MICHIGAN STATE UNIVERSITY
MATH 234 – SPRING 2024

LECTURE NOTES

16 Lecture 16 - Tangent plane review and min/max of functions with several variables

16.1 Tangent plane revisited

Given a surface with equation $F(x, y, z) = 0$, think of $x^2 + \frac{y^2}{9} + \frac{z^2}{9} = 1$ for example.

Question. Given a point $P_0(x_0, y_0, z_0)$ (think of $\left(\frac{1}{3}, 2, 2\right)$ for example) that lies on the surface, find the tangent plane to the surface at the point P_0.

Note. To answer this question, we need to find a normal vector to the surface at P_0.

Method 1.

1. Parametrize the surface by $\mathbf{r}(u, v)$.
2. Then solve for (u_0, v_0) that corresponds to $P_0(x_0, y_0, z_0)$.
3. Compute the normal vector by $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$, let's say it is (a, b, c).
4. The tangent plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Method 2.

1. The normal is given by $\nabla F(x_0, y_0, z_0) = (F_x, F_y, F_z) = (a, b, c)$.
2. The tangent plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

The second method is based on the fact that, if $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a curve in the surface passing through P_0, then

$$F(x(t), y(t), z(t)) = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}F(x(t), y(t), z(t)) = 0$$

Therefore

$$(F_x, F_y, F_z) \cdot (x'(t), y'(t), z'(t)) = 0.$$

Here $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ is a tangent vector to the curve, thus belongs to the tangent plane at P_0. In other words, ∇F is the normal vector to the tangent plane at P_0.

Proof using method 1. We can do

$$\begin{cases} x = \sin\phi\cos\theta \\ \frac{y}{3} = \sin\phi\sin\theta \\ \frac{z}{3} = \cos\phi \end{cases}$$

In other words,

$$r(\phi,\theta) = (\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi).$$

To solve for (ϕ,θ) at the point $P_0\left(\frac{1}{3},2,2\right)$ we solve

$$\begin{cases} \sin\phi\cos\theta & = \frac{1}{3} \\ \sin\phi\sin\theta & = \frac{2}{3} \\ \cos\phi & = \frac{2}{3} \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \sin\phi = \frac{\sqrt{5}}{3} \\ \cos\theta = \frac{1}{\sqrt{5}} \\ \sin\theta = \frac{2}{\sqrt{5}} \end{cases}$$

We have

$$\mathbf{r}_\phi = (\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi)$$
$$\mathbf{r}_\theta = (-\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0)$$

We compute the normal vector

$$\mathbf{n} = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix} = \left(9\sin^2\phi\cos\theta, 3\sin^2\phi\sin\theta, 3\sin\phi\cos\phi\right)$$
$$= \left(9\times\frac{5}{9}\times\frac{1}{\sqrt{5}}, 3\times\frac{5}{9}\times\frac{2}{\sqrt{5}}, 3\times\frac{\sqrt{5}}{3}\times\frac{2}{3}\right) = \left(\sqrt{5}, \frac{2\sqrt{5}}{3}, \frac{2\sqrt{5}}{3}\right).$$

We can simplify by choosing

$$\mathbf{n} = \left(1, \frac{2}{3}, \frac{2}{2}\right)$$

and the tangent plane at $P\left(\frac{1}{3},2,2\right)$ is

$$\boxed{\left(x-\frac{1}{3}\right) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2) = 0.}$$

□

Proof using method 2. We have

$$\nabla F(x,y,z) = \left(2x, \frac{2y}{9}, \frac{2z}{9}\right) \qquad \Longrightarrow \qquad \mathbf{n} = \nabla F\left(\frac{1}{3},2,2\right) = \left(\frac{2}{3}, \frac{4}{9}, \frac{4}{9}\right).$$

We can choose the parallel vector

$$\mathbf{n} = \left(1, \frac{2}{3}, \frac{2}{3}\right)$$

and thus at $P\left(\frac{1}{3},2,2\right)$ we get the tangent plane

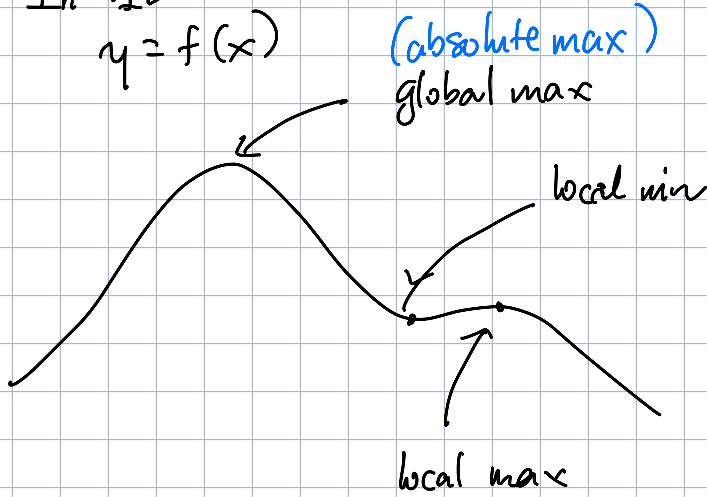
$$\boxed{\left(x-\frac{1}{3}\right) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2) = 0.}$$

□

16.2 Critical points, local min, local max and saddle points

To find min / max of a function $f(x,y)$ of 2 variables

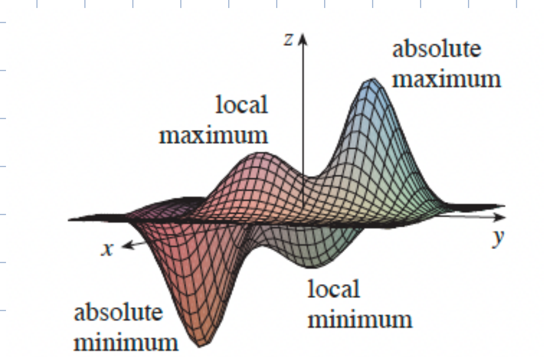
In 1D
$y = f(x)$



Procedure:

- Critical points:
  solves for x s.t. $f'(x) = 0$
- For a critical points: x_0
  - $\to f''(x_0) > 0$: local min
  - $\to f''(x_0) < 0$, local max

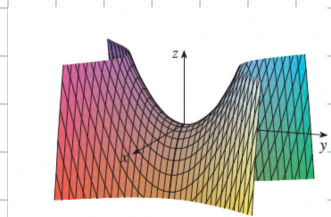
In 2D
$z = f(x,y)$



Definition:

local max : $f(x,y) \leq f(x_0, y_0)$
for (x,y) near (x_0,y_0)

absolute max or global max
$f(x,y) \leq f(x_0, y_0)$ for all
(x,y) in the domain of f



saddle point

Procedure,

- Critical points, point (x_0, y_0) where

  $\nabla f(x_0,y_0) = 0$ or $\nabla f(x_0,y_0)$ does not exist.

  $(f_x(x_0,y_0), f_y(x_0,y_0)) = (0,0)$

- Compute second derivatives

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Find the determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

(recall $f_{xy} = f_{yx}$ for nice functions)

$\longrightarrow D < 0$: saddle point
(neither max ~~nor~~ nor min)

$\longrightarrow D > 0$
- $\to f_{xx} > 0$, local min
- $\to f_{xx} < 0$: local max

Example 1: $f(x,y) = y^2 - x^2$

Step 1. $\nabla f(x,y) = \left(f_x(x,y), f_y(x,y)\right) = (-2x, 2y)$

Solve: $\begin{cases} -2x = 0 \\ 2y = 0 \end{cases} \Rightarrow (x,y) = (0,0)$

Step 2: $f_{xx} = -2 \qquad f_{xy} = 0$

$f_{yx} = 0 \qquad f_{yy} = 2$

$$D = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0 \Rightarrow \text{saddle point at } (0,0)$$

Example 2: $f(x,y) = x^3 + 3xy + y^3$

Step 1. $\nabla f(x,y) = \left(f_x(x,y), f_y(x,y)\right) = (3x^2 + 3y, 3y^2 + 3x)$

Solve: $\begin{cases} 3x^2+3y=0 \\ 3y^2+3x=0 \end{cases} \Leftrightarrow \begin{cases} x^2+y=0 \\ y^2+x=0 \end{cases} \Rightarrow y = -x^2$

$(-x^2)^2 + x = 0$

$x^4 + x = 0$

$x(x^3+1) = 0$

$x = 0 \Rightarrow y = 0$

$x = -1 \quad y = -1$

$(y = -x^2)$

2 points: $(0,0)$ and $(-1,-1)$

Step 2

$f_{xx}(x,y) = 6x \qquad f_{xy}(x,y) = 3$

$f_{yx}(x,y) = 3 \qquad f_{yy}(x,y) = 6y$

$$D = \begin{vmatrix} 6x & 3 \\ 3 & 6y \end{vmatrix} = 36xy - 9$$

At $(0,0)$: $D = -9 < 0 \rightsquigarrow (0,0)$ is a saddle point

At $(-1,-1)$: $D = 27 > 0$, $f_{xx} = -6 < 0 \rightsquigarrow (-1,-1)$ is a local max

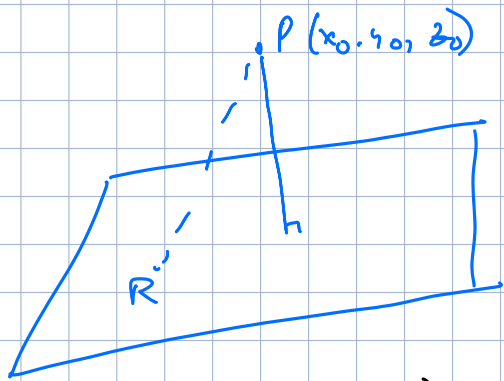
Example 3: Find the shortest distance from the point $(1,0,2)$ to the plane $x+2y+z=4$

Proof 1. use formula $P(x_0,y_0,z_0)$
$ax+by+cz+d=0$ plane.

$$\boxed{d=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}}=\frac{|1+2.0+2-4|}{\sqrt{1^2+2^2+1^2}}$$

$$=\frac{1}{\sqrt{6}}$$

the shortest distance is the perpendicular (projection)



Proof 2 Take any point $R(x,y,z)$ in the plane $x+2y+z=4$ $P(1,0,2)$

Compute the distance

$$|PR| = \sqrt{(x-1)^2+(y-0)^2+(z-2)^2}$$

Q: find $(x,y,z) \in$ plane s.t. $\sqrt{(x-1)^2+(y-0)^2+(z-2)^2}$ is minimum
then the answer is this min value.

Q. find $(x,y,z) \in (x+2y+z=4)$
s.t. $(x-1)^2+y^2+(z-2)^2$ is minimum

Q. find minimum of $(x-1)^2+y^2+(z-2)^2$

subjected to $x+2y+z=4$

Q. use $z = 4-x-2y$

find minimum of

$$f(x,y) = (x-1)^2+y^2+(4-x-2y-2)^2$$

$$f(x,y) = (x-1)^2+y^2+(2-x-2y)^2$$

Step 1. $f_x(x,y) = 2(x-1)+2(2-x-2y)(-1)$

$= 2x-2-2(2-x-2y)$

$= 2x-2-4+2x+4y$

$$\boxed{f_x(x,y) = 4x+4y-6}$$

$f_y(x,y) = 2y+2(2-x-2y)(-2)$

$= 2y-4(2-x-2y)$

$= 2y-8+4x+8y$

$$\boxed{f_y(x,y) = 4x+10y-8}$$

Solve for: $\begin{cases} 4x+4y-6=0 \\ 4x+10y-8=0 \end{cases} \Rightarrow -6y+2=0$

$y = \frac{1}{3}$

$x = -y+\frac{3}{2} = -\frac{1}{3}+\frac{3}{2} = \frac{7}{6}$

$(x,y) = \left(\frac{7}{6}, \frac{1}{3}\right)$

Step 2 : $f_{xx} = 4$ $\quad f_{xy} = 4$

$f_{yx} = 4$ $\quad f_{yy} = 10$

$$D = \begin{vmatrix} 4 & 4 \\ 4 & 10 \end{vmatrix} = 40 - 16 = 24 > 0$$

$f_{xx} = 4 \Rightarrow \left(\frac{7}{6}, \frac{1}{3}\right)$

is a minimum

We conclue that min achived at $\left(\frac{7}{6}, \frac{1}{3}\right)$

with min value is

$$f(x,y) = (x-1)^2 + y^2 + (2 \cdot x - 2y)^2$$

$$= \left(\frac{7}{6} - 1\right)^2 + \left(\frac{1}{3}\right)^2 + \left(2 - \frac{7}{6} - \frac{2}{3}\right)^2$$

$$= \left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2 = \frac{6}{36} = \frac{1}{6}$$

the distance$_{min}$ $= \frac{1}{\sqrt{6}}$ as proof 1