\cdot \underbrace{g'(y)}_{\text{Jacobian in higher dimension.}}\,dy$$

Definition: A transformation T from (u,v)-plane to (x,y)-plane is a set of functions:



here:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}.$$

If $\underbrace{(x,y)}_{\text{image}} = T\underbrace{(u,v)}_{\text{pre-image}}$

$$\boxed{\begin{array}{l}\text{We will consider } T \text{ that is } C^1 \\ \text{and one-to-one, that is} \\ \text{"no 2 points have the same image"}\end{array}}$$

Example 1. $\begin{cases} x = 3u + 2v \\ y = v + 1 \end{cases}$

a) What is the (x, y) point that corresponds to $(u, v) = (-3, 4)$?

$$(x,y) = (3(-3) + 2\cdot 4,\ 4+1) = (-1, 5)$$

b) What is the (u,v) point that corresponds to $(x,y) = (-3,4)$?

We solves for (u,v) : $\begin{cases} 3u + 2v = x = -3 \\ v + 1 = y = 4 \end{cases}$

then $\begin{cases} 3u + 2v = -3 \\ v = 3 \end{cases} \Rightarrow 3u = -3 - 2v = -9$

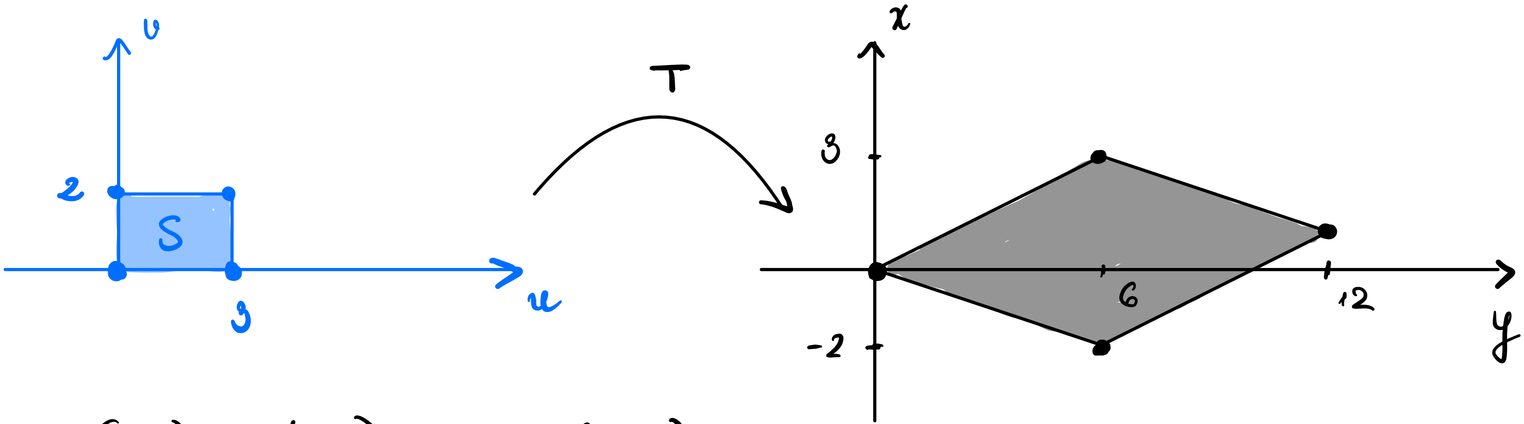
$\Rightarrow u = -3$

thus $(u,v) = (-3,3)$.

Example 2. Consider the set $S = \{ (u,v) : u \in [0,3], v \in [0,2] \}$ and the transformation

$$T(u,v) = (2u + 3v, u - v)$$

a) Find the image of S under the transformation T. Sketch S and its image in 2 separate planes.



(u,v): $(0,0) \to (0,0)$

$(3,0) \to (6,3)$

$(3,2) \to (12,1)$

$(0,2) \to (6,-2)$

b) Compute T^{-1} : solve for (u,v) in terms of (x,y).

$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} 2u + 3v = x \\ u - v = y \end{cases}$$

$$+ \begin{cases} 2u + 3v = x \\ 3u - 3v = 3y \end{cases}$$

$$5u = x + 3y$$

$$u = \frac{x+3y}{5}$$

$$- \begin{cases} 2u + 3v = x \\ 2u - 2v = 2y \end{cases}$$

$$5v = x - 2y$$

$$v = \frac{x-2y}{5}$$

$$\boxed{T^{-1}(x,y) = \left(\frac{x+3y}{5}, \frac{x-2y}{5} \right) = (u,v)}$$

Change of variables formula

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Jacobian :

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

determinant

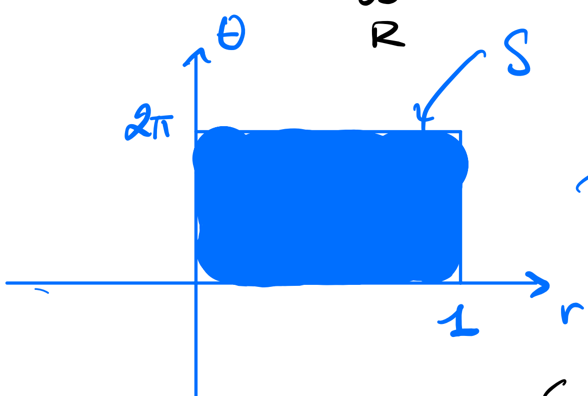
If $T : \underset{\substack{S \\ \text{(domain)}}}{(u,v)} \longrightarrow \underset{\substack{R \\ \text{(domain)}}}{(x,y)}$

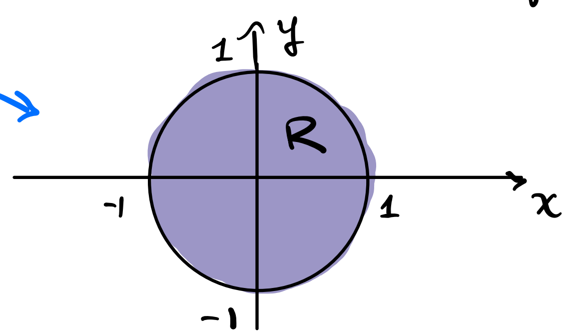
$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>note</u> : absolute value of the determinant

Example 3.

Evaluate $\iint_R 1\,dA$ where R is the circle of radius 1 centered at the origin





$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\iint_R 1\,dA = \iint_S 1 \cdot r\,dr\,d\theta = \int^{2\pi} \int^{1} r\,dr\,d\theta$$

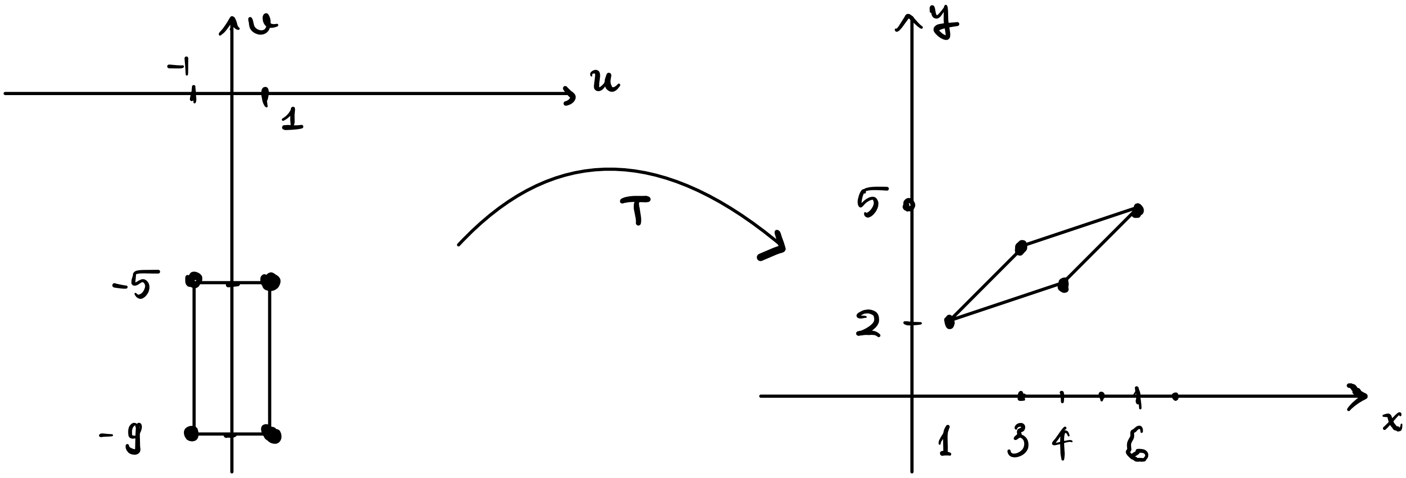
$$\frac{r^2}{2}\Big|_0^1$$

$$= 2\pi \times \frac{1}{2} = \boxed{\pi}.$$

Example 4. Evaluate

$$\iint_R (x-y)\, dA$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ by using the transformation $T(u,v) = \left(\frac{3u-v}{2}, \frac{u-v}{2}\right)$



Step 1. Solve for the inverse T^{-1}

$$\begin{cases} \dfrac{3u-v}{2} = x \\ \dfrac{u-v}{2} = y \end{cases} \Rightarrow \begin{cases} u = x-y \\ v = x-3y \end{cases}$$

Step 2. Compute (u,v) points

(x,y) place		(u,v) planes
$(1,2)$	$\longrightarrow$	$(-1,-5)$
$(4,3)$		$(1,5)$
$(3,4)$		$(-1,-9)$
$(6,5)$		$(1,9)$

Step 3. Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{-3}{4} + \frac{1}{4} = \frac{-1}{2}$$

absolute value : $\left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| = \dfrac{1}{2}$

Step 4. Compute

$$\iint_R (x-y)\, dA = \iint_S \left(\frac{3u-v}{2} - \frac{u-v}{2} \right) \cdot \underbrace{\boxed{\frac{1}{2}}}_{\text{Jacobian}} \, du\, dv$$

$$= \int_{-1}^{1} \int_{-9}^{-5} u \cdot \frac{1}{2} \, dv\, du$$

$$= \frac{1}{2} \left(\int_{-1}^{1} u \, du \right) \left(\int_{-9}^{-5} dv \right)$$

$$= \frac{1}{2} \cdot \frac{u^2}{2} \Big|_{-1}^{1} \cdot 4$$

$$= \frac{1}{2} \cdot \frac{1-(1)}{2} \cdot 4 = 0 .$$