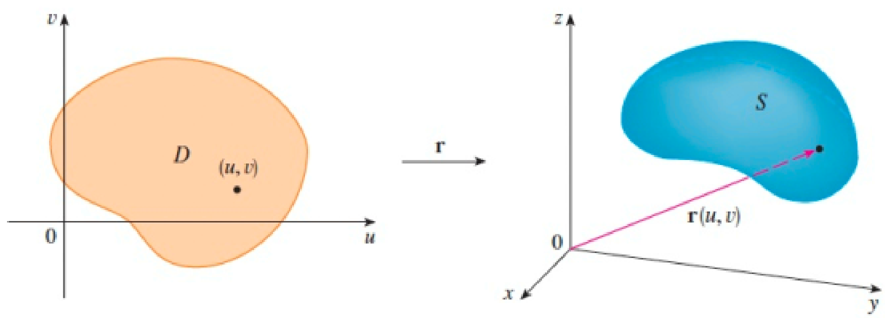
Michigan State University

Math 234 – Spring 2024

Lecture notes

1 Parametric surfaces

- A curve is a function with one parameter $\mathbf{r}() = (x(t), y(t), z(t))$.
  1. Example 1. $\mathbf{r}(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$, this is a circle in xy-plane ($z = 0$).
  2. Example 2. $\mathbf{r}(t) = (t, 2t, 3t)$, $t \in \mathbb{R}$, this is a line going through $(0, 0, 0)$ with direction $\mathbf{v} = (1, 2, 3)$.
- A parametric surface is a function with two parameter $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D$.



Example 1. $\boldsymbol{r}(u, v) = (u, v, 1 - u - v)$, $(u, v) \in \mathbb{R}^2$.

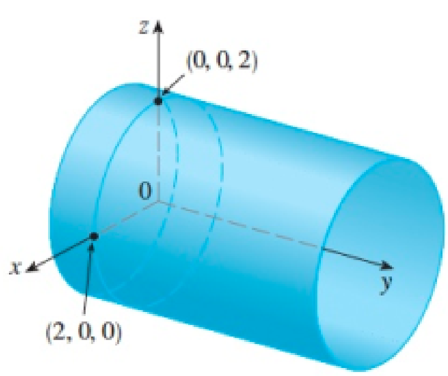
- *This is the plane* $x + y + z = u + v + (1 - u - v) = 1$.
- *We can also view it as*

  $$(u, v, 1 - u - v) = (0, 0, 1) + (u, 0, -u) + (0, v, -v) = (0, 0, 1) + u(1, 0, -1) + v(0, 1, -1), \quad (u, v) \in \mathbb{R}^2.$$

  In this way, the plane is the one containing $(0, 0, 1)$ *and all vectors in the planes generated by* $(1, 0, -1)$ *and* $(0, 1, -1)$.

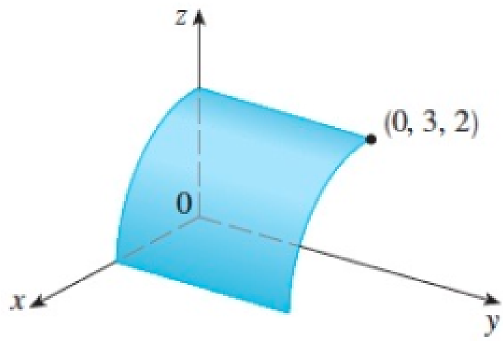
Example 2. $\boldsymbol{r}(u, v) = (2\cos u, v, 2\sin u)$, $u \in [0, 2\pi]$, $v \in \mathbb{R}$.

- *Look at* $x = 2\cos u, y = v, z = 2\sin u$, *thus* $x^2 + z^2 = 4$, *while* $y \in \mathbb{R}$. *This is a cylinder.*



Example 3. $\boldsymbol{r}(u,v) = (2\cos u, v, 2\sin u)$, $u \in \left[0, \frac{\pi}{2}\right]$, $v \in [0,3]$.

- *Look at* $x = 2\cos u, y = v, z = 2\sin u$, *thus* $x^2 + z^2 = 4$, *while* $y \in \mathbb{R}$. *This is a cylinder.*
- *Note the angle* θ *in the Oxz-plane is* $\pi/4$, *thus only a quarter of the Oxz-plane is covered.*



2 Parametrize a surface in x, y, z

Example 4. *Find a parametric equation for* $x^2 + y^2 = 4, 0 \leq z \leq 1$.

Proof. We can use polar coordinates $x = 2\cos\theta, y = 2\sin\theta$ and $0 \leq z \leq 1$, thus

$$r(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

The domain is $D = \{(\theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1\}$. □

Example 5. *Find a parametric equation for* $z = 2\sqrt{x^2 + y^2}, 0 \leq z \leq 1$.

Proof 1. We can just use the graph

$$r(x, y) = (x, y, z) = (x, y, 2\sqrt{x^2 + y^2}).$$

Note the condition $0 \leq z \leq 1$ means $0 \leq 2\sqrt{x^2 + y^2} \leq 1$, thus $x^2 + y^2 \leq \frac{1}{4}$. Therefore

$$D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq \frac{1}{4}\right\}.$$

□

Proof 2. We can use polar coordinates $x = r\cos\theta, y = r\sin\theta$ and $0 \leq z = 2r \leq 1$ which means $0 \leq r \leq \frac{1}{2}$, thus

$$\mathbf{r}(\theta, z) = (r\cos\theta, r\sin\theta, 2r)$$

The domain now is

$$D = \left\{(\theta, r) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq \frac{1}{2}\right\}.$$

□

3 Grid

For a parametric surface $\mathbf{r}(u,v)$, if we:

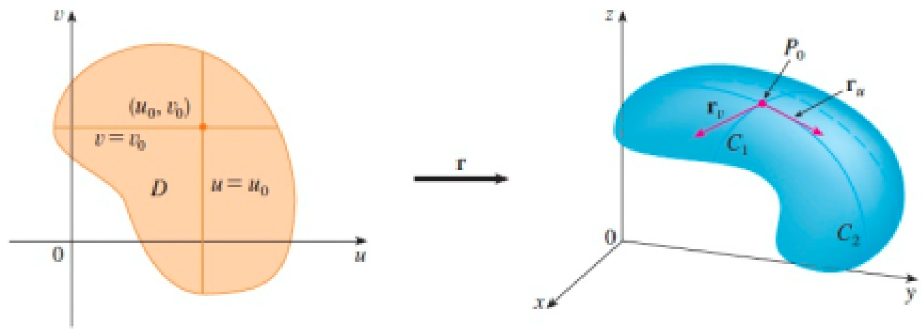
- Fix $u = u_0$, run v we get the images as a curvy grid on the surface
- Fix $v = v_0$, run u we get the images as a curvy grid on the surface

The two direction at each point $(x_0, y_0, z_0) = r(u_0, v_0)$ form a tangent plane at that point. The two directions here are the partial derivatives

$$\mathbf{r}_u \qquad \text{and} \qquad \mathbf{r}_v.$$

The normal vector of the tangent plane is

$$\mathbf{n} = r_{\mathbf{u}} \times r_{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}.$$



Example 6. *Find the tangent plane of $x = u^2, y = v^2, z = u + 2v$ at $(1,1,3)$.*

Proof.

- Step 1. Solve for (u,v):

$$\begin{cases} x = u^2 = 1 \\ y = v^2 = 1 \\ z = u + 2v = 3 \end{cases} \implies \begin{cases} u = \pm 1 \\ v = \pm 1 \\ u + 2v = 3 \end{cases} \implies \begin{cases} u = 1 \\ v = 1 \end{cases}$$

- Step 2. Compute the partial derivatives of $\mathbf{r}(u,v) = (u^2, v^2, u + 2v)$

$$\mathbf{r}_u = (2u, 0, 1)$$
$$\mathbf{r}_v = (0, 2v, 2).$$

- Step 3. Plug in the value $u = v = 1$ to get

$$\begin{cases} \mathbf{r}_u = (2,0,1) \\ \mathbf{r}_u = (0,2,2) \end{cases}$$

- Step 4. Compute the normal by cross product

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2, -4, 4)$$

- The tangent plane with normal $(-2,-4,-4)$ going through $(1,1,3)$ is

$$\boxed{-2(x-1)-4(y-1)-4(z-3)=0.}$$

□

Example 7. *Find the tangent plane of $x=u^2+1, y=v^3+1, z=u+v$ at $(5,2,3)$.*

Proof.

- Step 1. Solve for (u,v):

$$\begin{cases} x=u^2+1=5 \\ y=v^3+1=2 \\ z=u+v=3 \end{cases} \implies \begin{cases} u=2 \\ v=1. \end{cases}$$

- Step 2. Compute the partial derivatives of $\mathbf{r}(u,v)=(u^2+1,v^3+1,u+v)$

$$\mathbf{r}_u=(2u,0,1)$$
$$\mathbf{r}_v=(0,3v^2,1).$$

- Step 3. Plug in the value $u=2, v=1$ to get

$$\begin{cases} \mathbf{r}_u=(4,0,1) \\ \mathbf{r}_u=(0,3,1) \end{cases}$$

- Step 4. Compute the normal by cross product

$$\mathbf{n}=\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}=(-3,-4,12)$$

- The tangent plane with normal $(-3,-4,12)$ going through $(5,2,3)$ is

$$\boxed{-3(x-5)-4(y-2)+12(z-3)=0.}$$

□