Rate of convergence of vanishing viscosity

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Outline

History

Viscosity solutions

- A type of weak solution that is natural for the Hamilton-Jacobi equation introduced by Michael G. Crandall and Pierre-Louis Lions in 1983
- Applications in control theory, physics and economics
- One can obtain the viscosity solution through a procedure known as the vanishing viscosity procedure
- I consider the rate of convergence of this procedure
- This proof was given by Michael G. Crandall and Pierre-Louis Lions in the 80s

Setting

Problem

Consider the static Hamilton-Jacobi equation $u(x) + H(x, Du(x)) = 0$. The subject of the talk is the viscous Hamilton-Jacobi

$$u^\epsilon(x) + H(x, Du^\epsilon(x)) = \epsilon \Delta u^\epsilon(x).$$

Under nice assumption on H, the equation is a parabolic equation is is known to have smooth unique solutions.

Questions

1. What happens when we take $\epsilon \to 0$. Can we say $u^\epsilon \to u$ where u is the solution to the static problem?
2. If so, what is the rate of convergence? (Our focus)
3. We call this a viscosity solution, which is a type of weak solution.

Example

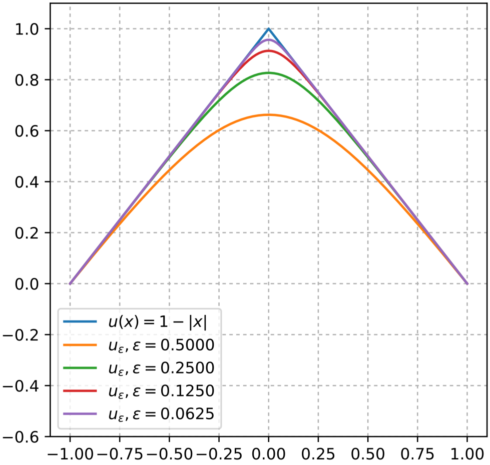

Figure: Solutions to the viscous 1D Eikonal equation: $|u^{\epsilon'}(x)| - 1 = \epsilon \Delta u^{\epsilon}(x)$

Viscosity solution: another way to understand

Definition (Viscosity solution)

(i) We define u to be a **viscosity subsolution** if for all test functions $\varphi \in C^1(\Omega)$, $u - \varphi$ has a max at x_0 in Ω, and $u(x_0) + H(x_0, D\varphi(x_0)) \leq 0$.

(ii) We define u to be a **viscosity supersolution** if for all test functions $\psi \in C^1(\Omega)$, $u - \psi$ has a minimum at x_0 in Ω, and $u(x_0) + H(x_0, D\psi(x_0)) \geq 0$.

A solution is said to be a viscosity solution if both conditions are satisfied.

Convergence of u^ϵ

- By the classical Bernstein's method, we are able to get

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}^n)} + \|Du^\epsilon\|_{L^\infty(\mathbb{R}^n)} < C.$$

  This gives us equicontinuity of the family u^ϵ (all members of the family are continuous with equal variation).
- Then, by Arzela Ascoli, there exists a subsequence $\{\epsilon_{k_j}\}$ such that $\{u^\epsilon_{k_j}\} \to u$.
- We deduce that u solves the static problem.

Convergence to u

- Our viscosity solution u is unique, which can be shown with the comparison principle.
- So we conclude that $u^\epsilon \to u$ as $\epsilon \to 0$.

Rate of Convergence of u^ϵ to u

Theorem (Convergence rate)

Assume that H satisfies

$$\begin{cases} H \in C^2(\mathbb{R}^n \times \mathbb{R}^n), \\ H, D_pH \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R)) \text{ for each } R > 0, \\ \lim_{|p|\to\infty} \inf_{x\in\mathbb{R}^n} \left(\frac{1}{2}H(x,p)^2 + D_xH(x,p)\cdot p\right) = +\infty \end{cases}$$

Assume further that $H \in Lip(\mathbb{R}^n \times B(0,R))$ for each $R > 0$. For each $\epsilon \in (0,1)$, let u^ϵ be the solution to the vanishing viscosity Hamilton Jacobi, and let u be the unique viscosity solution of the Hamilton Jacobi. Then there exists $C > 0$ such that

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}^n)} \le C\sqrt{\epsilon}.$$

We divide the proof into 5 steps.

Step 1: Test function from above

We use our favorite trick, the doubling variable technique.

$$\Phi^\delta(x,y) = u^\epsilon(x) - u(y) - \frac{|x-y|^2}{2\alpha} - \delta(\mu(x)+\mu(y)),$$

where we chose $\delta, \alpha > 0$ and $\mu \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\begin{cases} \mu(0) = 0, \mu(x) \geq 0 & \text{for all } x \in \mathbb{R}^n \\ \lim_{|x|\to\infty} \mu(x) = +\infty & \\ |D\mu(x)| + |D^2\mu(x)| \leq 1 & \text{in } \mathbb{R}^n. \end{cases}$$

Assume this has a max at (x_δ, y_δ):

Takeaway

Since $x \mapsto \Phi^\delta(x, y_\epsilon)$ attains a max at x_δ, so does $x \mapsto u^\epsilon(x) - (\frac{|x-y_\delta|^2}{2\alpha} + \delta\mu(x))$. We then get

$$u^\epsilon(x_\delta) + H(x_\delta, \frac{x_\delta - y_\delta}{\alpha} + \delta D\mu(x_\delta)) \leq \epsilon(\frac{n}{\alpha} + \delta\Delta\mu(x_\delta)) \leq \epsilon(\frac{n}{\alpha} + \delta).$$

Step 2: Test function from below

Since $y \mapsto \Phi^\delta(x_\delta, y)$ has a min at y_δ, this means that

$$y \mapsto u(y) + \frac{|x-y|^2}{2\alpha} + \delta\mu(y) = u(y) - (-\frac{|x-y|^2}{2\alpha} - \delta\mu(y))$$

.

Supersolution test

Using $(-\frac{|x-y|^2}{2\alpha} - \delta\mu(y))$ as our test function

$$u(y_\delta) + H(y_\delta, \frac{x_\delta - y_\delta}{\alpha} - \delta D\mu(y_\delta)) \geq 0.$$

What we have so far

$$\begin{cases} u^\epsilon(x_\delta) + H(x_\delta, \frac{x_\delta - y_\delta}{\alpha} + \delta D\mu(x_\delta)) \leq \epsilon(\frac{n}{\alpha} + \delta) \\ u(y_\delta) + H(y_\delta, \frac{x_\delta - y_\delta}{\alpha} - \delta D\mu(y_\delta)) \geq 0 \end{cases}$$

Two evaluations

- We get that $\Phi^\delta(x_\delta, x_\delta) \leq \Phi^\delta(x_\delta, y_\delta)$, which gives

$$\frac{|x_\delta - y_\delta|^2}{2\alpha} \leq u(x_\delta) - u(y_\delta) + \delta(\mu(x_\delta) - \mu(y_\delta)).$$

- $\Phi^\delta(y_\delta, y_\delta) \leq \Phi^\delta(x_\delta, y_\delta)$, which gives

$$\frac{|x_\delta - y_\delta|^2}{2\alpha} \leq u^\epsilon(x_\delta) - u^\epsilon(y_\delta) - \delta(\mu(x_\delta) - \mu(y_\delta)).$$

Result

$$\frac{|x_\delta - y_\delta|^2}{\alpha} \leq u(x_\delta) - u(y_\delta) + u^\epsilon(x_\delta) - u^\epsilon(y_\delta) \leq C|x_\delta - y_\delta| + C'|x_\delta - y_\delta| \leq 2C|x_\delta - y_\delta|.$$

This gives

$$|x_\delta - y_\delta| \leq C\alpha$$

Step 4: Lipschitzness of H

Since $H \in Lip(\mathbb{R}^n \times B(0,R))$, for each $R > 0$, pick δ:

$$H\left(y_\delta, \frac{x_\delta - y_\delta}{\alpha} - \delta D\mu\left(y_\delta\right)\right) - H\left(x_\delta, \frac{x_\delta - y_\delta}{\alpha} - \delta D\mu\left(y_\delta\right)\right) \leq C\alpha$$

$$H\left(x_\delta, \frac{x_\delta - y_\delta}{\alpha} - \delta D\mu\left(y_\delta\right)\right) - H\left(x_\delta, \frac{x_\delta - y_\delta}{\alpha} + \delta D\mu\left(x_\delta\right)\right) \leq C\delta$$

Result

$$H\left(y_\delta, \tfrac{x_\delta - y_\delta}{\alpha} - \delta D\mu\left(y_\delta\right)\right) - H\left(x_\delta, \tfrac{x_\delta - y_\delta}{\alpha} + \delta D\mu\left(x_\delta\right)\right) \leq C(\alpha + \delta)$$

Step 5: Combining

We get that for all x

$$\begin{aligned} u^\epsilon(x) - u(x) &\le u^\epsilon(x_\delta) - u(y_\delta) \\ &\le H(x_\delta, \frac{x_\delta - y_\delta}{\alpha} + \delta D\mu(x_\delta)) - H(y_\delta, \frac{x_\delta - y_\delta}{\alpha} - \delta D\mu(y_\delta)) + \epsilon(\frac{n}{\alpha} + \alpha) \\ &\le C(\delta + \alpha) + \epsilon(\frac{n}{\alpha} + \delta). \end{aligned}$$

Letting $\delta \to 0$, we conclude that

$$u^\epsilon(x) - u(x) \le C(\epsilon/\alpha + \alpha).$$

Choosing $\alpha = \epsilon$ (AM-GM) gives us one side of the desired inequality. Performing a symmetric arguments gets us the desired result.

Final

A symmetric argument completes the proof that

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}^n)} \le C\sqrt{\epsilon}$$

In optimal control theory, the minimum cost (called the value function) can be formulated through the Dynamic Programming Principle. **This function gives us the control that induces the minimum cost.**

Theorem

The value function u is a viscosity solution of the static Hamilton–Jacobi equation

$$\lambda u + H(x, Du) = 0 \quad \in \mathbb{R}^n,$$

where for $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$

$$H(x, p) = \sup_v \{-b(x, v) \cdot p - f(x, v)\}.$$

Future

Future

This was done as a reading course with Dr. Son Tu. The proof I presented was given by Crandall and Lions. In the summer, we will continue studying HJE by working on a paper titled "Uniqueness set for an ergodic problem with state-constraint."