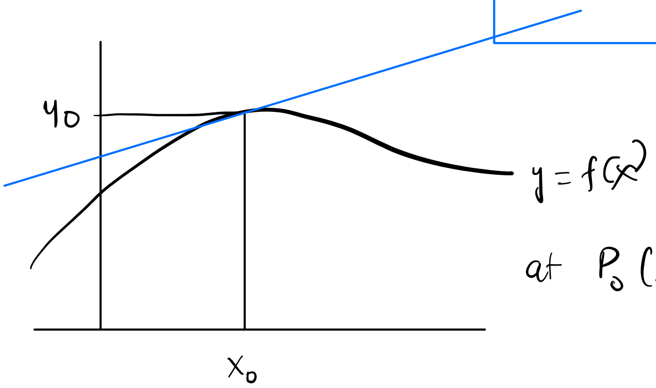
## Tangent line in 2D

$y = f(x)$

$$\boxed{\text{the line} \quad y - y_0 = f'(x_0)\,(x - x_0)}$$

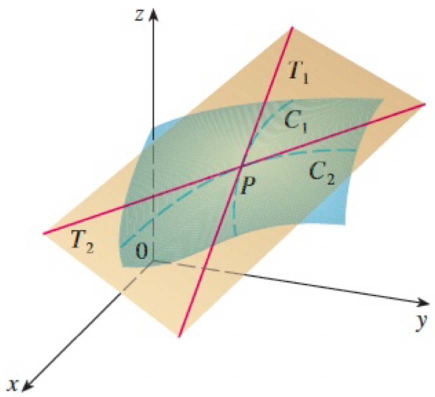



at $P_0\,(x_0, y_0)$

## Tangent plane in 3D

$z = f(x,y)$ at $P_0\,(x_0, y_0, z_0)$

$$\boxed{\text{the plane} \quad z - z_0 = f_x\,(x_0, y_0)\,(x - x_0) + f_y\,(x_0, y_0)\,(y - y_0)}$$




Definition: the plane contains both tangent lines to $C_1, C_2$, the curves of intersection between $z = f(x,y)$ between $x = x_0$ and $y = y_0$, respectively.

## Explanation of the formula

We can write $\quad f_x\,(x_0, y_0)\,(x - x_0) + f_y\,(x_0, y_0)\,(y - y_0) - (z - z_0) = 0$

$$\underbrace{\big(f_x\,(x_0, y_0),\ f_y\,(x_0, y_0),\ -1\big)}_{\substack{\vec{v} \\ \text{(normal)}}} \cdot \big(\underbrace{(x, y, z)}_{P} - \underbrace{(x_0, y_0, z_0)}_{P_0}\big) = 0$$

so : $\vec{v} \cdot (P - P_0) = 0$

set of all points $\perp$ normal, going through $P_0$ !

**Example 1.** *Find equation of the tangent plane of the surface $z = f(x,y) = 3y^2 - 2x^2 + x$ at the point $(2,-1,3)$.*

*Proof.*

- Step 1. We write the general form

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

- Step 2. Plug in $(x_0,y_0,z_0) = (2,-1,-3)$

$$z-(-3) = f_x(2,-1)(x-2) + f_y(2,-1)(y-(-1)).$$
$$z+3 = f_x(2,-1)(x-2) + f_y(2,-1)(y+1).$$

- Step 3. Compute the partial derivatives

$$f_x(x,y) = -4x+1, \qquad f_x(2,-1) = -7$$
$$f_y(x,y) = 6y, \qquad f_y(2,-1) = -6.$$

- Step 4. Final answer

$$z+3 = -7(x-2) - 6(y+1).$$

## Linear approximation

The linearization of $f(x,y)$ at $(x_0,y_0)$ is given by

$$L(x,y) = \underbrace{f(x_0,y_0)}_{z_0} + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Compare with tangent plane

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(z-z_0)$$

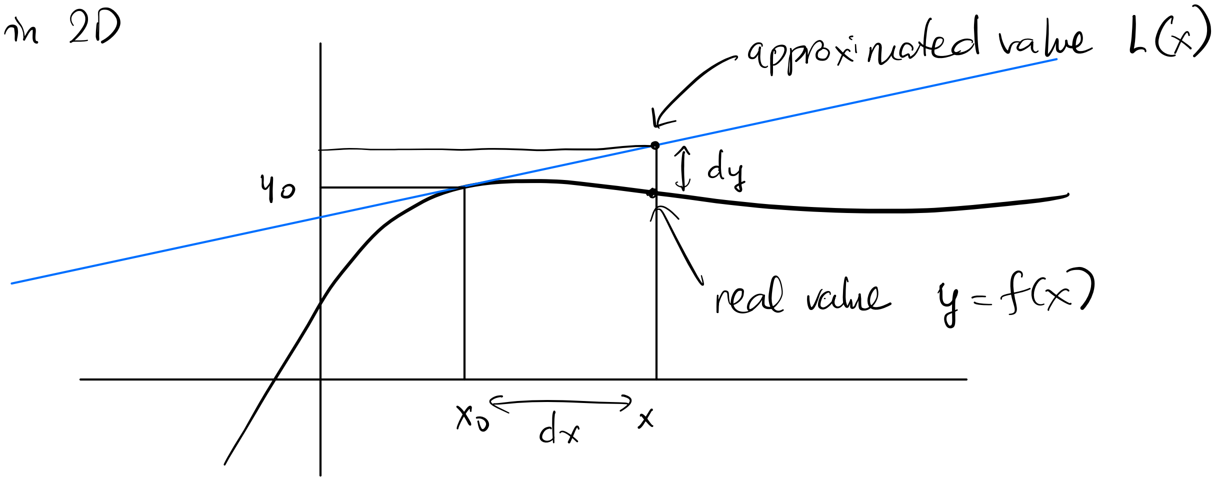
The idea is : $L(x,y)$ can be used as an approximation for the true value $f(x,y)$ if $(x,y)$ near $(x_0,y_0)$

$$f(x,y) \simeq L(x,y)$$

$\hookleftarrow$ the Linear approximation of $f$ at $(x_0,y_0)$

or the tangent plane approximation

picture in 2D




if $|x - x_0|$ small then $y = f(x) \approx L(x)$

**Example 2.** *Find the linear approximation of* $f(x,y) = \frac{2x+3}{4y+1}$ *at* $(0,0)$*. Use it to approximate* $f(0.1,-0.2)$ *and* $f(0.01,-0.02)$.

*Proof.* Find the tangent plane at $(x_0, y_0) = (0,0)$ with $z_0 = f(x_0, y_0) = 3$.

- Step 1. We write the general form

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Step 2. Plug in $(x_0, y_0, z_0) = (0,0,3)$

$$z - 3 = f_x(0,0)(x - 0) + f_y(0,0)(y - 0.$$
$$z - 3 = f_x(2,-1)x + f_y(2,-1)y$$

- Step 3. Compute the partial derivatives

$$f_x(x,y) = \frac{2}{4y+1}, \qquad f_x(0,0) = 2$$
$$f_y(x,y) = (2x+3)\frac{-4}{(4y+1)^2}, \qquad f_y(0,0) = -12$$

- Step 4. The tangent plane

$$z + 3 = 2x - 12y$$

- Step 5. The linear approximation

$$L(x,y) = 2x - 12y - 3$$

- Now plug in the value

$$L(0.1,-0.2) = 5.6 \qquad \text{while the true value} \qquad f(0.1,-0.2) = 16$$

  Here the change in $x, y$ are 0.1 and $-0.2$. Now

$$L(0.01,-0.02) = 3.26 \qquad \text{while the true value} \qquad f(0.01,-0.02) = 3.28$$

  Here the change in $x, y$ are 0.01 and $-0.02$.

We see that if the changes in $x$ and $y$ are small then the approximation is good! □

Total differential $z = f(x,y)$

call $dx$ the small change in $x$, $dy$ the small change in $y$

$(dx \approx x - x_0)$ $\qquad$ $dy \approx y - y_0$

then the total differential is

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

Meaning: $dz$ measures how much changes in $z = f(x,y)$ if $x,y$ change by $dx, dy$ (small).

**Example 3.** *Consider $z = f(x,y) = x^2 + 3xy - y^2$. Find $dz$. If $x$ changes from $2 \to 2.05$ and $y$ changes from $3 \to 2.96$, compare the value of $\Delta z$ (true differences) and $dz$ (the total differential).*

*Proof.*

- Step 1. Here $(x_0, y_0) = (2,3)$ and $dx = 0.05$, $dy = -0.04$.
- Step 2. Write the total differential formula $dz = f_x(2,3)dx + f_y(2,3)dy$.
- Step 3. Compute the partial derivatives

$$\begin{aligned} f_x(x,y) &= 2x + 3y, \qquad & f_x(2,3) &= 13 \\ f_y(x,y) &= 3x - 2y, \qquad & f_y(2,3) &= 0 \end{aligned}$$

- Step 4. Plug in to the formula $dz$

$$dz = 13 \times (0.05) + 0 \times (-0.04) = 0.65$$

- Step 5. The true value

$$\begin{cases} f(2.05, -2.96) = (2.05)^2 - 3 \times (2.05) \times (-2.96) - (-2.96)^2 = 13.6449 \\ f(2,3) = 2^2 - 3 \times \times 3 - 3^2 = 13 \end{cases} \implies \quad \Delta z = 0.06449$$

We see that $\Delta z \approx dz$, but $dz$ is much easier to compute.

**Example 4.** *The dimensions of a box are measure to be 10cm, 5cm, 8cm. If each measurement is correct within 0.2cm, approximate the largest possible error when the volume of the box is calculated from these measurements.*

*Proof.* We have $V(x,y,z) = xyz$, thus

$$\begin{aligned} dV &= V_x dx + V_y dy + V_z dz \\ &= yzdx + xzdy + xydz \\ &= 0.2(yz + xz + xy) = 0.2(10 \times 5 + t \times 8 + 8 \times 10) = 170 \times 0.2 = 34cm^3 \end{aligned}$$

The error is at most $34cm^3$. $\square$

**Example 4.11.** Consider the surface $x^2 + \frac{y^2}{9} + \frac{z^2}{9} = 1$ ellipsoid

(a) Find the equation of the tangent plane to this surface at the point $\left(\frac{1}{3}, 2, 2\right)$. $(x_0, y_0, z_0) = \left(\frac{1}{3}, 2, 2\right)$

$z = (9 - 9x^2 - y^2)^{1/2}$ → choose this since $z_0 = 2 > 0$

or

$z = -(9 - 9x^2 - y^2)^{1/2}$

$f_x = \frac{1}{2}(9 - 9x^2 - y^2)^{-1/2}(-18x)$ at $\left(\frac{1}{3}, 2\right)$ ⇝ $f_x = -3/2$

$f_y = \frac{1}{2}(9 - 9x^2 - y^2)^{-1/2}(-2y)$ ⇝ $f_y = -1$

$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

$$\boxed{z - 2 = -\frac{3}{2}\left(x - \frac{1}{3}\right) - 1(y - 2)}$$

(b) Find a point at which the tangent plane to this surface is horizontal. Are there any other such points?

tangent plane : $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

↳ tangent : $(f_x, f_y, -1)$

tangent plane // $Oxy$ (has tangent $Oz$) $\Rightarrow$ tangent $(f_x, f_y, -1)$ // $Oz = (0, 0, 1)$

solve $f_x = 0$, $f_y = 0$ $\Rightarrow \begin{cases} 1/2\,(9 - 9x^2 - y^2)^{-1/2}(-18x) = 0 \\ 1/2\,(9 - 9x^2 - y^2)^{-1/2}(-2y) = 0 \end{cases}$

↳ $(x, y) = (0, 0)$

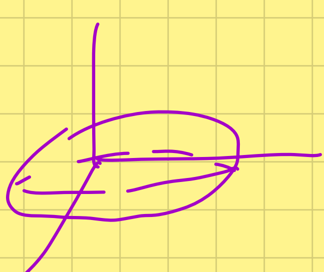
the point is $x^2 + \frac{y^2}{9} + \frac{z^2}{9} = 1 \Rightarrow \frac{z^2}{9} = 1 \Rightarrow z = 3$, $z = -3$

$\boxed{(0, 0, 3) \quad (0, 0, -3)}$

(c) Find a point at which the tangent plane to this surface is vertical. Are there any other such points?

as an ellipsoid, to have the tangent plane is vertical

$\Rightarrow$ any point on the equator work



let $z = 0 \Rightarrow x^2 + \frac{y^2}{9} = 1$

↳ parametrize : $(\cos t, 3\sin t, 0)$, $t \in [0, 2\pi]$

↳ all points on the equator