MICHIGAN STATE UNIVERSITY
MATH 234 – SPRING 2024

LECTURE 05
2024-01-19

Key takeaways:

- Equation of plane, normal vector of a plane, how to find a plane knowing 3 points.
- Angle between two planes
- Distance of a point to a plane

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1 Equation of plane

If $\mathbf{n} = (a, b, c)$ is a given vector, $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the position vector for $P_0(x_0, y_0, z_0)$, then the plane going through (P) going through P_0 and is perpendicular to $\mathbf{n}$ is given by

- **Vector form** $\mathbf{r} = (x, y, z) \in (P)$ if
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$
- **Parametric form** $\mathbf{r} = (x, y, z) \in (P)$ if
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \qquad \Longrightarrow \qquad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Other parametric form** sometime we write
$$ax + by + cz - \underbrace{(ax_0 + by_0 + cz_0)}_{d} = 0, \qquad \text{or} \qquad ax + by + cz = d.$$

Example. Find an equation of a plane orthogonal to (1,4,-2) and contains (0,-3,1)

$$1 \cdot (x-0) + 4 \cdot (y - (-3)) + (-2)(z-1) = 0$$

$$x + 4y + 12 - 2z + 2 = 0$$

$$\boxed{x + 4y - 2z + 14 = 0}$$

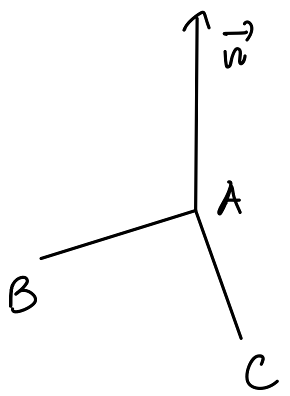
Example. Find where the line $\vec{r}(t) = (3-t, 2+t, 5t)$ intersects the plane $x - y + 2z = 6$

Write
$$\overset{x}{(3-t)} - \overset{y}{(2+t)} + 2\overset{z}{(5t)} = 6$$
$$3 - t - 2 - t + 10t = 6$$
$$1 + 8t = 6 \Rightarrow 8t = 5 \Rightarrow t = \frac{5}{8}$$

thus the intersection is $\left(3 - \frac{5}{8}, 2 + \frac{5}{8}, 5 \cdot \frac{5}{8}\right)$

Example. Find an equation of the plane that contains $\underset{A}{(1,0,0)}, \underset{B}{(0,1,0)}, \underset{C}{(0,1,3)}$

- need to find the normal vector and one point (can be A, B or C)



By definition:

$\vec{AB} \in$ plane $\Rightarrow \vec{AB} \perp \vec{n}$

$\vec{AC} \in$ plane $\Rightarrow \vec{AC} \perp \vec{n}$ $\Rightarrow$ choose $\vec{n} = \vec{AB} \times \vec{AC}$

(any other ordering is fine)

$\vec{AB} = (-1, 1, 0)$

$\vec{AC} = (-1, 1, 3)$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - j \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix} + k \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= (3, 3, 0)$$

Pick A at a point: $(1,0,0)$

$$3(x-1) + 3(y-0) + 0(z-0) = 0$$
$$\Rightarrow 3x + 3y - 3 = 0$$
$$\Rightarrow \boxed{x + y - 1 = 0}$$

$$\boxed{\text{Angle between two planes is the angle between their normal vectors}}$$

Example. (P_1) $x+y = 2z+4$, (P_2) : $4z-2x = 2y+5$

$x+y-2z=4$ $\qquad$ $-2x-2y+4z=5$

$\vec{n_1} = (1,1,-2)$ $\qquad$ $\vec{n_2} = (-2,-2,4)$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|} = \frac{(1,1,-2)\cdot(-2,-2,4)}{\sqrt{6}\cdot\sqrt{24}} = \frac{-2-2-8}{\sqrt{144}} = \frac{-12}{12} = -1$$

thus $\theta = \pi$, or $-180°$

i.e., $\overset{\longleftarrow}{\vec{n_1}}\ \circ\ \overset{\longrightarrow}{\vec{n_2}}$ $\qquad$ thus (P_1), (P_2) are parallel.

Another way to see: $-2\vec{n_1} = \vec{n_2}$

they are parallel, so are (P_1), (P_2)

Theorem. If (P_1) and (P_2) are non-parallel planes with normal vectors $\vec{n_1}$ and $\vec{n_2}$, then their line of intersection has direction vector $\vec{v} = \vec{n_1}\times\vec{n_2}$

How to find the intersection line: $\qquad$ $\vec{n_1} = (1,1,1)$ $\qquad$ $\vec{n_2} = (1,2,2)$

Example: Find the intersection for (P_1) : $x+y+z=1$, (P_2) : $x+2y+2z=1$

line is defined by a direction vector, a point

Compute direction vector

$$\vec{v} = \vec{n_1}\times\vec{n_2} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \boxed{(0,-1,1)}$$

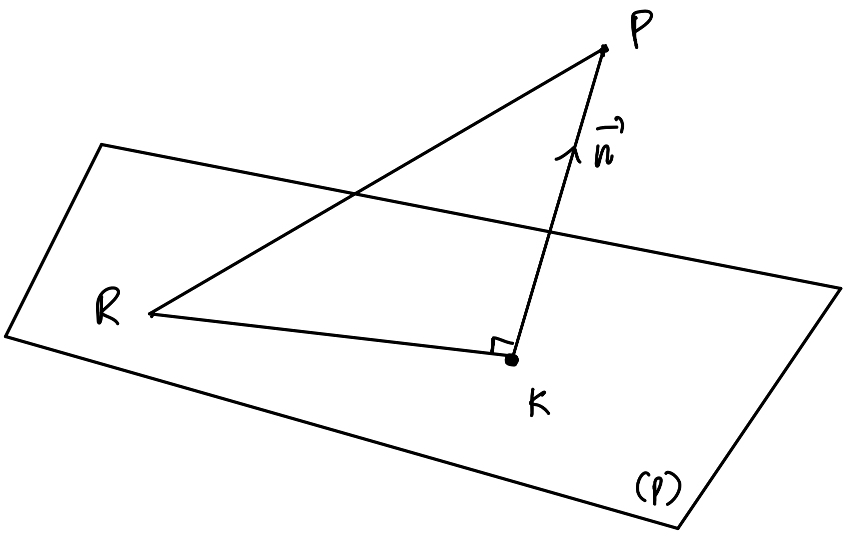
How to choose a point (x,y,z) : $\hookrightarrow$ it must satisfy

$$\begin{cases} x+y+z=1 \\ x+2y+2z=1 \end{cases}, \quad \text{I can pick } (1,0,0) \quad \text{(there are others)}$$

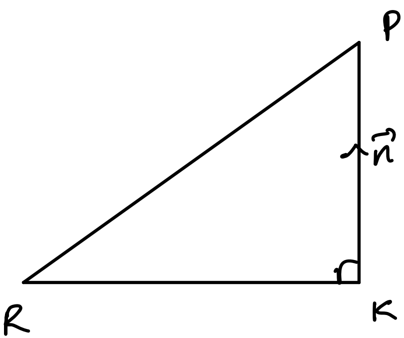
So : the line is $\boxed{\vec{r}(t) = (1,0,0) + t\,\boxed{(0,-1,1)}}$

Distance from a point to a plane

The distance from a point $P(x,y,z)$ to a plane (P) with normal $\vec{n}$ going through a point R



in the right triangle



$$\text{dist}(P,(P)) = |\vec{PK}| = |\text{proj}_{\vec{n}}(\vec{RP})|$$

recall from lecture 2

$$\frac{|\vec{n}\cdot\vec{RP}|}{|\vec{n}|}$$

(compare with distance to a line $\to$ cross product)

<u>Theorem</u> : The distance between the point P to a plane containing R with $\vec{n}$ normal is

$$\text{dist} = \frac{|\vec{RP}\cdot\vec{n}|}{|\vec{n}|}$$

Example Find the distance between $P(1,-2,4)$ to $3x+2y+6z=5$

$\vec{n} = (3,2,6)$

pick a point on $3x+2y+6z=5$

$R(1,2,0)$ (you can choose $(\frac{5}{3},0,0)$ for example)

then $$d = \frac{|\vec{RP} \cdot \vec{n}|}{|\vec{n}|}$$

$\vec{RP} = (1,-2,4) - (1,2,0) = (0,-4,4)$

$$= \frac{|(0,-4,4)\cdot(3,2,6)|}{|(3,2,6)|}$$

$$= \frac{|-8+24|}{\sqrt{3^2+2^2+6^2}} = \frac{16}{\sqrt{49}} = \frac{16}{7}$$

Proof of another formula. (P) $ax+by+cz+d=0$

(P_0) (x_0, y_0, z_0)

take $R(x,y,z) \in (P) \Rightarrow ax+by+cz+d=0 \Rightarrow$ $ax+by+cz=-d$

then $\vec{RP_0} = (x-x_0, y-y_0, z-z_0)$

thus

$$\vec{n}\cdot\vec{RP_0} = a(x-x_0)+b(y-y_0)+c(z-z_0)$$

$$= (ax+by+cz) - (ax_0+by_0+cz_0)$$

$$= -d - (ax_0+by_0+cz_0)$$

therefore

$$\text{dist} = \frac{|\vec{n}\cdot\vec{RP_0}|}{|\vec{n}|} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example 5.28. Consider the line $\mathbf{r}(t) = \langle \overset{x}{7+4t}, \overset{y}{2t}, \overset{z}{1-t} \rangle$ and the plane $x - y + 2z = 5$. If they intersect, find the point of intersection. If they don't intersect find the distance between them.

Check if they intersect, solve for t

$$x - y + 2z = (7+4t) - (2t) + 2(1-t) = 5$$
$$= 7 + 4t - 2t + 2 - 2t = 9 \neq 5$$

So the equation $7 + 4t - 2t + 2(1-t) = 5$ does not have any solution

$\Rightarrow$ do not intersect

take $t = 0 \Rightarrow P = (7, 2, 1)$, compute distance $(x - y + 2z - 5, \; x = 7, \; y = 2, \; z = 1)$

$$d = \frac{|7 - 2 + 2.1 - 5|}{\sqrt{1^2 + 1^2 + 2^2}} = \boxed{\frac{2}{\sqrt{6}}}$$

Example 5.29. Consider the planes $x + y = 2z + 4$ and $4z - 2x = 2y + 5$. If they intersect, find the line of intersection. If they don't intersect find the distance between them.

$x + y - 2z - 4 = 0$ $\qquad$ $-2x - 2y + 4z - 5 = 0$

$\vec{n}_1 = (1, 1, -2)$ $\qquad$ $\vec{n}_2 = (-2, -2, 4)$

$$\vec{n_1} = -\frac{1}{2}\vec{n_2}$$

two planes are parallel, do not intersect

take a point $P = (4, 0, 0)$ on $x + y = 2z + 4$

Compute

$$d = \frac{|-2.(4) - 2(0) + 4(0) - 5|}{\sqrt{(-2)^2 + (-2)^2 + 4^2}}$$