2 13.1/13.3 – Parametrizing Curves and Arc Length

2.1 Tangent Lines and Intersections – Video Before Class

Objective(s):

- Parametrize tangent lines to vector functions.
- Find where a curve and surface intersect.
- Find where two curves intersect.

From **Theorem 1.12** we saw that we can obtain a tangent vector. With this would could also find the parametric equations of a tangent line.

Example 2.1. Find an equation of a tangent line to $\mathbf{r}(t) = \left\langle t^2, \sqrt{t}, te^{t-1} \right\rangle$ at $t = 1$

Equation : $\underbrace{\vec{r}(1)}_{\text{point}} + t\underbrace{\vec{r}'(1)}_{\text{direction vector}} = \boxed{(1,1,1) + t\left(2, \frac{1}{2}, 2\right)}$

$r'(t) = \left(2t, \frac{1}{2\sqrt{t}}, t^{e-1} + te^{t-1}\right) \Rightarrow r'(1) = \left(2, \frac{1}{2}, 2\right)$

Theorem 2.2. An equation of the tangent line to $\mathbf{r}(t)$ at $t = a$ is given by

$\mathbf{L}(t) = \vec{r}\,(a) + t\,\vec{r}\,'(a)$ for $t \in (-\infty, \infty)$

Now switching over to intersections we eventually will want to get to get to finding the curve of intersection between two surfaces. This is an upgrade of what we did in 12.5 when we found the line of intersection of two planes. For now though let's do some easier situations, starting with a curve and a surface.

Example 2.3. At what point(s) does the curve $\mathbf{r}(t) = t\mathbf{j} + (4 - 3t)\mathbf{k}$ intersect the surface $z = x + y^2$?

$r(t) = (\underset{x}{0}, \underset{y}{t}, \underset{z}{4-3t})$

Therefore $z = x + y^2 \Rightarrow 4 - 3t = 0 + t^2 \Rightarrow t^2 + 3t - 4 = 0$

$\Rightarrow (t-1)(t+4) = 0$

$t = 1$ one intersection is $r(1) = (0, 1, 1)$

$t = -4$, one intersection is $r(-4) = (0, -4, 16)$

Now let's do a problem a bit more difficult, finding where two curves intersect... and the angle!

Example 2.4. Consider the curves

$$\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle \quad \text{and} \quad \mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$$

(a) At what point do the curves intersect?

$\boxed{t = 3-s}$

$1-t = s-2$

$3+t^2 = s^2$

$3 + (3-s)^2 = s^2 \Rightarrow 3 + 9 - 6s + \cancel{s^2} = \cancel{s^2}$

$\Rightarrow 12 = 6s \Rightarrow s = 2$

$s=2$,

the point of intersection is

$\vec{r}_2(s) = (3-2, 2-2, 2^2) = (1, 0, 4)$

note: $s = 2 \rightsquigarrow \boxed{t = 3-s} = 1$

(b) Find the angle of intersection.

$\vec{r}_1(t) = (t, 1-t, 3+t^2) \Rightarrow \vec{v}_1 = \vec{r}_1\,'(t) = (1, -1, 2t) \underset{t=1}{=} (1, -1, 2)$

$\vec{r}_2(s) = (3-s, s-2, s^2) \Rightarrow \vec{v}_2 = \vec{r}_2\,'(s) = (-1, 1, 2s) \underset{s=2}{=} (-1, 1, 4)$

angle $\theta \Rightarrow \cos\theta = \dfrac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \, |\vec{v}_2|}$

Theorem 2.5. The angle of intersection between two parametrized curves is given by the angle between the tangent lines when the curves intersect.

(*Think back to 12.3 and finding the angle between vectors*)

2.2 Upgrading Arc Length to 3 Dimensions – Video Before Class

Objective(s):

- Upgrade our calculus 2 arc length formula so it can be used in 3 dimensions.
- Define the arc length function.
- Get a little practice calculating arc length.

Note if you are reading out of the book we are skipping **Curvature** and **The Normal and Binormal Vectors** portions of this section. We will focus on arc length.

Definition(s) 2.6. A parametrization $\mathbf{r}(t)$ is called smooth on an interval I if $\mathbf{r}'(t)$ is continuous and $r'(t) \neq 0$ on I. A curve is called smooth if it has a smooth parametrization.

We will use smooth parametrizations later in the course. Now recall:

Old Theorems

(a) (Calc 1 Theorem): If we have the velocity function $v(t) = r'(t)$ (notice this is not a vector function) then the total distance traveled is given by:

$$D = \int_a^b |r'(t)|\, dt$$

(b) (Calc 2 Theorem): If a curve C is parameterized by $(x(t), y(t))$ then the length of C (denoted $L(C)$) is given by the formula:

$$L(C) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$$

(c) (Calc 2 Theorem) **REVISED**: If a curve C is parameterized by ______________ then the length of C (denoted $L(C)$) is given by the formula:

$$L(C) =$$

Now let's make up a Length formula for a 3 dimensional curve!

Theorem 2.7. If a curve C is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ then the length of C (denoted $L(C)$) is given by the formula:

$$L(C) = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2 + |z'(t)|^2}\,dt$$

Example 2.8. Find the arc length of $\mathbf{r}(t) = \langle \cos t, \sin t, 2t\rangle$ from $t = 0$ to $t = 5$.

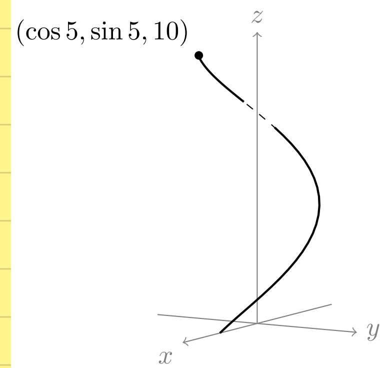
$r'(t) = (-\sin t, \cos t, 2)$

$$\int_0^5 |r'(t)|\,dt = \int_0^5 \sqrt{(-\sin t)^2 + (\cos t)^2 + 2^2}\,dt$$

(use $\sin^2 x + \cos^2 x = 1$) $= \int_0^5 \sqrt{5}\,dt = \sqrt{5}\int_0^5 1\,dt$

$= \sqrt{5}\,t\Big|_0^5$

$= \sqrt{5}(5-0) = 5\sqrt{5}$



So we want to realize that we can treat b as a variable to get an arc length function that we will call _______ (for now).

Theorem 2.9 (**Arc Length Function** with Base Point a)**.**

$$s(b) =$$

Most people don't like to use b as a variable so lets rename a few things to get our variable to be t.

Theorem 2.10 (**Arc Length Function** with Base Point a)**.**

$$s(t) = \int_a^t |r'(u)|\,du$$

arc length ⇝ go the same curve but with constant velocity

2.3 Parametrizing Curves – During Class

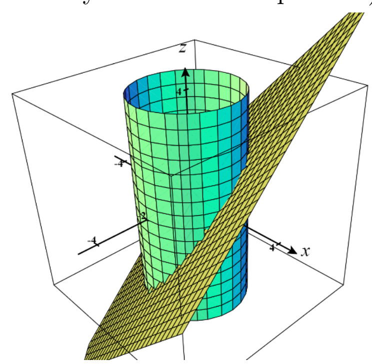
Objective(s):

- Parametrize curves of intersections between two surfaces.
- Understand why not all parametrizations are created equal.

In our pre-class video we talked about finding where two curves intersect and also where one curve and one surface intersect. These were a bit easier because the intersection is just a point (or a set of points) however as we tackle where two surfaces intersect the answer is typically a full curve in space. Think back to 12.5... we found that two planes intersected in a line in space.

Again our goal here is to find a vector function, $\mathbf{r}(t)$, that is on both surfaces (that is it must satisfy both surface equations).

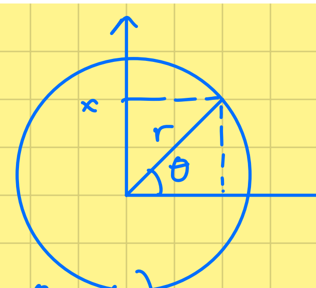
Example 2.11. Show that the curve $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 2\cos(t) + 2\sin(t) - 1\rangle$ for $t \in [0, 2\pi]$ is the curve of intersection for the surfaces



$$\boxed{4 = x^2 + y^2}$$
$$z = x + y - 1$$

polar coordinate :

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}, \quad \theta \in [0, 2\pi]$$



$$z = 2\cos\theta + 2\sin\theta - 1$$

thus,

$$r(t) = (x, y, z) = (2\cos\theta, 2\sin\theta, 2\cos\theta + 2\sin\theta - 1)$$

Suppose another student said they found the curve of intersection to be $\mathbf{r}(t) = \langle 2, 0, 1\rangle$. Is this correct? Why / Why not?

just a point
not the whole intersection

In general it can be somewhat difficult to find the curve(s) of intersection between two surfaces. Luckily in our class we stick to relatively nice examples that can be reasonably done by hand.

Technique 2.12 (A decent technique for parametrizing curves).

(a) If you believe the curve of intersection is closed (starts and ends at the same point (like a circle or ellipse)) try using ______________ in your parametrization

(b) Otherwise try one of

- ________ OR ________ OR ________

Example 2.13. Find a vector function that represents the curve of intersection of the surfaces $y = x^2 z$ and $x^2 + z^2 = 9$

$x = 3\cos t$ $\quad t \in [0, 2\pi]$

$z = 3\sin t$

$y = x^2 z = (3\cos t)^2\, 3\sin t = 27\cos^2 t \sin t$

thus intersection $(x, y, z) = (3\cos t,\ 27\cos^2 t \sin t,\ 3\sin t),\ t \in [0, 2\pi]$

Example 2.14. Find a set of parametric equations that represents the curve of intersection of the surfaces $\boxed{z = \sqrt{x^2 + y^2}}$ and $\boxed{z = 1 + y}$.

$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ $t \in [0, 2\pi]$ goal: write r in t

note: $x^2 + y^2 = r^2$

$\boxed{z = \sqrt{r^2} = r}$ (as $r > 0$)

then

$\boxed{z = 1 + y} \Rightarrow r = 1 + r\sin t \Rightarrow r(1 - \sin t) = 1$

$\Rightarrow r = \dfrac{1}{1 - \sin t}$

thus

$(x, y, z) = (r\cos t, r\sin t, r) = \left(\dfrac{\cos t}{1 - \sin t}, \dfrac{\sin t}{1 - \sin t}, \dfrac{1}{1 - \sin t}\right)$ $t \in [0, 2\pi]$

2.4 Additional Practice with Arc Length – During Class

Objective(s):

- Calculate the length of more curves!
- Calculate the arc length function and use it to solve some problems!

Example 2.15. Find the length of the curve: $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle, \quad t \in [0, 1]$

$$r'(t) = (2, 2t, t^2), \qquad |r'(t)| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4}$$

$$= \sqrt{(t^2+2)^2} = t^2 + 2$$

$$L = \int_0^1 |r'(t)|\, dt = \int_0^1 (t^2+2)\, dt = \frac{t^3}{3}\Big|_0^1 + 2t\Big|_0^1$$

$$= \frac{1}{3} + 2 = \boxed{\frac{7}{3}}$$

Example 2.16. Find the length of the curve: $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad 0 \le t \le 2$

$$r(t) = (1, t^2, t^3)$$

$$r'(t) = (0, 2t, 3t^2), \qquad |r'(t)| = \sqrt{(2t)^2 + (3t^2)^2}$$

$$= \sqrt{4t^2 + 9t^4}$$

$$= \sqrt{t^2(4 + 9t^2)}$$

$$= t\sqrt{4 + 9t^2}$$

$$L = \int_0^2 t\sqrt{4 + 9t^2}\, dt$$

$$u = 9t^2 + 4 \qquad t = 0,\ u = 4$$
$$du = 18t\, dt \qquad t = 2,\ u = 40$$

$$\underbrace{\sqrt{4+9t^2}}_{\sqrt{u}}\ \underbrace{(t\,dt)}_{\frac{du}{18}} \qquad (u\text{-substitution})$$

$$\frac{1}{18}\int_4^{40} u^{1/2}\, du = \frac{1}{18} \cdot \frac{u^{3/2}}{3/2}\Big|_4^{40} = \boxed{\frac{2}{3} \cdot \frac{1}{18} \cdot \left(40^{3/2} - 4^{3/2}\right)}$$

Example 2.17. Consider the vector function $\mathbf{r}(t) = \langle \cos(-t), \sin(-t), t+1 \rangle$.

(a) Find the arc length function with the initial point $\langle 1, 0, 1 \rangle$. $\longrightarrow$ solve for t

$\hookrightarrow t = 0$

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

$$r(u) = (\cos(-u), \sin(-u), u+1)$$

$$r'(u) = (\sin(-u), -\cos(-u), 1)$$

$$|r'(u)| = \sqrt{(\sin(-u))^2 + (-\cos(-u))^2 + 1^2}$$

$$= \sqrt{2}$$

thus

$$s(t) = \int_0^t \sqrt{2}\,dt = \sqrt{2}\,t$$

is the arc-length function

(b) Find the time in which a particle has traveled 7 units along the curve from $\langle 1, 0, 1 \rangle$ in the positive direction.

Use the arclength :

arclength $= 7 \Rightarrow s(t) = \sqrt{2}\,t = 7$

$$\Rightarrow t = \frac{7}{\sqrt{2}}$$

(c) Find the location of the particle in (b).

$$r(t) = (\cos(-t), \sin(-t), t+1)$$

$t = \frac{7}{\sqrt{2}} \Rightarrow$ point is $\left(\cos\left(\frac{-7}{\sqrt{2}}\right), \sin\left(\frac{-7}{\sqrt{2}}\right), \frac{7}{\sqrt{2}}+1 \right)$