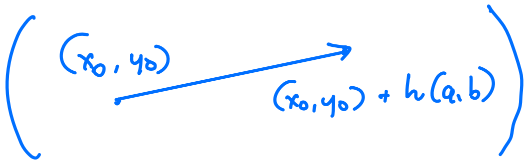
Directional derivatives

Def: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of the unit vector $\vec{u}=(a,b)$ is

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha,\ y_0+hb) - f(x_0,y_0)}{h}$$



if this limit exists

$\boxed{\text{Example. } z = x^2+y^2 \text{ at } (1,1) \text{ in } \vec{u}=(1,2)}$,

$f(x,y) = x^2+y^2$

$$D_{\vec{u}} f(1,1) = \lim_{h\to 0} \frac{f((1,1)+h(1,2)) - f((1,1))}{h} = \lim_{h\to 0} \frac{f(1+h,1+2h) - f(1,1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^2 + (1+2h)^2 - 1^2 - 1^2}{h} = \lim_{h\to 0} \frac{(h^2+2h)+(h^2+4h)}{h}$$

$$= 2+4 = \boxed{6}$$

Def. The gradient of f at (x_0,y_0) is a vector

$$\boxed{\nabla f(x_0,y_0) = \left(f_x(x_0,y_0),\ f_y(x_0,y_0)\right)}$$

partial derivative

Note: $\vec{v} = (1,0)$

$D_{\vec{v}} f(x,y) = f_x(x,y)$

$\vec{v} = (0,1)$

$D_{\vec{v}} f(x,y) = f_y(x,y)$

Theorem:

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u}$$

dot product

$\boxed{\text{Example. } z = x^2+y^2 \text{ at } (1,1) \text{ in } \vec{u}=(1,2)}$

$f(x,y) = x^2+y^2$

$\nabla f(x,y) = (2x, 2y)$

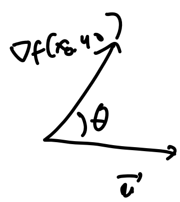
$\nabla f(1,1) = (2,2)$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1)\cdot(1,2)$$

$$= (2,2)\cdot(1,2)$$

$$= 2+4 = \boxed{6}$$

Question: What is the direction we should go to maximize / minimize the directional derivative (the function increases / decreases the most)?



Theorem: u is unit

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u} = |\nabla f(x_0,y_0)|\cdot\underbrace{|\vec{u}|}_{1}\cdot\cos\theta$$

max : $|\nabla f(x_0,y_0)|$ when $\vec{u} = \dfrac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$ (unit)

min : $-|\nabla f(x_0,y_0)|$ when $\vec{u} = -\dfrac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$.

($\vec{u}$ is the same direction of $\nabla f(x_0,y_0)$)

Example: $f(x,y,z) = \sin(xy) + z$ at $(1,0,1)$, what is the max rate of change?

$$\nabla f(x,y,z) = (\cos(xy)\cdot y,\ \cos(xy)\cdot x,\ 1)$$
$$\nabla f(1,0,1) = (\cos(0)\cdot 0,\ \cos(0)\cdot 1,\ 1) = (0,1,1)$$

max rate $= |\nabla f(1,0,1)| = |(0,1,1)| = \sqrt{2}$ when direction $\vec{v} = \dfrac{(0,1,1)}{\sqrt{2}}$.

Theorem: If f is differentiable, the function sees no changes when the directional derivative is zero
$D_u f(x_0,y_0) = 0$, i.e., $\vec{u} \perp \nabla f(x_0,y_0)$

Example: $f(x,y) = \dfrac{y^2}{x}$, the point is $P(1,2)$

a) Find the gradient of f at $(1,2)$: $\nabla f(x,y) = \left(-\dfrac{y^2}{x^2},\ 2\dfrac{y}{x}\right)$

$\nabla f(1,2) = (-4, 4)$

b) Find the rate of change of f at $(1,2)$ in the direction of $\vec{v} = (2,\sqrt{5})$

$$D_u f(1,2) = \nabla f(1,2)\cdot\vec{u} = (-4,4)\cdot(2,\sqrt{5}) = -8 + 4\sqrt{5}$$

c) Find a direction $\vec{v}$ in which f neither increases nor decreases

Find $\vec{v} = (a,b)$ (unit) s.t. $D_u f(1,2)\cdot\vec{v} = 0$

$(-4,4)\cdot(a,b) = 0 \Rightarrow -4a + 4b = 0$
$\Rightarrow -4(a-b) = 0$

Choose $(a,b) = (1,1)$ or $(-1,1)$

$\vec{v} = \dfrac{(1,1)}{\sqrt{2}}$ or $-\dfrac{(1,1)}{\sqrt{2}}$

Application: finding tangent line to level surface

"normal vector of $f(x,y,z) = f(x_0,y_0,z_0)$ at (x_0,y_0,z_0)"

is given by $\nabla f(x_0,y_0,z_0)$

Example : $x^2 + \frac{y^2}{9} + \frac{z^2}{9} = 1$ at $\left(\frac{1}{3}, 2, 2\right)$

take $\vec{n} = \nabla f(x,y,z) = \left(2x, \frac{2y}{9}, \frac{2z}{9}\right)$

at $\left(\frac{1}{3}, 2, 2\right)$: $\vec{n} = \left(\frac{2}{3}, \frac{4}{9}, \frac{4}{9}\right) \upuparrows (6, 4, 4)$

thus the tangent plane:

$$\boxed{6\left(x - \frac{1}{3}\right) + 4(y-2) + 4(z-2) = 0}$$

(this is easier than using parametric surfaces)

Normal line : to a surface at a point (x_0, y_0, z_0)

is $\vec{r}(t) = (x_0, y_0, z_0) + t \cdot \vec{n}$

normal to the tangent plane at that point.