5 Equations of Lines and Planes (Part A)

5.1 Introduction to Lines in Space – Video Before Class

Objective(s):

- Define lines in space several different ways and learn some basic terminology.
- Determine when lines are parallel.
- Determine when lines intersect or not.

Theorem 5.1. Vectors **v** and **w** are parallel if and only if $\vec{v} = k\vec{w}$ for some scalar k.

Alternatively if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ then **v** and **w** are parallel if and only if

$$\begin{cases} v_1 = k w_1 \\ v_2 = k w_2 \\ v_3 = k w_3 \end{cases} \quad \text{or} \quad \vec{v} \times \vec{w} = \vec{0} \quad \text{or} \quad |\vec{v} \times \vec{w}| = 0$$

Definition(s) 5.2.

(a) A **vector equation** for the line L through $P_0(x_0, y_0, z_0)$ parallel to **v** is given by:

$$P(t) = \vec{r_0} + t\vec{v} \qquad -\infty < t < \infty$$

direction vector

$(\vec{r_0} = \overrightarrow{OP_0})$

where $\mathbf{r_0}$ is the position vector of P_0.

(b) t is called the parameter.

(c) Alternatively if $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$. we can write a line in **parametric equation**

$(x(t), y(t), z(t))$

$$P(t) = (a, b, c) + t(x_0, y_0, z_0) \qquad -\infty < t < \infty$$

(parametric form)

(d) a, b, and c are called direction numbers of L.

(e) Finally we can chose to solve each of the parametric equations for t to get the symmetric form of L given by:

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Definition(s) 5.3.

(a) Suppose $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ and $\mathbf{s}(t) = \mathbf{s}_0 + t\mathbf{w}$ intersect. Then the angle between them is the same as the angle between $\vec{v}, \vec{w}$. Likewise $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are **parallel** if and only if $\vec{v} \parallel \vec{w}$.

(b) Lines that do not intersect and are not parallel are called skew.

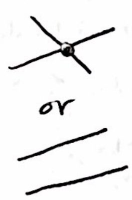
We don't really have skew lines in 2D though. Let's take a closer look at them here: `https://tinyurl.com/mth234-002`

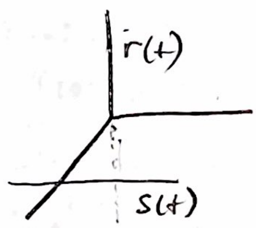
Example 5.4. Consider each pair of lines. Determine if

(i) They intersect (by finding the point of intersection).

(ii) They don't intersect and are parallel.

(iii) They don't intersect and are skew.





skew $r(\cdot), s(\cdot)$

(a) $\mathbf{r}_1(t) = \langle 3, 1, 0\rangle + t\langle 2, 0, 1\rangle$ and $\mathbf{r}_2(s) = \langle 1, -2, 5\rangle + s\langle -1, 3, -2\rangle$

- they are not parallel since $(2,0,1)$ is not $\parallel$ $(-1,3,-2)$
- intersect? $(3+2t, 1, t) = (1-s, -2+3s, 5-2s)$

$$\begin{cases} 3+2t = 1-s \\ 1 = -2+3s \Rightarrow s = 1 \\ t = 5-2s \end{cases} \quad \begin{matrix} \nearrow t = -3/2 \\ \searrow t = 3 \end{matrix} \quad \text{has no solution}$$

$\Rightarrow$ • skew

(b) $\mathbf{r}_1(t) = \langle 1+2t, 9-5t, t\rangle$ and $\mathbf{r}_2(s) = \langle 3-s, 3+5s, 2s\rangle$ — not parallel

$$\begin{cases} 1+2t = 3-s \\ 9-5t = 3+5s \\ t = 2s \end{cases} \quad \begin{cases} 1+4s = 3-s \\ 9-10s = 3+5s \end{cases} \Rightarrow \begin{cases} 5s = 2 \\ 6 = 15s \end{cases} \Rightarrow \begin{cases} s = 2/5 \\ s = 2/5 \end{cases}$$

there is one intersection : $r_2\left(\frac{2}{5}\right) = \left(3-\frac{2}{5}, 3+5\cdot\frac{2}{5}, 2\cdot\frac{2}{5}\right)$

(c)
L_1: $x(t) = t+3$, $y(t) = -2t-5$, $z(t) = 4$
L_2: $x(s) = 3-2s$, $y(s) = 4s$, $z(s) = -9$ → can't intersect

L_1: $(3,-5,4) + t(1,-2,0)$ $\quad\parallel$
L_2: $(3,0,-9) + s(-2,4,0)$

$$(1,-2,0) = \left(\frac{-1}{2}\right)\cdot(-2,4,0)$$

parallel

5.2 Parametrizations, Line Segments, and More Examples – During Class

Objective(s):

- Determine when two parametrizations describe the same line.
- Create a way to parametrize a piece of a line.
- Gain more exposure to types of line problems that can be asked.

Example 5.5. One small annoyance with parametrizing lines is that the parametrization is not unique. Using a graphing utility show that

$$L_1: \quad x = 1 + 2t \quad y = 5 - 2t \quad z = 6t$$

$$L_2: \quad x = 3 - s \quad y = 3 + s \quad z = 6 - 3s$$

are the same line

Theorem 5.6. Two parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ describe the same line if they are parallel and going through one common point

Now let's use **Theorem 5.6** to show that L_1 and L_2 describe the same line in **Example 5.5.**

$L_1: \ (1,5,0) + t(2,-2,6)$ $\qquad (2,-2,6) = -2\,(1,1,-3)$

$L_2: \ (3,3,6) + s(-1,1,-3)$ $\qquad$ they are parallel

then for $s = 2$: $\quad L_2(2) = (1,5,0) \in L_1$

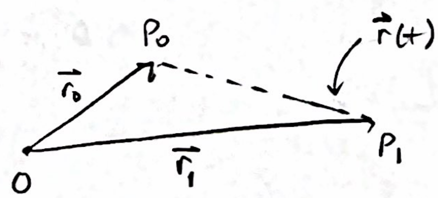
therefore they are the same line.

Theorem 5.7 (Equation of a line segment)**.** The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is give by the vector equation

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$$

Example 5.8. Find an equation for the line segment from $\overset{P_0}{(1,2,3)}$ to $\overset{P_1}{(5,2,0)}$. $\quad$ (recall: $\vec{r_0} = \overrightarrow{OP_0}$)

$$\vec{r}(t) = (1-t)(1,2,3) + t(5,2,0)$$
$$= (1,2,3) + t\big((5,2,0) - (1,2,3)\big)$$
$$= (1,2,3) + t(4,0,-3)$$



Example 5.9.

(a) Find parametric equations of the line that passes through the points $A(2,3,4)$ and $B(1,0,-1)$.

$$\vec{r}(t) = (1-t)(2,3,4) + t(1,0,-1)$$
$$= (2,3,4) + t\big((1,0,-1) - (2,3,4)\big)$$
$$= (2,3,4) + t(-1,-3,-5)$$

(b) At what point does the line intersect the xy-plane.

$\hookrightarrow z = 0$

$$4 - 5t = 0 \rightarrow t = \frac{4}{5}$$

point of intersection : $\vec{r}\left(\frac{4}{5}\right) = (2,3,4) + \frac{4}{5}(-1,-3,-5)$

Example 5.10. The lines $\mathbf{r}_1(t) = \langle 1+t, 1-t, 2t \rangle$ and $\mathbf{r}_2(s) = \langle 2-s, s, 2 \rangle$ intersect at $(2,0,2)$. Determine the angle between the lines.

$$\vec{r}_1(t) = (1,1,0) + t\underbrace{(1,-1,2)}_{\vec{u}}$$
$$\vec{r}_2(t) = (2,0,2) + s\underbrace{(-1,1,0)}_{\vec{v}}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{1\cdot(-1) + (-1)\cdot 1 + 2\cdot 0}{\sqrt{1^2+1^2+2^2}\cdot\sqrt{1^2+1^2+0^2}} = \frac{-2}{\sqrt{6}\cdot\sqrt{2}} = \frac{-2}{\sqrt{12}}$$
$$= \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

$$\theta = \arccos\left(\frac{-1}{\sqrt{3}}\right)$$

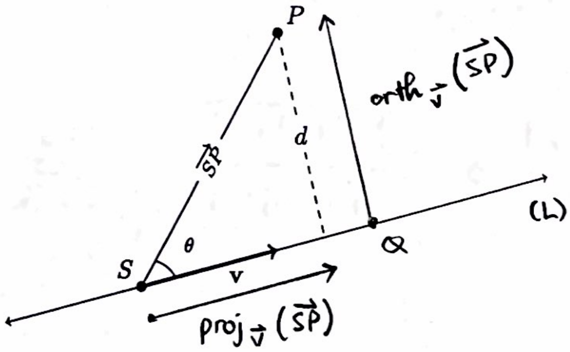
5.3 Distance from a Point to a Line – During Class

Objective(s):

- Develop a formula to determine the distance from a point to a line.
- Utilize the newly developed formula to calculate the distance from a point to a line in space!

Now we have lines, we have points, lets talk about distance!

Here is a pretty picture



$$\text{distance}(P, (L)) = |PQ| = |d|$$

$$\vec{SP} \times \vec{v} = |\vec{SP}| \cdot |\vec{v}| \cdot \sin\theta$$

↳ here θ is acute (projection)

thus

$$|\vec{PQ}| = |\vec{SP}| \cdot \sin\theta$$

$$= \cancel{|\vec{SP}|} \cdot \frac{|\vec{SP} \times \vec{v}|}{\cancel{|\vec{SP}|} \cdot |\vec{v}|}$$

thus

$$|d| = \frac{|\vec{SP} \times \vec{v}|}{|\vec{v}|}$$

Theorem 5.11. The distance from a Point P to a line through S parallel to $\mathbf{v}$ is given by

$$\frac{|\vec{SP} \times \vec{v}|}{|\vec{v}|}$$

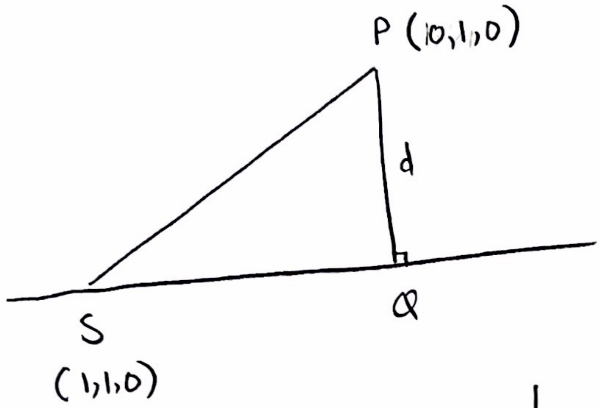
And this is a perfectly good Theorem but that triangle looks like something we have seen before when we were talking about projections. So in fact....

Theorem 5.12. The distance from a Point P to a line through S parallel to $\mathbf{v}$ is given by

$$|\text{orth}_{\vec{v}}(\vec{SP})| = \left(|\vec{SP}|^2 - |\text{proj}_{\vec{v}}(\vec{SP})|^2\right)^{1/2}$$

Example 5.13. Find the distance between the point $(0, 1, 0)$ and the line containing the points $(1, 1, 0)$ and $(2, -4, 1)$.

(a) By using **Theorem 5.11**



$$\vec{v} = (2,-4,1) - (1,1,0)$$

$$\vec{v} = (1,-5,1)$$

$$\vec{SP} = (0,1,0) - (1,1,0) = (-1,0,0)$$

$$d = \frac{|\vec{SP} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{0^2+1^2+5^2}}{\sqrt{1^2+5^2+1^2}} = \frac{\sqrt{26}}{\sqrt{27}}$$

$$\vec{SP} \times \vec{v} = \begin{vmatrix} i & j & k \\ -1 & 0 & 0 \\ 1 & -5 & 1 \end{vmatrix}$$

$$= (0, 1, 5)$$

(b) By using **Theorem 5.12**

$$\text{proj}_{\vec{v}}(\vec{SP}) = \left(\frac{\vec{SP} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \vec{v}$$

$$= \frac{(-1,0,0) \cdot (1,-5,1)}{|\vec{v}|^2} \cdot \vec{v}$$

$$= \frac{-1}{|\vec{v}|^2} \cdot \vec{v}$$

$$|\text{proj}_{\vec{v}}(\vec{SP})| = \frac{1}{|\vec{v}|} = \frac{1}{\sqrt{27}}$$

thus $$d = \sqrt{|v|^2 - |\text{proj}_v(\vec{SP})|^2} = \left(27 - \frac{1}{27}\right)^{1/2}$$

$$= \sqrt{\frac{26}{27}}.$$