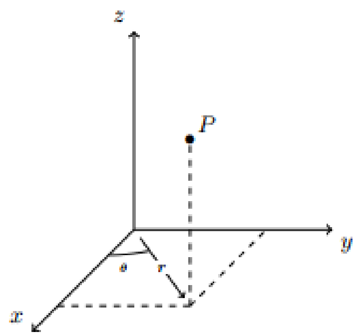
Michigan State University
Math 234 – Spring 2024

Lecture notes

1 Cylindrical coordinates

- Cylindrical coordinates represent a point $P(x, y, z)$ in space by ordered triples (r, θ, z) in which (r, θ) is the polar coordinate of (x, y).
- z remains unchanged.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{and} \quad \begin{cases} r^2 = x^2 + y^2 \\ \dfrac{y}{x} = \tan\theta. \end{cases}$$



- **Example.** Change $(x, y, z) = (-1, 1, 1)$ into cylindrical coordinates.

  Proof. $r^2 = x^2 + y^2 = 2$, thus $r = \sqrt{2}$. Then $\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$, thus $\theta = \frac{3\pi}{4}$. Hence

$$(-1, 1, 1) \mapsto \left(\sqrt{2}, \frac{3\pi}{4}, 1\right).$$

□

- **Example.** Change $(\sqrt{2}, 3\pi/4, 2)$ to Cartesian coordinates.

  Proof. We have $x = r\cos\theta = \sqrt{2} \times \left(-\frac{1}{\sqrt{2}}\right) = -1$ and $y = r\sin\theta = \sqrt{2} \times \left(\frac{1}{\sqrt{2}}\right) = 1$. Thus

$$(\sqrt{2}, 3\pi/4, 2) \mapsto (1, -1, 2).$$

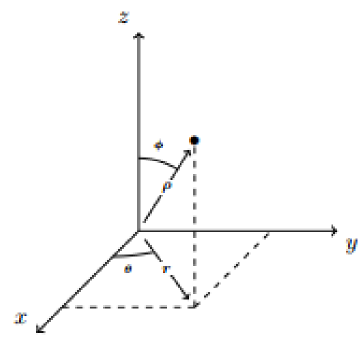
□

2 Spherical coordinates

- $(x, y, z) \mapsto (\rho, \theta, \phi)$, where basically we repeat the polar coordinate first, and the *height* z is tracked via the variable ϕ, the angle with Oz. Note that the order is sometime written as (r, ϕ, θ). **Pay attention to the order!**

- The relations, still introducing an extra variable r as in polar coordinates (it will be very useful)

$$\begin{cases} x = (\rho \sin\phi) \cos\theta \\ y = (\rho \sin\phi) \sin\theta \\ z = \rho \cos\phi \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ r = \rho \sin\phi \\ \frac{r}{z} = \tan\phi \end{cases}, \quad \theta \in [0, 2\pi], \phi \in [0, \pi]$$



- If $\phi > \frac{\pi}{2}$ then $z < 0$, the angle make P lies below the Oxy-plane.

- **Example.** Convert $(1, 1, 0)$ into spherical coordinate.

  Proof. $\rho^2 = x^2 + y^2 + z^2 = 2$, thus $\rho = \sqrt{2}$. Now $z = \rho\cos\phi$ implies $0 = \sqrt{2}\cos\phi$, thus $\phi = \frac{\pi}{2}$. Finally $\tan\theta = \frac{y}{x} = 1$, thus $\theta = \frac{\pi}{4}$ (since $x > 0, y > 0$). We conclude

$$(1, 1, 0) \mapsto \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right) = (r, \theta, \phi).$$

  □

- **Example.** True/False: Consider the point with spherical coordinates $(\rho, \theta, \phi) = (4, \frac{3\pi}{4}, \frac{5\pi}{7})$. The product of the Cartesian coordinates, xyz, is positive.

  Proof. **True**. We see that $\phi = \frac{5\pi}{7} > \frac{\pi}{2}$, thus $z < 0$. Now $\theta = \frac{3\pi}{4}$, thus $x > 0, y < 0$ (draw a picture). Therefore $xyz > 0$. □

3 Practice

- **Example.** Convert the equation $z = \sqrt{x^2 + y^2}$ into cylindrical coordinates and spherical coordinates.

  Proof.

  - Cylindrical: $z = r$.
  - Spherical: $\rho\cos\phi = r = \rho\sin\phi$, thus $\tan\phi = 1$, thus $\phi = \frac{\pi}{4}$ is the equation of the cone!

  □

- **Example.** Identify the surface whose equation is $z = 4 - r^2$ in cylindrical coordinate.

Proof. We have $z = 4 - x^2 - y^2$, thus this is a elliptical paraboloid (one term of 1st order, two terms of second order having the same sign). □

- **Example.** Convert to x, y, z the surface: $\rho = \sin\phi\cos\phi$.

  Proof. We can do

  $$(x^2 + y^2 + z^2)^{\frac{3}{2}} = \rho^3 = (\rho\sin\phi)(\rho\cos\phi) = rz = z\sqrt{x^2 + y^2}.$$

  The answer is $(x^2 + y^2 + z^2)^{\frac{3}{2}} = z\sqrt{x^2 + y^2}$. □

- **Example.** Identify the surface whose equation is: $\rho = \sin\phi\cos\theta$.

  Proof. We can do

  $$x^2 + y^2 + z^2 = \rho^2 = (\rho\sin\phi)\cos\theta = r\cos\theta = x$$

  Therefore

  $$\left(x - \frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4}$$

  This is a sphere centered at $(\frac{1}{2}, 0, 0$ with radius $\frac{1}{2}$, this is a *ellipsoid*. □