MICHIGAN STATE UNIVERSITY
MATH 234 – SPRING 2024

LECTURE NOTES

17 Maximum and Minimum of functions on a domain with boundary

Given $f(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R}$. To find max/min on D we follows the following steps (mostly from the previous lecture).

1. Find critical points (points where $\nabla f(x,y) = (0,0)$ or it does not exists)

$$\text{Solve for } \nabla f(x,y) = 0.$$

2. Classify local max/min or saddle point (*This step can be skipped if we only care about absolute max/min of the function*)

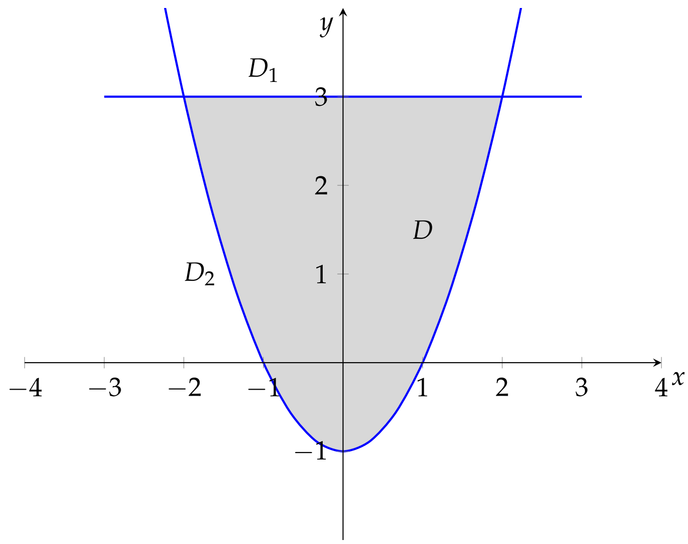
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 \qquad \Longrightarrow \qquad \begin{cases} D < 0: & \text{saddle point} \\ D = 0: & \text{inconclusive} \\ D > 0 & \begin{cases} f_{xx} > 0 & \text{local min} \\ f_{xx} < 0 & \text{local max.} \end{cases} \end{cases}$$

3. Find max/min on the boundary.
   (a) Draw a graph to identify parts of the boundary.
   (b) On each part, write the relation between x and y, use that to simplify $f(x,y)$ into a function of one variable $g(x)$ or $g(y)$, together with the domain of x and y correspondingly.
   (c) Use Cal 2 to find max/min of this one-variable function.
4. Compare all the max/min values found in Step 3 and the values of all critical points in Step 1. The biggest is the absolute max, the smallest is the absolute min.

Notes. When the question is finding absolute maximum or absolute minimum, we can ignore step 2 completely.

Example 1. *Find the absolute max/min values for $f(x,y) = x + y$ on the region D given by the picture below below.*

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 \le y \le 3\}.$$



Proof.

1. Finding the intersection of $y = x^2 - 1$ and $y = 3$, we have $x^2 - 1 = 3$, thus $x^2 = 4$, hence $x = 2$ or $x = -2$.

2. Critical points: $\nabla f(x,y) = (1,1)$. Solving for $\nabla f(x,y) = (0,0)$ does not yield any solution as $(1,1) = (0,0)$ has no solution (x,y). We conclude that there is no critical point.

3. Find max/min on the boundary: there are two parts of the boundary:

   - On D_1, we have $y = -3$ and $-2 \le x \le 2$, and

   $$f(x,y) = x + y = x + 3 \qquad \Longrightarrow \qquad \begin{cases} \text{max } 5 \text{ at } (2,3) \\ \text{min } 1 \text{ at } (-2,3). \end{cases}$$

   - On D_2, we have $y = x^2 - 1$ and $-2 \le x \le 2$, and

   $$f(x,y) = x + y = x + x^2 - 1 = g(x).$$

   This is now a separate question: $\boxed{\text{Find max/min } g(x) = x^2 + x - 1 \text{ over } x \in [-2,2].}$

     – $g'(x) = 0$ iff $2x + 1 = 0$, iff $x = -\frac{1}{2}$. The corresponding $y = x^2 - 1 = \left(-\frac{1}{2}\right)^2 - 1 = -\frac{3}{4}$. We compute $g\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 1 = -\frac{5}{4}$, and $(x,y) = \left(-\frac{1}{2}, -\frac{3}{4}\right)$.
     – If $x = -2$ then $y = 3$, $g(-2) = (-2)^2 + (-2) - 1 = 1$, at $(-2,3)$.
     – If $x = 2$ then $y = 3$, $g(2) = (2)^2 + (2) - 1 = 5$, at $(2,3)$.

4. Comparing all points, we conclude

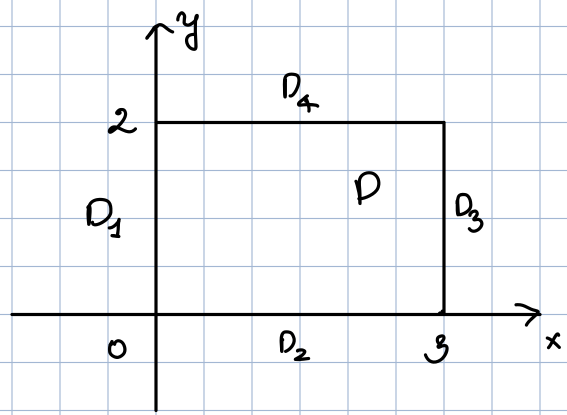
   - Absolute max $= 5$ at $(2,3)$.
   - Absolute min $= -\frac{5}{4}$ at $\left(-\frac{1}{2}, -\frac{3}{4}\right)$.

$\square$

Example. Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle

$$D = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$$

Proof:



- $\nabla f(x,y) = (2x - 2y, -2x + 2) = (0,0)$

  $\begin{aligned} x &= y \\ x &= 1 \end{aligned} \Rightarrow (x,y) = (1,1)$

  $f(1,1) = 1$

- On D_1: $x = 0$, $0 \le y \le 2$

  $g(y) = f(\underset{0}{x}, y) = 2y$ — max 4 at $(0,2)$; min 0 at $(0,0)$

- On D_2: $y = 0$, $0 \le x \le 3$

  $g(x) = f(x, \underset{0}{y}) = x^2$ — max 9 at $(3,0)$; min 0 at $(0,0)$

- On D_3: $x = 3$, $0 \le y \le 2$

  $g(y) = f(\underset{3}{x}, y) = 9 - 6y + 2y = 9 - 4y$ — max 9 at $(3,0)$; min 1 at $(3,2)$

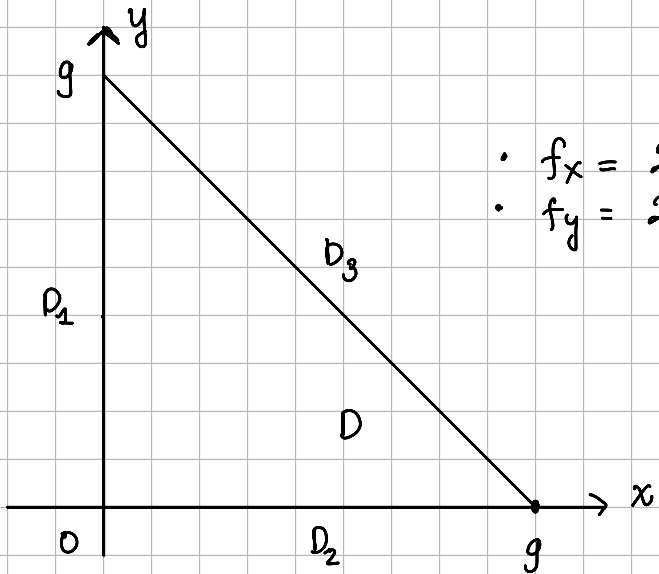
- On D_4: $y = 2$, $0 \le x \le 3$

  $g(x) = f(x, \underset{2}{y}) = x^2 - 4x + 4 = (x-2)^2$

  $g'(x) = 2(x-2) = 0$ —
  $g(2) = 0$ at $(2,2)$
  $g(0) = 4$ at $(0,2)$
  $g(3) = 1$ at $(3,2)$

Thus, the absolute max $= 9$ at $(3,0)$
absolute min $= 0$ at $(0,0)$ or $(2,2)$

Example. $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by and including $y = 9 - x$



- $f_x = 2 - 2x = 0 \Rightarrow x = 1$
- $f_y = 2 - 2y = 0 \Rightarrow y = 1$

$(x,y) = (1,1)$

$f(1,1) = 4$

On D_1: $x = 0$, $0 \le y \le 9$

$g(y) = f(0,y) = 2 + 2y - y^2$

$g'(y) = 2 - 2y = 0$
$\hookrightarrow y = 1$

$g(0) = 2$ at $(0,0)$
$g(1) = 3$ at $(0,1)$
$g(9) = -61$ at $(0,9)$

On D_2: $0 \le x \le 9$, $y = 0$

$g(x) = f(x,0) = 2 + 2x - x^2$

$g'(x) = 2 - 2x = 0$
$\hookrightarrow x = 1$

$g(0) = 2$ at $(0,0)$
$g(1) = 3$ at $(1,0)$
$g(9) = -61$ at $(9,0)$

On D_3: $0 \le x \le 9$, $y = 9 - x$

$g(x) = f(x,y) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$g(x) = 2 + \cancel{2x} + 18 - \cancel{2x} - x^2 - (x^2 - 18x + 91)$

$g(x) = 20 - 2x^2 + 18x - 91$

$g(x) = -2x^2 + 18x - 61$

$g'(x) = -4x + 18 = 0$

$\hookrightarrow x = \frac{18}{4} = \frac{9}{2}$

$g(0) = -61$ at $(0,9)$
$g(\frac{9}{2}) = -\frac{41}{2}$ at $(\frac{9}{2}, \frac{9}{2})$
$g(9) = -61$ at $(9,0)$

Absolute max $= 3$ at $(0,1)$ or $(1,0)$
Absolute min $= -61$ at $(0,9)$ or $(9,0)$

Example: $f(x,y) = 2x^2 - y^2 + 6y$ on the disk D given $x^2+y^2 \leq 16$

a) Find all critical points of f

$$\begin{cases} f_x(x,y) = 4x = 0 \\ f_y(x,y) = -2y + 6 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 3 \end{cases} \qquad (x,y) = (0,3)$$

$$f(0,3) = -9 + 18 = 9 \quad \text{at } (0,3)$$

b) Find the function values of on the bondary of D

$\hookrightarrow x^2+y^2 = 16$

$\hookrightarrow x^2 = 16 - y^2$

$\hookrightarrow -4 \leq y \leq 4$

c)

$$g(y) = f(x,y) = 2(16-y^2) - y^2 + 6y$$

$$g(y) = -3y^2 + 6y + 32$$

$$g'(y) = -6y + 6$$

$$g'(y) = 0 \Rightarrow y = 1$$

$g(-4) = 16 - 24 = -8 \quad (0,-4)$

$g(1) = 35 \quad (\sqrt{15}, 1)$

$g(4) = 8 \quad (0,4)$