Equations of lines and planes

Theorem: Vectors $\vec{v}$ and $\vec{w}$ are parallel if and only if $\vec{v} = k\vec{w}$ for some $k \in \mathbb{R}$ (scaler)
Or, if
$$\vec{v} \times \vec{w} = \vec{0} \iff |\vec{v} \times \vec{w}| = 0$$

Equation of line

Given a point $P_0(x_0, y_0, z_0)$ we identify P_0 with its position vector $\vec{OP_0} = \vec{r_0} = (x_0, y_0, z_0)$

→ describe vector position of an arbitrary point $P(x, y, z) = \vec{OP} = \vec{r}$

a) Vector form. the line (L) through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v}$ is given by:

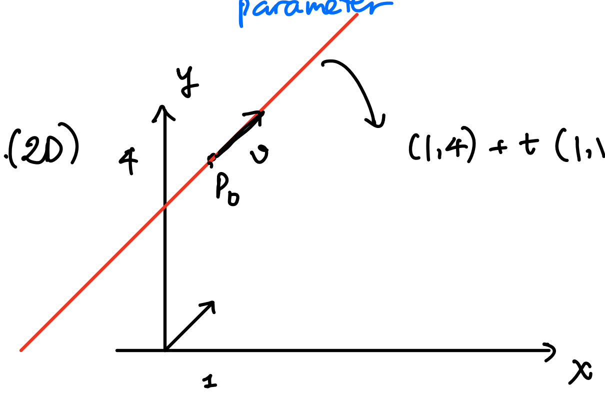
$$\vec{r} = \underbrace{\vec{r_0}}_{\text{one point}} + \underbrace{t\vec{v}}_{\text{direction vector}}, \quad -\infty < t < \infty$$

one point $P_0 \in (L)$, $\vec{r_0} = \vec{OP_0}$

direction vector

parameter

Example (2D)



the line going through $P_0(1,4)$
direction vector $\vec{v} = (1,1)$

is $(1+t, 4+t)$, $t \in \mathbb{R}$

b) Parametric form: from $\underset{(x,y,z)}{\vec{r}(t)} = \vec{r_0} + t\vec{v}$, if $\vec{r_0} = (x_0, y_0, z_0)$
$\vec{v} = (a, b, c)$ - direction numbers
then
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad t \in \mathbb{R}$$
, this is the parametric form

c) Symmetric form (only if $a, b, c \neq 0$)

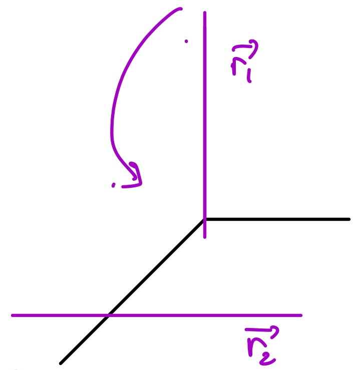
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t \in \mathbb{R}$$

Skew, intersect or parallel.

Suppose $\vec{r}_1(t) = \vec{r}_0 + t\vec{v}$
$\vec{r}_2(s) = \vec{s}_0 + s\vec{w}$

in 2D: they can either
- intersect
- parallel

in 3D
- intersect
- parallel
- skew



$\boxed{\begin{array}{l}\text{Parallel if } \vec{v} \uparrow\uparrow \vec{w} \text{ (2 direction are parallel)}\\ \text{Intersect if } \exists!\,(x,y,z) \text{ belongs to both lines}\\ \text{Skew if neither}\end{array}}$

Example. $\vec{r}_1(t) = (3,1,0) + t(2,0,1) = (3+2t,\ 1,\ 2t)$
$\vec{r}_2(s) = (1,-2,5) + s(-1,3,-2) = (1-s,\ -2+3s,\ 5-2s)$

- they are not parallel, as $(2,0,1)$ and $(-1,3,-2)$ are not parallel to each other (can not multiply by a constant)
- intersect? if $\exists\,(x,y,z)$ belongs to both lines:

$$\begin{cases} x = 3+2t = 1-s \\ y = 1 = -2+3s \\ z = 2t = 5-2s \end{cases} \Rightarrow s = 1 \quad \begin{array}{l} 3+2t = 0 \Rightarrow t = -3/2 \\ \qquad\qquad\qquad\quad \neq \\ 2t = 3 \Rightarrow t = 3/2 \end{array}$$

thus no solution

$\Rightarrow$ no intersection

$\hookrightarrow$ $\vec{r}_1$ and $\vec{r}_2$ are skew!

Example: $\vec{r}_1(t) = (1+2t,\ 9-5t,\ t) = (1,9,0) + t(2,-5,1)$
$\vec{r}_2(t) = (3-s,\ 3+5s,\ 2s) = (3,3,0) + s(-1,5,2)$

- they are not parallel $(2,-5,1) \not\parallel (-1,5,2)$
- intersect? solve

$$\begin{cases} 1+2t = 3-s \\ 9-5t = 3+5s \\ t = 2s \end{cases}$$

replace $t = 2s$: $\begin{cases} 1+4s = 3-s \\ 9-10s = 3+5s \end{cases}$

$$\begin{cases} 5s = 2 \\ 15s = 6 \end{cases} \Rightarrow s = \frac{2}{5}, \quad t = \frac{4}{5}$$

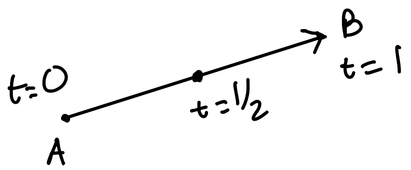
thus the intersection is $(3-s,\ 3+5s,\ 2s)\big|_{s=2/5}$

$$= \left(3-\frac{2}{5},\ 3+5.\frac{2}{5},\ 2.\frac{2}{5}\right)$$

Equations of a line going through two points, and line segment

given A with position vector $\vec{r_1}$ (E.g. $A = (1,2,3)$, $r_1 = \vec{OA} = (1,2,3)$)

B with position vector $\vec{r_2}$



direction : $\vec{v} = \vec{B} - \vec{A} = \vec{r_2} - \vec{r_1}$

$$\vec{r}(t) = \vec{r_1} + t\vec{v}$$

$$= \vec{r_1} + t(\vec{r_2} - \vec{r_1}) = (1-t)\vec{r_1} + t\vec{r_2}$$

Look $\vec{r}(t) = (1-t)\vec{r_1} + t\vec{r_2}$

$t = 0 : \vec{r_1}$

$t = 1 : \vec{r_2}$

$t = 1/2 : \dfrac{\vec{r_1} + \vec{r_2}}{2}$

> thus $\vec{r}(t) = (1-t)\vec{r_1} + t\vec{r_2}$
>
> $t \in [0,1]$
>
> is the line segment from $\vec{r_1}$ to $\vec{r_2}$

Example :
- Find equation of the line going through $A(2,3,4)$ and $B(1,0,-1)$
- Find the intersection with the xy-plane.

$$\vec{r}(t) = (1-t)(2,3,4) + t(1,0,-1) = (2(1-t)+t,\ 3(1-t)+0,\ 4(1-t)-t)$$

$$= (2-t,\ 3-3t,\ 4-5t),\ t \in \mathbb{R}$$

Intersection with xy-plane : $z = 0$ $\hookrightarrow$ $4-5t = 0 \Rightarrow t = \frac{4}{5}$

$\longrightarrow$ point is $\left(2 - \frac{4}{5},\ 3 - 3.\frac{4}{5},\ 0\right)$

Angle between two lines

$\vec{r_1}(t) = \vec{r_1} + t\vec{v}$

$\vec{r_2}(s) = \vec{r_2} + s\vec{w}$

is the angle between 2 direction vectors

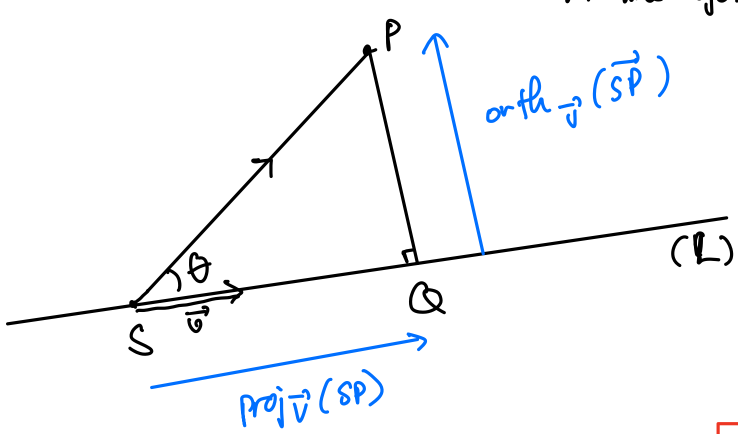
$= \text{angle}(\vec{v}, \vec{w}) = \theta$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|.|\vec{w}|.}$$

Distance to a line

Given a point $P(x, y, z)$

A line going through $S(x_0, y_0, z_0)$ with direction vector $\vec{v}(a, b, c)$



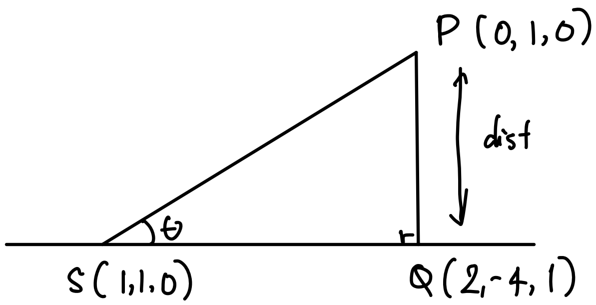
$$\text{distance}(P, L) = |\vec{PQ}|$$

$$|\vec{PQ}| = |\vec{SP}| \sin\theta$$
$$= |\vec{SP}| \cdot \left(\frac{|\vec{SP} \times \vec{v}|}{|\vec{SP}| \cdot |\vec{v}|} \right)$$

Recall
$$\vec{SP} \times \vec{v} = |\vec{SP}| \cdot |\vec{v}| \cdot \sin\theta$$

$$\boxed{\text{dist}(P, L) = |\vec{PQ}| = \frac{|\vec{SP} \times \vec{v}|}{|\vec{v}|}}$$

Example. Find the distance between $(0,1,0)$ and the line containing $(1,1,0)$ and $(2,-4,1)$



$$\text{dist} = \frac{|\vec{SP} \times \vec{v}|}{|\vec{v}|} = \frac{|(-1,0,0) \times (1,-5,1)|}{|(1,-5,1)|}$$

$$\begin{vmatrix} i & j & k \\ -1 & 0 & 0 \\ 1 & -5 & 1 \end{vmatrix} = (0, 1, 5)$$

Choose $\vec{v} = \vec{SQ} = (2-1, -4-1, 1-0)$
$\vec{v} = (1, -5, 1)$

and $\vec{SP} = (0-1, 1-1, 0-0) = (-1, 0, 0)$

$$\frac{|(0,1,5)|}{|(1,-5,1)|} = \frac{\sqrt{26}}{\sqrt{27}}.$$