Lecture 02 : vectors, dot-products, projections of vectors

Take away :

- A vector is determined by
  - length
  - direction

$\vec{v} = |\vec{v}| \cdot \left(\frac{\vec{v}}{|\vec{v}|}\right)$

length

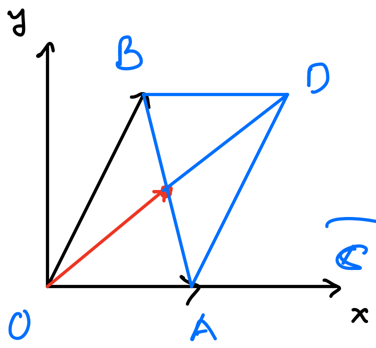
unit vector represents the direction

- dot product
- angle
- projection $\quad proj_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$
- orthogonal

$$\vec{b} = proj_{\vec{a}}(\vec{b}) + orth_{\vec{a}}(\vec{b})$$

$\vec{a} = (x_1, y_1, z_1)$ then $\vec{a} + \vec{b} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\vec{b} = (x_2, y_2, z_2)$ if $\alpha \in \mathbb{R}$ then $\alpha\vec{a} = (\alpha x_1, \alpha x_2, \alpha x_3)$



$\vec{OB} = (2, 4)$, $\vec{OA} + \vec{OB} = (5, 4)$

$\vec{OA} = (3, 0)$

parallelogram

$2\vec{OA} = \vec{OC} = (0, 6)$

<u>Example</u> $\vec{a} = (1, 3)$, $\vec{b} = (-3, 2)$

a) Compute $\vec{a} + \vec{b}$

b) $-3\vec{a} + 2\vec{b}$

c) Sketch $\vec{a}, \vec{b}, \vec{a} + \vec{b}, \vec{a} - \vec{b}$

<u>Theorem.</u> $|\alpha\vec{a}| = |\alpha| \cdot |\vec{a}|$,

Proof : $|\alpha\vec{a}| = |(\alpha x, \alpha y, \alpha z)| = \sqrt{(\alpha x)^2 + (\alpha y)^2 + (\alpha z)^2} = \sqrt{|\alpha|^2 (x^2 + y^2 + z^2)}$

let $\vec{a} = (x, y, z)$ $= |\alpha| . \sqrt{x^2 + y^2 + z^2} = |\alpha| |\vec{a}|$.

- Unit vector : given a vector $\vec{a} \neq \vec{0}$, $\dfrac{\vec{a}}{|\vec{a}|}$ is the unit vector
  is of length 1

<u>Ex</u> : $\vec{a} = (1, 2, 3)$, $|\vec{a}| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\dfrac{\vec{a}}{|\vec{a}|} = \left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right)$ is a unit vector (same direction as $\vec{a}$)

Properties

Theorem 2.12 (Properties of Vector Operations).

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors and c, d be scalars:

(a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(b) $\mathbf{a} + \mathbf{0} = \mathbf{a}$

(c) $0\mathbf{a} = \mathbf{0}$

(d) $c(d\mathbf{a}) = (cd)\mathbf{a}$

(e) $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

(f) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

(g) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

(h) $1\mathbf{a} = \mathbf{a}$

(i) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

Components $\vec{i}, \vec{j}, \vec{k}$

$\vec{i} = (1,0,0)$ $\quad$ $\vec{j} = (0,1,0)$ $\quad$ $\vec{k} = (0,0,1)$

$\vec{a} = (1,2,3)$ then

$\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ is the same as writing $\vec{a} = (1,2,3)$

Notes: 0 as a number is just zero

$\vec{0}$ as a vector: 2D: $\vec{0} = (0,0)$

3D: $\vec{0} = (0,0,0)$

Lecture 2 - dot product

* Given $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$ then the dot-product $\vec{a} \cdot \vec{b}$ is a real number (a scalar)

$$\boxed{\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2}$$

Note: $\vec{a} \cdot \vec{b}$ is NOT a vector

Example 3.4. Find $\mathbf{a} \cdot \mathbf{b}$ for the following vectors. Determine if the vectors are perpendicular or not.

(a) $\mathbf{a} = \langle 1, 0, 1 \rangle$ and $\mathbf{b} = \langle -1, 3, 1 \rangle$

$\vec{a} \cdot \vec{b} = 1 \cdot (-1) + 0 \cdot 3 + 1 \cdot 1 = 0$

$\hookrightarrow$ peependicular

(b) $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 4\mathbf{k}$

$\vec{a} = (3, -2, 1)$. $\vec{b} = (0, 2, 4)$

$\vec{a} \cdot \vec{b} = 3 \cdot 0 + (-2) \cdot 2 + 1 \cdot 4 = 0$

$\hookrightarrow$ peependicular

(c) $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{CB}$ where $A = (3, -2)$, $B = (5, 6)$, and $C = (-1, -3)$

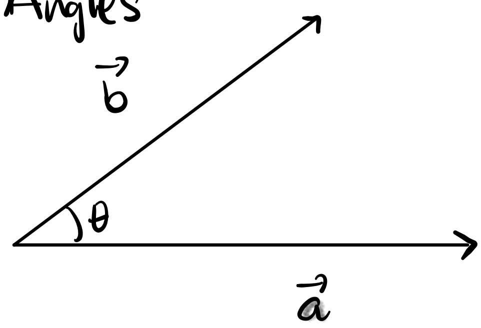
$\vec{a} = \overrightarrow{AB} = (5-3, 6-(-2))$ (end - start) or $B - A$

$\vec{b} = \overrightarrow{CB} = (5-(-1), 6-(-3))$

$\vec{a} = (2, 8)$ $\quad$ $\vec{a} \cdot \vec{b} = 2 \cdot 6 + 8 \cdot 9 = 12 + 72 = 84$

$\vec{b} = (6, 9)$ $\quad$ $\hookrightarrow$ not peependicular

* Angles



$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

magnitude (or length of $\vec{a}$)

$\vec{a} = (x, y, z)$

$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

$$\boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}}$$

If $\vec{a} \cdot \vec{b} = 0$ then $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} + 2k\pi$ or $-\frac{\pi}{2} + 2k\pi$

We say $\vec{a}$ and $\vec{b}$ are orthogonal, or perpendicular to each other

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Example 3.6. Find the angle between the following vectors.

(a) $\mathbf{a} = \langle 1, 0, 1 \rangle$ and $\mathbf{b} = \langle -1, 3, 2 \rangle$

$|\vec{a}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$,

$|\vec{b}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{1 \cdot (-1) + 0 \cdot 3 + 1 \cdot 2}{\sqrt{2} \cdot \sqrt{14}} = \frac{1}{\sqrt{28}}$$

(b) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 4\mathbf{k}$

$\vec{a} = (1, -1, 3)$, $|\vec{a}| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$

$\vec{b} = (0, 2, 4)$, $|\vec{b}| = \sqrt{0^2 + 2^2 + 4^2} = \sqrt{20}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{1 \cdot 0 + (-1) \cdot 2 + 3 \cdot 4}{\sqrt{11} \cdot \sqrt{20}} = \frac{10}{\sqrt{11} \cdot \sqrt{20}}$$

$\longrightarrow\ast$ Properties: $c \in \mathbb{R}$, $\vec{a}, \vec{b}$ are vectors

a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

b) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

c) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

d) $\vec{0} \cdot \vec{a} = 0$ ← 0 is the number 0

zero vector $\vec{0} = (0,0,0)$

e) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

Proof: $\vec{a} = (x,y,z)$, $|\vec{a}| = \sqrt{x^2+y^2+z^2}$

$$\vec{a} \cdot \vec{a} = x \cdot x + y \cdot y + z \cdot z = x^2 + y^2 + z^2 = |\vec{a}|^2$$

$\longrightarrow\ast$ Work

$$W = \vec{F} \cdot \vec{D}$$

force displacement vector

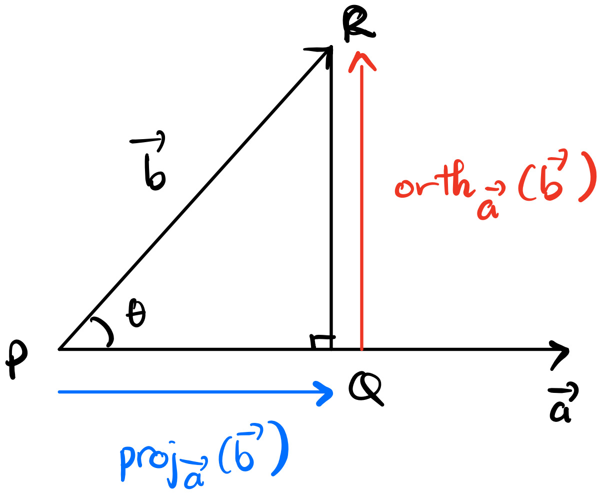
Example 3.8. A box is pushed with a constant force of $\mathbf{F} = \langle 1, 2, 3 \rangle$ Newtons. How much work is done in moving the box from $(1, 0, 1)$ to $(2, 1, 1)$?

$\vec{D} = (\text{end}) - (\text{start})$

$= (2,1,1) - (1,0,1) = (1,1,0)$

$W = \vec{F} \cdot \vec{D} = (1,2,3) \cdot (1,1,0) = 1 + 2 + 0 = \boxed{3}$

* Projection



projection of $\vec{b}$ onto $\vec{a}$ is a vector that has the same direction as $\vec{a}$

$$\boxed{comp_{\vec{a}}(\vec{b}) = |proj_{\vec{a}}(\vec{b})|}$$

↳ the magnitude of the projection $= |PQ|$

Remember:

$$\vec{b} = proj_{\vec{a}}(\vec{b}) + orth_{\vec{a}}(\vec{b})$$

Note:

in the right-triangle PQR:

$$|PQ| = |PR| . \cos\theta$$

$$comp_{\vec{a}}(\vec{b}) = \cancel{|\vec{b}|} \cdot \left(\frac{\vec{a} . \vec{b}}{|\vec{a}| . \cancel{|\vec{b}|}} \right) = \frac{\vec{a} . \vec{b}}{|\vec{a}|}$$

(you can cancel scalar $|\vec{b}|$)

$$\boxed{comp_{\vec{a}}(\vec{b}) = \frac{\vec{a} . \vec{b}}{|\vec{a}|}} = \text{magnitude } |proj_a(b)|$$

Now: if $\vec{v}$ is a vector then $\vec{v} = |\vec{v}| \cdot \frac{\vec{v}}{|\vec{v}|}$

magnitude

unit vector having the same direction as $\vec{v}$

thus:

$$\text{proj}_{\vec{a}}(\vec{b}) = |\text{proj}_{\vec{a}}(\vec{b})| \cdot \frac{\vec{a}}{|\vec{a}|} = \text{comp}_{\vec{a}}(\vec{b}) \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\text{proj}_a(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\boxed{\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\right) \vec{a} \quad \text{or} \quad \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right) \vec{a}}$$

and $\text{orth}_{\vec{a}}(\vec{b}) = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$

Example 3.13. Consider the vector $\mathbf{u} = \langle 5, 3 \rangle$ and the vector $\mathbf{v} = \langle 2, -1 \rangle$.

(a) Sketch **a** on the axes to the right.

(b) Calculate $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and sketch it on the axes as well.

(c) Calculate $\text{orth}_{\mathbf{v}}(\mathbf{u})$ and sketch it too.

$$\text{proj}_v(u) = \left(\frac{u \cdot v}{v \cdot v}\right) v = \frac{(5,3) \cdot (2,-1)}{(2,-1) \cdot (2,-1)} (2,-1)$$

$$= \frac{10-3}{5} (2,-1) = \frac{7}{5} (2,-1) = \left(\frac{14}{5}, \frac{-7}{5}\right)$$

$$\text{orth}_v(u) + \text{proj}_v(u) = u$$

$$\hookrightarrow \quad u - \text{proj}_v(u) = (5,3) - \left(\frac{14}{5}, \frac{-7}{5}\right)$$