Fundamental theorem of calculus

1D : $\int_a^b f'(x)\,dx = f(b) - f(a)$

2D or 3D : $\underbrace{\int_a^b \nabla f(r(t)) \cdot r'(t)\,dt}_{}$ $= f(r(b)) - f(r(a))$ → simplify line integral

where $\vec{F} = \nabla f$ $\int_C \vec{F} \cdot d\vec{r}$

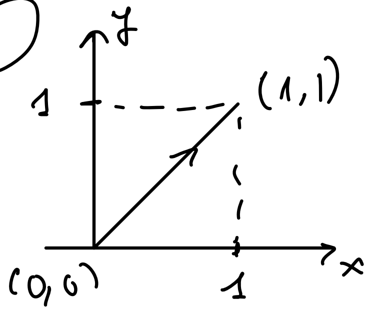
Example. Calculate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0)$ to $(1,1)$

$\vec{F}(x,y) = (2x, 2y)$

Proof.

There are many paths that connect $(0,0)$ and $(1,1)$

(1)



$\vec{r}(t) = (t,t)$ $t \in [0,1]$ → $r'(t)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2t, 2t) \cdot (1,1)\,dt$$

$$= \int_0^1 4t\,dt = 4\frac{t^2}{2}\Big|_0^1 = 2$$

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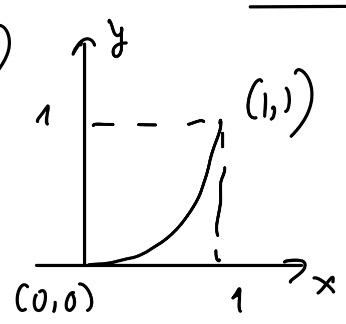
(2)



$r(t) = (t, t^2)$ $t \in [0,1]$ → $r'(t)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2t, 2t^2) \cdot (1, 2t)\,dt$$

$$= \int_0^1 2t + 4t^3\,dt$$

$$= (t^2 + t^4)\Big|_0^1 = 1 + 1 = 2$$

(3)

Using the fundamental theorem

for $f(x,y) = x^2 + y^2$ then $\nabla f(x,y) = (2x, 2y) = \vec{F}$

therefore

$$\int_C \vec{F} \cdot d\vec{r} = f((1,1)) - f((0,0))$$
$$= (1^2+1^2) - (0^2+0^2) = 2.$$

Note:

- Choose a "potential" $f(x,y) = x^2+y^2$

  any other potential fcn $\hat{f}(x,y) = x^2+y^2+2$

  will work as well

- How to find a potential?

(1) Find $f(x,y)$ s.t. $\quad \vec{F} = (P,Q) = \left(P(x,y), Q(x,y)\right)$

$$\frac{\partial f}{\partial x} = P(x,y)$$

$$\Rightarrow \quad f(x,y) = \int P(x,y)\,dx \quad \text{function of } y$$
$$= \hat{P}(x,y) + C(y)$$

(2) Do $\dfrac{\partial f}{\partial y} = Q(x,y) \Rightarrow \dfrac{\partial}{\partial y}\left(\hat{P}(x,y) + C(y)\right) = Q(x,y)$

Solve for $C(y)$

(3) Answer $\boxed{f(x,y) = \hat{P}(x,y) + C(y)}$

Example: $\vec{F}(x,y) = (2x, 2y)$, find $f(x,y)$

(1) $\dfrac{\partial f}{\partial x} = 2x \Rightarrow f(x,y) = \int 2x = x^2 + C(y)$

(2) $\dfrac{\partial f}{\partial y} = 2y \Rightarrow \dfrac{\partial}{\partial y}\left(x^2 + C(y)\right) = 2y$

$$\Rightarrow \quad C'(y) = 2y \Rightarrow C(y) = y^2$$

(3) Conclude: $\boxed{f(x,y) = x^2 + C(y) = x^2 + y^2}$

Notes:

It is not always possible to do this!

$F = (P, Q) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\Rightarrow \quad P = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$\Rightarrow \quad Q = \frac{\partial f}{\partial y} \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

thus $\vec{F} = \nabla f \Longleftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Ex: $\vec{F} = (\underset{P}{2x}, \underset{Q}{2y}) \Rightarrow \frac{\partial P}{\partial y} = 2 = \frac{\partial Q}{\partial x}$ ✓

Ex: $\vec{F} = (\underset{P}{2y}, \underset{Q}{x}) \Rightarrow \frac{\partial P}{\partial y} = 2 \neq 1 = \frac{\partial Q}{\partial x}$ ✗

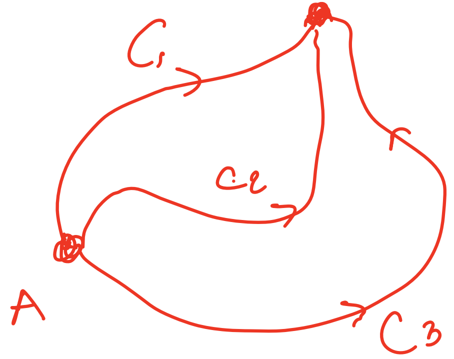
$\hookrightarrow$ cannot find f scalar s.t. $\vec{F} = \nabla f$

i.e., $\vec{F} = (2y, x)$ is not a conservative vector field.

Meaning: Conservative vector fields $\vec{F}$

$\Leftarrow \int_C \vec{F} \cdot d\vec{r}$ DOES NOT depend on C

B $\qquad \vec{F} = \nabla f$



$$\int_{C_i} \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

no matter what path

as long as $\vec{F}$ is well-defined

in the curve along $A \to B$.

<u>Simply connected domain.</u>

<u>Example</u>:

$$\vec{F} = (\underbrace{3+2xy}_{P}, \underbrace{x^2 - 3y^2}_{Q})$$

find $\int_C \vec{F} \cdot d\vec{r}$, C is given by

$\vec{r}(t) = (e^t \sin t, e^t \cos t)$

$0 \le t \le \pi$

① . Check

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(3+2xy) = 2x$$

$$\|$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x$$

$\Rightarrow$ we can proceed to find $f(x,y)$ s.t. $\nabla f = (P, Q) = \vec{F}$

② $\frac{\partial f}{\partial x} = P = 3+2xy$

$\Rightarrow f(x,y) = \int (3+2xy)\, dx$

$= 3x + x^2 y + c(y)$

(3) $\frac{\partial f}{\partial y} = Q \Rightarrow \underbrace{\frac{\partial}{\partial y}\left(3x + x^2 y + C(y)\right)}_{x^2 + C'(y)} = x^2 - 3y^2$

$$x^2 + C'(y) = x^2 - 3y^2$$

$$\Rightarrow C'(y) = -3y^2$$

$$\Rightarrow C(y) = -y^3$$

(4) $$\boxed{f(x,y) = 3x + x^2 y - y^3}$$

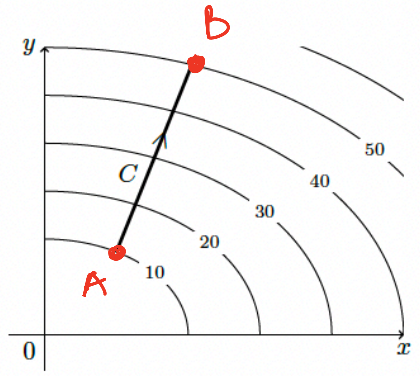
thus

$$\int_C \vec{F} \cdot d\vec{r} = f\left(e^{\pi}\sin\pi,\ e^{\pi}\cos\pi\right) - f\left(e^{0}\sin 0,\ e^{0}\cos 0\right)$$

$$= f(0, -e^{\pi}) - f(0, 1)$$

$$= e^{3\pi} - 1.$$

Example 3.14. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



$f(B) - f(A)$

$50 \quad 10$

$= 40$

A. 10

B. 40

C. 50

D. 60

E. 500