1 Vector Fields

1.1 A Vector at Every Point - Video Before Class

Objective(s):

- Sketch a given vector field
- Recognize that vector fields fill in the missing gaps

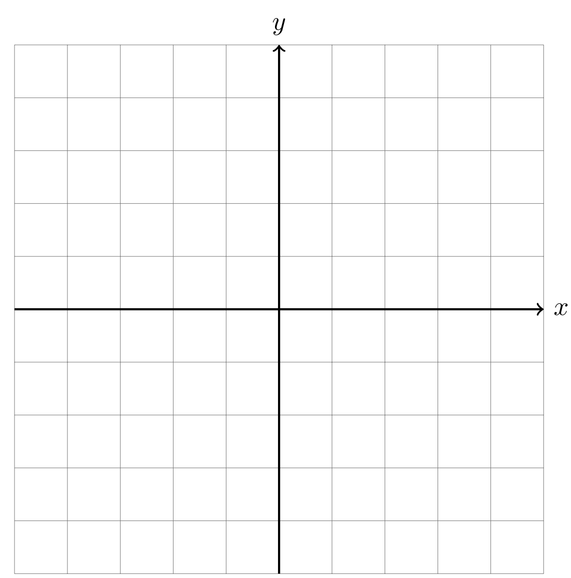
Definition(s) 1.1.

(a) Let D be a set in $\mathbb{R}^2$ (plane region). A ______________________________ is a function ___ that assigns to each point (x, y) in D a two-dimensional vector ____________.

(b) Let E be a set in $\mathbb{R}^3$. A ______________________________ is a function ___ that assigns to each point (x, y, z) in E a three-dimensional vector ________________.

Let's practice by sketching a vector field.

Example 1.2. Sketch the vector field: $\mathbf{F}(x, y) = \langle -y, x \rangle$ on the graph below.



Vector's calculus

Concerning functions whose outputs are vectors ($\mathbb{R}^2$ or $\mathbb{R}^3$)

vector fields

<u>Examples:</u>

$$F(x,y) = (y,x) \quad : \quad \mathbb{R}^2 \to \mathbb{R}^2$$

or

$$F(x,y,z) = (x, y^2, z+1) \quad : \quad \mathbb{R}^3 \to \mathbb{R}^3$$

or

$$F(x,y) = (x, y, 1) \quad : \quad \mathbb{R}^2 \to \mathbb{R}^3$$

or (parametrization)

$$r(t) = (\cos t, \sin t, 1) \quad \mathbb{R} \to \mathbb{R}^3$$

A vector field can be generated by gradient of a scalar function (output in $\mathbb{R}^1$)

<u>Ex</u>

$$f(x,y) = x^2 + y^2$$

$$\nabla f(x,y) = (2x, 2y) \quad \text{is a vector field } \mathbb{R}^2 \to \mathbb{R}^2$$

<u>Ex</u> :

$$f(x,y,z) = x$$

$$\nabla f(x,y,z) = (1,0,0) \quad \text{is a vector field } \mathbb{R}^3 \to \mathbb{R}^3$$

Conservative vector field:

need to find

F is conservative if $F = \nabla f$ for some scalar f

given

(f is the potential of F)

Example 1.4. Which of the vector field describes the plot to the right?

A. $\langle x, x-y \rangle$

B. $\langle y, x-y \rangle$

C. $\langle x, x+y \rangle$

D. $\langle y, x+y \rangle$

E. None of the above

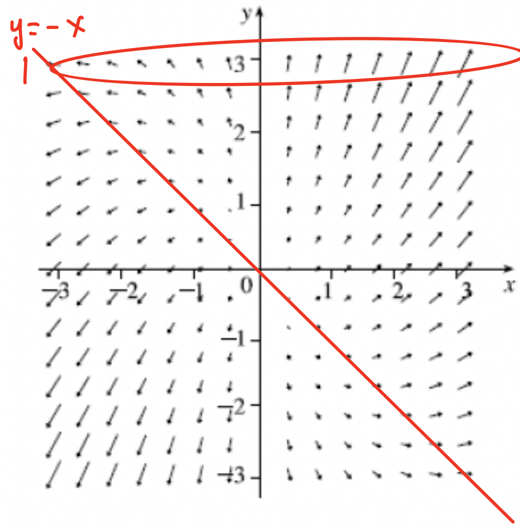
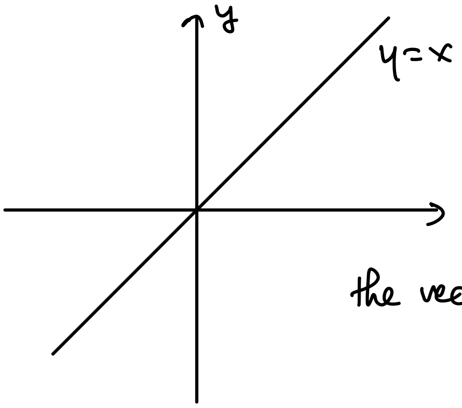


Figure may be scaled down

Look at the line $y = x$



the vector field becomes

A. $\langle x, 0 \rangle$
B. $\langle y, 0 \rangle$

these are vectors of "horizontal direction"

$\rightarrow$ or $\leftarrow$, clearly not true

can't be A or B

Look at the line $y = -x$

C. $\langle x, 0 \rangle$
D. $\langle y, 0 \rangle$

these are also horizontal vectors

however, if D. $\langle y, 0 \rangle$ $\Rightarrow$ the vector is longer if we go further up or down

this is not the case

so:

C: $\langle x, x+y \rangle$ is the answer.

Example 1.5. Find the gradient vector field of $f(x,y) = 2xy + 3x - e^{-xy}$

$$F = \nabla f(x,y) = \left(2y + 3 - e^{-xy}(-y),\ 2x - e^{-xy}(-x)\right)$$

$$F(x,y) = \left(2y + 3 + ye^{-xy},\ 2x + xe^{-xy}\right)$$

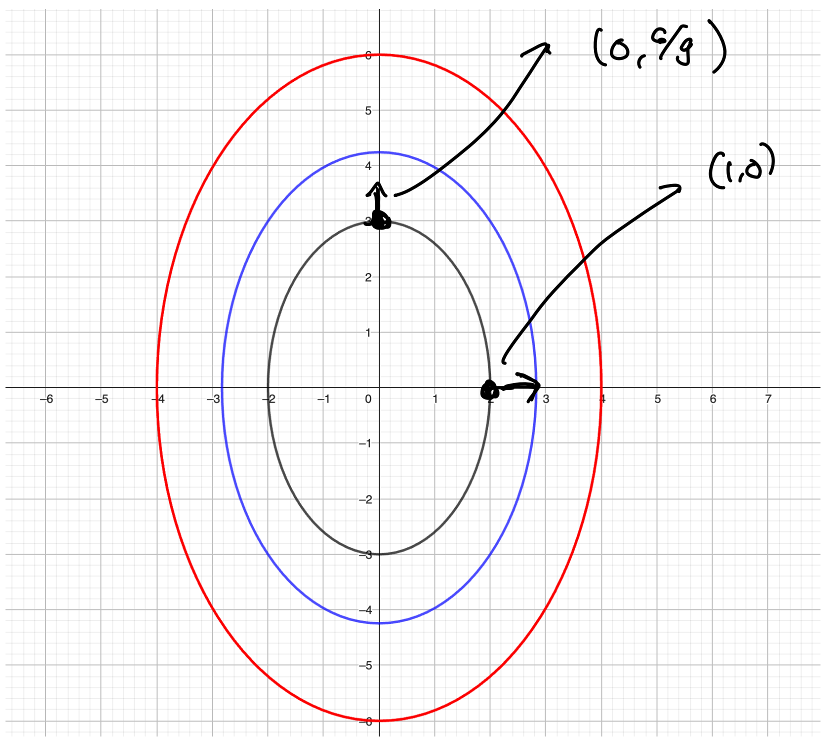
Example 1.6. For each of the following functions, draw level curves $f(x,y) = k$ for the indicated values of k. Then compute the gradient vector field, and sketch it at one or two points on each level curve.

(a) $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}$; $k = 1, 2, 4$

$$\nabla f(x,y) = \left(\frac{2x}{4}, \frac{2y}{9}\right) = \left(\frac{x}{2}, \frac{2y}{9}\right)$$

at $(0,3)$: $\left(\frac{x}{2}, \frac{2y}{9}\right) = \left(0, \frac{6}{9}\right)$

at $(2,0)$: $\left(\frac{x}{2}, \frac{2y}{9}\right) = (1, 0)$



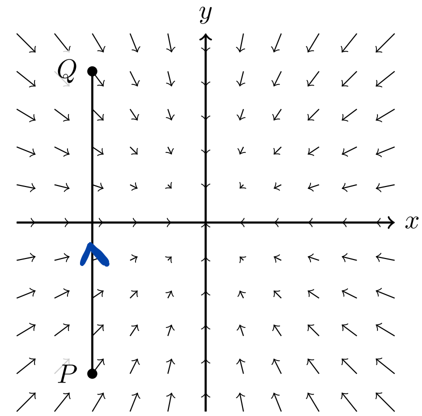
Looking ahead it will be important to be able to answer the following questions.

Example 1.7. Consider the vector field **F** to the right. Suppose particles are moving from P to Q along the curve.

A. **F** is helping push particles from P to Q along the curve C.

B. **F** is making it harder for particles to move from P to Q along the curve C.

C. Neither.

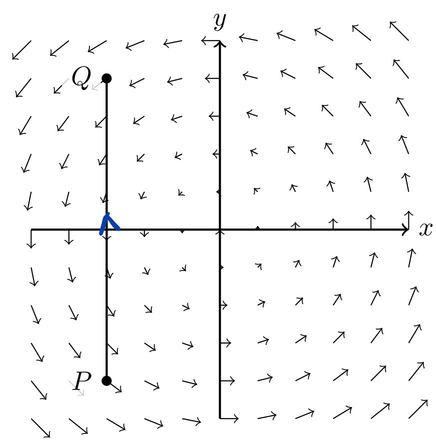


Example 1.8. Consider the vector field **F** to the right. Suppose particles are moving from P to Q along the curve.

A. **F** is helping push particles from P to Q along the curve C.

B. **F** is making it harder for particles to move from P to Q along the curve C.

C. Neither.



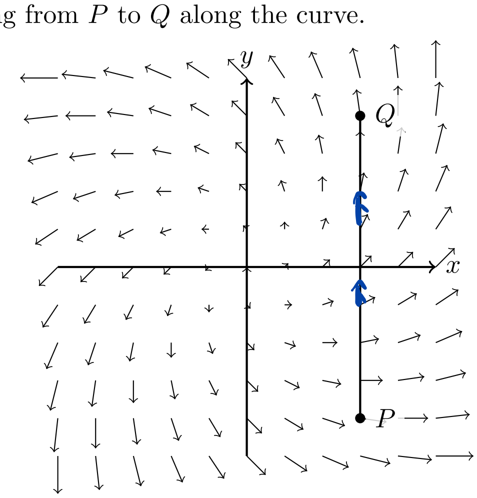
Example 1.9. Consider the vector field **F** to the right. Suppose particles are moving from P to Q along the curve.

A. **F** is helping push particles from P to Q along the curve C.

B. **F** is making it harder for particles to move from P to Q along the curve C.

C. Neither.

$\theta < \frac{\pi}{2}$



Line integral $\rightsquigarrow$ integral (sum of all value) of a function along a curve C

if C is parametrized by a curve $r(t)$, $t \in [a,b]$

(could be
$\vec{r}(t) = (x(t), y(t))$
$\vec{r}(t) = (x(t), y(t), z(t))$)

then $\vec{r}(t) = (x(t), y(t), z(t))$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,|\vec{r}'(t)|\,dt$$

scalar

norm of the velocity $r'(t)$

Example 2.5. Suppose you are a whale who is eating plankton as he/she swims through the ocean. The plankton are spread all throughout the ocean with a function $p(x,y,z) = \boxed{-\frac{1}{\pi}(10+z+x)}$ You (the whale) are chilling out at $(1,0,-12)$ are about to swim around in a circular curve; $C:\ x^2+y^2=1,\ z=-12$. How many plankton do you eat?

parametrization: $r(t) = (\cos t, \sin t, -12)$ $t \in [0, 2\pi]$

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = -12 \end{cases}$$

$$r'(t) = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$
$$= \sqrt{\sin^2 t + \cos^2 t + 0}$$
$$= 1$$

thus $\displaystyle\int_0^{2\pi} \frac{-1}{\pi}\left(10 + (-12) + \cos t\right) \cdot \cancel{|r'(t)|}^{\,1}\, dt = \int_0^{2\pi} \frac{-2+\cos t}{-\pi}\, dt.$

Example 2.6. A portion of a wall can be parametrized by $\mathbf{r}(t) = \langle 2\sin t, 2\cos t\rangle \quad t \in [0, \pi]$ where the height is given by $H(x, y) = xy^2$ *meters*. Each brick has a cross-sectional area of 100 cm^2. How many bricks are needed to build this portion of the great wall of china?

Compute the total area first, then $/100$ to get # bricks.

$$|r'(t)| = \sqrt{(2\cos t)^2 + (-2\sin t)^2} = \sqrt{4} = 2$$

$$\int_0^{\pi} \underbrace{(2\sin t)}_{x(t)} \underbrace{(2\cos t)^2}_{y(t)^2} \cdot \underbrace{2}_{|r'(t)|} \, dt$$

$$= 16 \int_0^{\pi} \sin t \cos^2 t \, dt$$

$u = \cos t$

$du = -\sin t \, dt$

t	0	π
u	1	-1

$$= 16 \int_1^{-1} u^2 (-du) = 16 \int_{-1}^{1} u^2 \, du = 16 \cdot \frac{u^3}{3} \Big|_{-1}^{1}$$

$$= \frac{16}{3} \left(1^3 - (-1)^3\right)$$

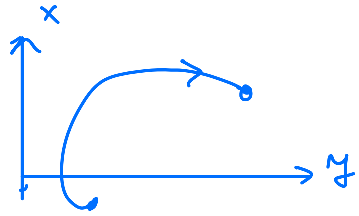
$$= \frac{32}{3} \quad (m^2)$$

$$\hookrightarrow \frac{32}{3} \times 100^2 \quad (cm^2)$$

$$\Rightarrow \# \text{ bricks} \approx \frac{32}{3} \times 100 \text{ bricks}$$

$\boxed{\text{Line integral}}$

Distinguish : $ds = |r'(t)|\,dt = \sqrt{|x'(t)|^2 + |y'(t)|^2}$



$\vec{r} : [a,b] \to \mathbb{R}^2$ or $\mathbb{R}^3$

$dx = x'(t)\,dt$

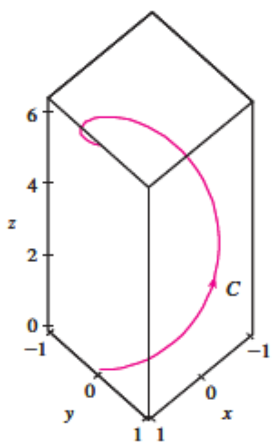
$dy = y'(t)\,dt$

Example

1. Evaluate $\int_C y \sin z \, ds$ and $\int_C y \sin z \, d$

where C is the circular helix

$$x(t) = \cos t$$
$$y(t) = \sin t \qquad 0 \le t \le 2\pi$$
$$z(t) = t$$



Proof: $ds = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\,dt = \sqrt{2}\,dt$

$$\int_C y \sin z \, ds = \int_0^{2\pi} \sin t \, \sin t \, \sqrt{2}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} \sin^2 t \, dt = \sqrt{2}\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt$$

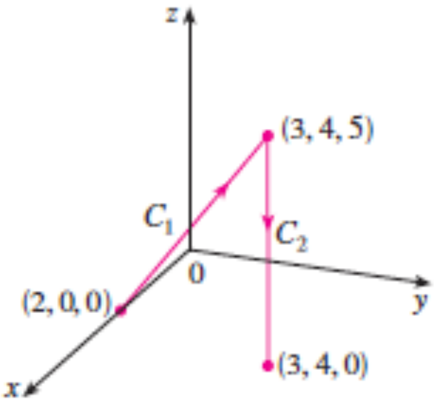
$$= \frac{\sqrt{2}}{2} \cdot \left[2\pi - \frac{\sin 2t}{2}\Big|_0^{2\pi} \right] = \sqrt{2}\pi$$

$dz = z'(t)\,dt = dt$

$$\int_C y \sin z \, dz = \int_0^{2\pi} \sin t \, \sin t \, dt = \pi$$

2. Evaluate $\int_C y\,dx + z\,dy + x\,dz$ C: line $C_1: (2,0,0) \to (3,4,5)$

$C_2: (3,4,5) \to (3,4,0)$



C_1: $\vec{r}(t) = (1-t)(2,0,0) + t(3,4,5)$

$= (2+t, 4t, 5t)$ $\quad 0 \le t \le 1$

$x(t) = 2+t$ $\quad x'(t) = 1$
$y(t) = 4t$ $\quad y'(t) = 4$
$z(t) = 5t$ $\quad z'(t) = 5$

$$\int_C y\,dx + z\,dy + x\,dz = \int_0^1 4t \cdot dt + 5t \cdot 4dt + (2+t) \cdot 5\,dt$$

$$= 24.5$$

C_2: $\vec{r}(t) = (1-t)(3,4,5) + t(3,4,0)$ $\quad 0 \le t \le 1$
$= (3, 4, 5-5t)$

$x(t) = 3$ $\quad x'(t) = 0$
$y(t) = 4$ $\quad y'(t) = 0$
$z(t) = 5-5t$ $\quad z'(t) = -5$

$$\int_C y\,\underbrace{\cancel{dx}}_{0} + z\,\underbrace{\cancel{dy}}_{0} + x\,dz = \int_0^1 3 \cdot (-5)\,dt = -15$$

thus $\boxed{\int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5}$

Line integral of vector fields

dx, dy, dz

$\int_C f\,ds$ v.s. $\int_C \vec{F} \bullet d\vec{r}$

scalar

$r'(t)$
not taking norm

vector field

length of curves

sum of all values along the curve

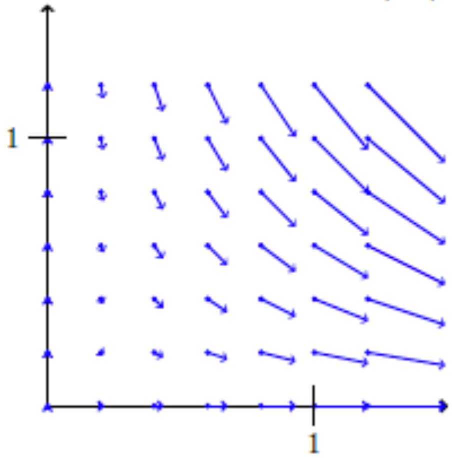
represent the total amount of work done by a vector field along the curve

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$$

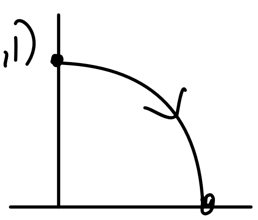
dot product

Ex 1.

Example 2.10. A picture of the force field $\mathbf{F}(x, y)$ is given below. Determine if the work in moving a particle along the quarter circle $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$ is positive or negative using the picture.



should be > 0



as the direction of $\vec{r}$ (tangent) and the vector field align

Example 2.10 (again). Find the work done by the force field $\mathbf{F}(x,y) = \langle x^2, -xy\rangle$ in moving a particle along the quarter circle $x^2+y^2=1$ from $(0,1)$ to $(1,0)$.

Parametrize $\vec{r}(t) = (\cos t, \sin t)$ $\qquad \vec{r}\,'(t) = (-\sin t, \cos t)$

t goes from: $\frac{\pi}{2} \to 0$

$(0,1) \to (1,0)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\frac{\pi}{2}}^{0} (x^2, -xy)\cdot \vec{r}\,'(t)\,dt$$

$$= \int_{\pi/2}^{0} (\cos^2 t, -\sin t\cos t)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_{\pi/2}^{0} (-\sin t\cos^2 t - \sin t\cos^2 t)\,dt$$

$$= -2\int_{\pi/2}^{0} \sin t\cos^2 t\,dt = 2\int_0^{\pi/2} \sin t\cos^2 t\,dt$$

$u = \cos t$

$du = -\sin t\,dt$

t	0	$\pi/2$
u	1	0

$$-2\int_1^0 u^2\,du = 2\int_0^1 u^2\,du = 2\,\frac{u^3}{3}\Big|_0^1 = \frac{2}{3}.$$

Example 2.11.

(a) Evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

$r(t) = (2+t, 4t, 5t)$

$\int_0^1 4t \cdot 1\,dt$

$\boxed{9.5}$

(b) What is the calculation in (a) telling you (in terms of Work)?

$\vec{F}(x,y,z) = (y, z, x)$

$\int_C \vec{F} \cdot d\vec{r}$

$r = (x,y,z)$ ce a para netrat

$d\vec{r} = (dx, dy, dz)$

$\int_C y\,dx + z\,dy + x\,dz = (y, z, x) \cdot (dx, dy, dz)$

$\int_C \vec{F} \cdot d\vec{r}$

Example 2.7. Suppose I have a nice spring that seems to follow the curve $\mathbf{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$ with $t \in [0, 8\pi]$ which happens to have a density function of $\delta(x, y, z) = 10 - x - y$ g/cm. How heavy is the spring?

scalar (no dot product)

$$\int_C \delta(x,y,z)\, ds \longrightarrow |r'(t)|\, dt$$

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$x = 3\sin t$
$y = 3\cos t$
$z = 4t$

$$\int_0^{8\pi} (10 - 3\sin t - 3\cos t)\sqrt{(3\cos t)^2 + (-3\sin t)^2 + 4^2}\, dt$$

$$\int_0^{8\pi} (10 - 3\sin t - 3\cos t)\overbrace{\sqrt{3^2 + 4^2}}^{5}\, dt$$

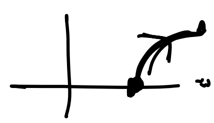
$$= 5\int_0^{8\pi} (10 - 3\sin t - 3\cos t)\, dt$$

vector field, dot product.

40. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

$F(x,y) = (-x^2, ye^x)$

$x = y^2 + 1$



$$\int_C \vec{F} \cdot d\vec{r} \longrightarrow r'(t)\, dt$$

$r(t) = (t^2+1, t)$
$t \in [0,1]$

$$\int_0^1 (-x^2, ye^x) \cdot (x', y')$$

$$(-(t^2+1)^2,\ te^{t^2+1})\ (2t, 1)\, dt$$

$$\int_0^1 \left(-2t(t^2+1)^2 + te^{t^2+1}\right) dt$$

lengthy integral