MICHIGAN STATE UNIVERSITY
MATH 234 – SPRING 2024

LECTURE NOTES

18 Lagrange multipliers

To find max/min of $f(x,y,z)$ with a given constraint $g(x,y,z) = 0$.

1. We find solutions for (x,y,z) and any possible λ that satisfy

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 0. \end{cases}$$

2. Among all (x,y,z) we found, find the biggest or smallest values of $f(x,y,z)$. They are potential absolute max/min of f given the constraint g.

Example 1. *Find max/min $f(x,y) = x^2 + y^2$ subjected to $xy = 1$.*

Proof. Here $f(x,y) = x^2 + y^2$ and $g(x,y) = xy - 1$.

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \implies \begin{cases} (2x,2y) = \lambda(y,x) \\ xy = 1 \end{cases} \implies \begin{cases} 2x &= \lambda y \\ 2y &= \lambda x \\ xy &= 1 \end{cases}$$

If we multiply the two equations together side by side, then

$$4xy = \lambda^2 xy \implies \lambda^2 = 4.$$

- If $\lambda = 2$ then $x = y$ and $xy = 1$, thus $(x,y) = (1,1)$ or $(-1,-1)$.
- If $\lambda = -2$ then $x = -y$, then $-x^2 = 1$ has no solution.

We conclude that the minimum of f is 2, at $(1,1)$ or $(-1,-1)$. Here f has no max since if we choose

$$(x,y) = \left(n, \frac{1}{n}\right) \implies f(x,y) = n^2 + \frac{1}{n^2} \geq n^2 \to \infty$$

if we let $n \to \infty$. □

Example 2. *Find max/min $f(x,y) = x^2 + 2y^2$ subjected to $x^2 + y^2 = 1$.*

Proof. Here $f(x,y) = x^2 + 2y^2$ and $g(x,y) = x^2 + y^2$.

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \implies \begin{cases} (2x,4y) = \lambda(2x,2y) \\ x^2 + y^2 = 1 \end{cases} \implies \begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

Look at the first equation, we have $2x(1 - \lambda) = 0$.

- If $x = 0$ then $y^2 = 1$, thus $(x,y) = (0,1), (0,-1)$.

- If $x \neq 0$ then $\lambda = 1$, then the second equation reads $4y = 2y$, thus $y = 0$ and hence $x^2 = 1$, thus $(x, y) = (1, 0), (-1, 0)$.

Comparing the values, we have f is max 2 at $(0, 1)$ or $(0, -1$, and f is min 1 at $(1, 0)$ or $(-1, 0)$. □

Example 3. *A rectangular box without a lid is to be made from* $12\ \mathrm{m}^2$ *of cardboard. Find the maximum volume of such a box.*

Proof. Let x, y be the measurements of the two sides on the bottom, and z be the height of the box. Here the volume is

$$f(x, y, z) = xyz,$$

and the area of the box without the lid is $xy + 2xz + 2yz = 12$, thus the constraint is

$$g(x, y, z) = xy + 2xz + 2yz - 12 = 0.$$

The system is

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = 0 \end{cases} \implies \begin{cases} (yz, xz, xy) = \lambda(y + 2z, x + 2z, 2x + 2y) \\ xy + 2xz + 2yz = 12. \end{cases}$$

Therefore

$$\begin{cases} yz = \lambda(y + 2z) \\ xz = \lambda(x + 2z) \\ xy = \lambda(2x + 2y) \\ xy + 2xz + 2yz = 12. \end{cases}$$

If $\lambda = 0$ then $yz = xz = xy = 0$, which does not satisfy $xy + 2xz + 2yz = 12$. We can safely assume $x, y, z \neq 0$ as they are dimensions of the box. Thus by dividing the equations side by side we have

$$\begin{cases} x \times yz = x \times \lambda(y + 2z) \\ y \times xz = y \times \lambda(x + 2z) \\ z \times xy = z \times \lambda(2x + 2y) \\ xy + 2xz + 2yz = 12. \end{cases} \implies \begin{cases} xyz = \lambda(xy + 2xz) \\ xyz = \lambda(xy + 2yz) \\ xyz = \lambda(2xz + 2yz) \\ xy + 2xz + 2yz = 12. \end{cases}$$

Therefore, from the 1st and 2nd equation (use $\lambda \neq 0$ and $z \neq 0$)

$$xy + 2xz = xy + 2yz \implies 2xz = 2yz \implies x = y.$$

from the 2nd and 3rd equation (use $\lambda \neq 0$ and $z \neq 0$)

$$xy + 2yz = 2xz + 2yz \implies xy = 2xz \implies y = 2x.$$

Therefore

$$x = y = 2z.$$

Use this □