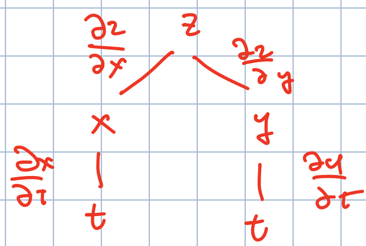
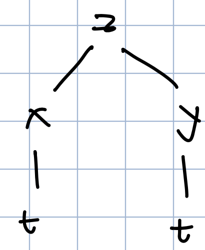
The chain rule

$$\boxed{z = f(x,y), \quad x = x(t),\ y = y(t) \quad \text{then} \quad \frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}}$$



Example:

$$z = x^2 + y^2 + xy \qquad x = \sin t, \quad y = e^t$$

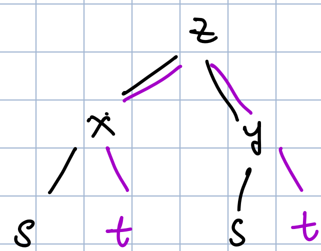


$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$(2x+y)\cdot(\cos t) + (2y+x)\cdot(e^t)$$

Def:

$$z = f(x,y) \qquad x = x(s,t), \quad y = y(s,t)$$

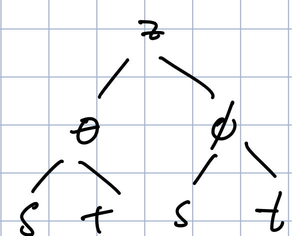


$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Example

$$z = \sin\theta\cos\phi \qquad \theta = st^2 \qquad \phi = s^2 t$$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\cdot\frac{\partial \phi}{\partial s}$$

$$= (\cos\theta\cos\phi)\cdot(t^2) + (-\sin\theta\sin\phi)(2st)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\cdot\frac{\partial \phi}{\partial t}$$

$$= (\cos\theta\cos\phi)(2st) + (-\sin\theta\sin\phi)(s^2)$$

Example $z = f(x,y)$, f is differential

$x = x(t)$ | $y = y(t)$

$x(3) = 2$ | $y(3) = 7$

$x'(3) = 5$ | $y'(3) = -4$

$f_x(2,7) = 6$ | $f_y(2,7) = -8$

Find $\frac{dz}{dt}$ when $t = 3$ Note : $t = 3 \Rightarrow (x(t), y(t)) = (2,7)$

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$f_x(2,7)\cdot x'(3) + f_y(2,7)\cdot y'(3)$$

$$6 \cdot 5 + (-8)\cdot(-4) = 30 + 32 = 62.$$

Implicit differentiation :

Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

treat y as a function of x : (x is the variable)

$y' = \frac{dy}{dx}$

$$3x^2 + 3y^2\cdot y' = 6(xy)'$$

$$= 6(x')y + 6x\cdot y'$$

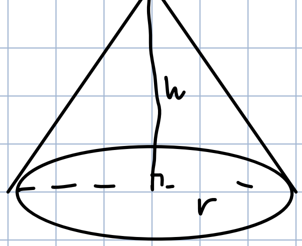
$$= 6y + 6xy'$$

$$3x^2 - 6y = 6xy' - 3y^2y' = y'(6x - 3y^2)$$

$$\Rightarrow \quad y' = \boxed{\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}}$$

Example : The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s

At what rate is the volume of the cone changing when the radius is 120 in. height is 140 in.



$V(h,r) = \frac{1}{3} r^2 h$ $\qquad$ $r = r(t)$, $h = h(t)$

$$\frac{dV}{dt} = \frac{\partial V}{\partial h} \cdot \frac{\partial h}{\partial t} + \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial t}$$

$$= \left(\frac{1}{3} r^2\right) \cdot h'(t) + \left(\frac{2}{3} r h\right) \cdot r'(t)$$

$$\text{at } \overline{(r,h)} = (120, 140) = \left(\frac{1}{3} \cdot 120^2\right) \times (-2.5) + \left(\frac{2}{3} 120 \times 140\right) \times (1.8)$$

Example.

One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the are of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

Ex. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x,y,z) = 0$

$$x^3 + y^3 + z^3 + 6xyz = 1$$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

$F(x,y) = 0$ $\qquad$ $F(x,y,z) = 0$.