Vector functions

- a) $\vec{r}(t) = (t, 2t, 3t) \quad t \in \mathbb{R}$
  $\hookrightarrow (0,0,0) + t(1,2,3)$
  (line)

  b) $\vec{r}(t) = (t, 2t, \sin t),\ t \in \mathbb{R}$
  $\hookrightarrow$ curve in 3D

  vector functions $\rightarrow$ outputs are vectors

- We can write $\vec{r}(t) = \underbrace{(x(t), y(t), z(t))}_{\text{component functions of } \vec{r}(t)} = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

  Example: $\vec{r}(t) = \underbrace{(t^2+1,\ \sin t,\ \sqrt[3]{1-t})}_{\text{components}}$

  $x(t) = t^2+1,\ y(t) = \sin t,\ z(t) = \sqrt[3]{1-t}$

- Limits If $\vec{r}(t) = (x(t), y(t), z(t))$ then

$$\boxed{\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} x(t),\ \lim_{t\to a} y(t),\ \lim_{t\to a} z(t)\right)}$$

Recall: $f: \mathbb{R} \to \mathbb{R}$ is continuous at a if and only if $\lim_{t\to a} f(t) = f(a)$

Example Compute

$$\lim_{t\to 0}\left\langle 1+3t^2, te^{-3t}, \frac{\sin 2t}{t}\right\rangle = \left(1, 0, \lim_{t\to 0}\frac{\sin 2t}{t}\right)$$

$\rightarrow$ form $\frac{0}{0}$

L'Hospital rule

$$\lim_{t\to 0}\frac{\sin 2t}{t} = \lim_{t\to 0}\frac{(\sin 2t)'}{t'} = \lim_{t\to 0}\frac{2\cos 2t}{1} = 2$$

thus the answer is $\boxed{(1,0,2)}$.

Example:

$$\mathbf{r}(t) = \begin{cases} \langle (1-t)^2, \sin t, \sqrt{9+t^2}\rangle & \text{if } t \le 0 \\ \langle \cos t, te^t, 1+2t\rangle & \text{if } t > 0 \end{cases}$$

Is $\vec{r}(t)$ continuous at $t=0$?

$\lim_{t\to 0^-} \vec{r}(t) = (1,0,3)$ and $\lim_{t\to 0^+} \vec{r}(t) = (1,0,1)$

$\neq$ thus $\vec{r}(t)$ is not continuous at $t=0$.

- **Derivatives** $r(t) = (x(t), y(t), z(t))$ then

$r'(t) = (x'(t), y'(t), z'(t))$ is the derivative of $\vec{r}(t)$

- **Integral** $\displaystyle\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt,\ \int_a^b z(t)\,dt \right)$

is the definite integral of $\vec{r}(t)$

Example: $\vec{r}(t) = (t^3, \sin(t)\ e^{3t})$

$\vec{r}'(t) = (3t^2, \cos t, 3e^{3t})$

$$\int_0^\pi \vec{r}'(t)\,dt = \left(\int_0^\pi t^3\,dt,\ \int_0^\pi \sin t\,dt,\ \int_0^\pi e^{3t}\,dt \right)$$

$$= \left(\frac{t^4}{4}\Big|_0^\pi,\ -\cos t\Big|_0^\pi,\ \frac{e^{3t}}{3}\Big|_0^\pi \right)$$

$$= \left(\frac{\pi^4}{4}, 2, \frac{e^{3\pi}-1}{3} \right)$$

- **Properties:** given $\vec{u}(t), \vec{v}(t)$ vector functions $(\mathbb{R} \to \mathbb{R}^3)$

c is scalar, f: real-valued fuctions $(\mathbb{R} \to \mathbb{R})$

(a) $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

(b) $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

(c) $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

(d) $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

(e) $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

(f) $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Example: $f(1) = 3$, $f'(1) = -4$

$\vec{u}(1) = (3, 0, -5)$

$\vec{u}'(1) = (0, 1, 2)$

Then

$$\frac{d}{dt}\left(f(t)\,\vec{u}(t) \right)\Big|_{t=1}$$

$$= f'(1)\,\vec{u}(1) + f(1)\,\vec{u}'(1)$$

$$= (-4)\cdot(3, 0, -5) + 3\cdot(0, 1, 2)$$

$$= (-12, 3, 26)$$

- **Tangents** the direction vector of $\vec{r}(t)$ at t is $\vec{r}'(t)$

the unit tangent $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Example: $\vec{r}'(5) = (2,1,2)$ then $\vec{T}(5) = \dfrac{(2,1,2)}{\sqrt{4+1+4}} = \dfrac{(2,1,2)}{3}$

Example: $r(t) = (te^{-t}, \arctan t, 2e^{t})$.

$r'(t) = (e^{-t} - te^{-t}, \dfrac{1}{1+t^2}, 2e^{t})$

$r'(0) = (1, 1, 2) \Rightarrow T(0) = \dfrac{(1,1,2)}{\sqrt{6}}$

Example, $\vec{r}'(t) = (2, 2e^{t}, \sqrt{3t+1})$, $\vec{r}(0) = (4,2,3)$

u-sub

$$r(t) = \left(2t + C_1, 2e^{t} + C_2, \frac{2}{9}(3t+1)^{3/2} + C_3\right)$$

$$r(0) = \left(C_1, 2 + C_2, \frac{2}{9} + C_3\right) = (4,2,3)$$

$$C_1 = 4, \quad C_2 = 0, \quad C_3 = 3 - \frac{2}{9}$$

Example

$\vec{r}'(t) = \left(\dfrac{5}{t}, 2te^{t}, \dfrac{1}{t^2+1}\right)$ $\quad \vec{r}(1) = (4,2,3)$

integration by parts

$$\vec{r}(t) = \left(5\ln t + C_1, 2(te^{t} - e^{t}) + C_2, \arctan t + C_3\right)$$

$$\vec{r}(1) = \left(C_1, C_2, \frac{\pi}{4} + C_3\right) = (4,2,3)$$

solve $C_1 = 4$

$C_2 = 2$

$C_3 = 3 - \dfrac{\pi}{4}$