4 The Cross Product

4.1 Review and Introduction of the Cross Product – Video Before Class

Objective(s):

- Define the cross product
- Review how to find determinants of 3x3 matrices.
- Use the cross product to find a vector that is perpendicular to two other vectors.

The goal of this class is given two non-zero non-parallel vectors **a**,**b** to be able to find a vector **n** that is

perpendicular to both a and b.

Definition(s) 4.1. The Cross Product of $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is given by the determinant:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example 4.2. Evaluate $\langle 1, 2, 3\rangle \times \langle -2, 1, 0\rangle$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -2 & 1 & 0 \end{vmatrix} = i\underbrace{\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}}_{(-3)} - j\underbrace{\begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}}_{6} + k\underbrace{\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}}_{5}$$

$$(-3, -6, 5)$$

Theorem 4.3.

(a) The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(b) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \le \theta \le \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |a| \cdot |b| \sin\theta$$

(c) Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if number (but $\vec{a} \times \vec{b}$ = vector)

$$|\mathbf{a} \times \mathbf{b}| = 0$$

Heart of the Proofs:

(a) Apply dot product with both $\mathbf{a}$ and $\mathbf{b}$ and see that they are 0.

(b) Expand $|\mathbf{a} \times \mathbf{b}|^2$ carefully and group cleverly to get $|\mathbf{a} \times \mathbf{b}|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

(c) Comes from (b).

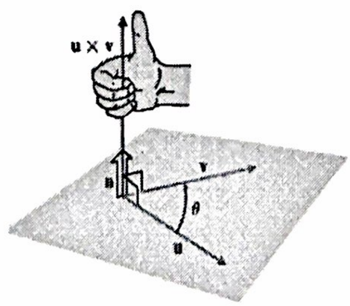
Example 4.4. Find a vector $\mathbf{u}$ that satisfies $\mathbf{u} \cdot \langle 9, 3, 1\rangle = 0$ and $\mathbf{u} \cdot \langle -2, 4, 0\rangle = 0$

Choose $\vec{u} = (9,3,1) \times (-2,4,0)$

$$= \begin{vmatrix} i & j & k \\ 9 & 3 & 1 \\ -2 & 4 & 0 \end{vmatrix} = i\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} - j\begin{vmatrix} 9 & 1 \\ -2 & 0 \end{vmatrix} + k\begin{vmatrix} 9 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= (-4, -2, 42)$$

Theorem 4.5 (Direction of the Cross Product). Take two non-zero non-parallel vectors $\mathbf{a}$, $\mathbf{b}$. Then the direction of $\mathbf{a} \times \mathbf{b}$ is determined by the right-hand-thumb-rule That is: the way your right thumb points when your right-hand fingers curl through the angle θ from $\mathbf{a}$ to $\mathbf{b}$.



4.2 Properties and Applications of the Cross Product – During Class

Objective(s):

- Determine and utilize properties of the cross product.
- Use the cross product to calculate areas of triangles and parallelograms.
- Apply the cross product to the physical application of torque.

Example 4.6. Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle 2, 4, 6 \rangle$.

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = i\underbrace{\begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix}}_{} - j\underbrace{\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}}_{} + k\underbrace{\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}}_{}$$

$$= \quad 10 \qquad 8 \qquad 2$$

$$= (10, -8, 2)$$

$$(a \times b) \cdot a = (10, -8, 2) \cdot (1, 1, -1) = 10 - 8 - 2 = 0$$

$$(a \times b) \cdot b = (10, -8, 2) \cdot (2, 4, 6) = 20 - 32 + 12 = 0$$

Theorem 4.7 (Properties of the Cross Product). Let **a**, **b**, **c** be vectors and r, s are scalars:

(a) $\mathbf{a} \times \mathbf{b} =$ $-(b \times a)$

(b) $(r\mathbf{a}) \times (s\mathbf{b}) =$ $(rs)(a \times b)$

(c) $\mathbf{0} \times \mathbf{a} =$ $\vec{0}$ vector

(d) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) =$ $a \times b + a \times c$

(e) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} =$ $b \times a + c \times a$

Example 4.8. Given that $\langle 1, 1, 0 \rangle \times \langle 3, 4, -2 \rangle = \langle -2, 2, 1 \rangle$ quickly calculate the following:

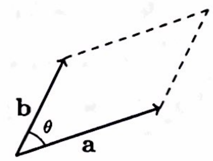
(a) $\langle 3, 4, -2 \rangle \times \langle 1, 1, 0 \rangle$ $= (2, -2, -1)$

(b) $\langle 4, 4, 0 \rangle \times \langle 3, 4, -2 \rangle$ $= (-8, 8, 4)$

Recall from geometry that a parallelogram with side lengths x and y with the angle θ between its sides has area: $A = xy\sin\theta$. Translated into MTH 234 notation:

Theorem 4.9. The parallelogram formed by vectors **a** and **b** with angle θ between them is given by:

Area of $\|$-ogram = $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\theta$



Example 4.10. Find the area of the parallelogram generated by $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 3\mathbf{k}$.

$u = (1,-1,0) \qquad v = (0,1,3)$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = i\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= (-3,-3,1)$$

$$\text{area} = |(-3,-3,1)| = \sqrt{3^2+3^2+1^2} = \sqrt{19}$$

Example 4.11. Find the area of the triangle with vertices $P(1,0,1)$, $Q(-2,1,3)$, and $R(4,2,5)$.

$\overrightarrow{PQ} = (-2,1,3) - (1,0,1) = (-3,1,2)$ end - start

$\overrightarrow{PR} = (4,2,5) - (1,0,1) = (3,2,4)$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = i\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} - j\begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix} + k\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (0,18,-9)$$

$$\text{Area} = \frac{1}{2}\sqrt{0^2+18^2+9^2} =$$

triangle

Example 4.12. Find two unit vectors orthogonal to both $\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$.

$u = (0, 1, -1)$

$v = (1, 1, 0)$

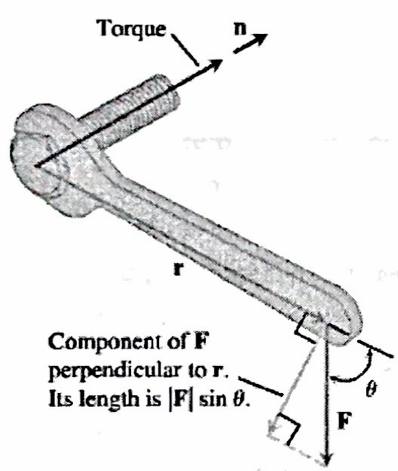
$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} - j\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (1,-1,-1) \quad \text{magnitude } \sqrt{3}$$

two vectors are

$$\frac{(1,-1,-1)}{\sqrt{3}} \quad \text{and} \quad -\frac{(1,-1,-1)}{\sqrt{3}}$$

So our application to the real world of the day is Torque! Here is the picture



Recall from you favorite physics class that

Torque = (*Force*)(*Distance* from pivot).

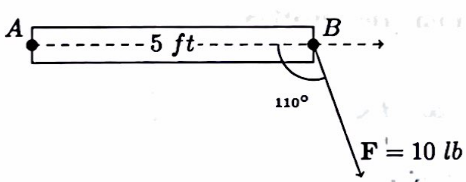
So long as the force is being applied perpendicular to the distance vector. But what if its not?

The magnitude of the torque vector is = $|\vec{r} \times \vec{F}|$

And what about direction? $\perp(\vec{r}$ and $\vec{F})$

The torque vector is of course given by: $\vec{r} \times \vec{F}$

Example 4.13. Find the magnitude of the torque generated by force $\mathbf{F}$ at the pivot point A in the figure below



$$|F| \cdot |r| \cdot \sin\theta = 10 \times 5 \times \sin(70^\circ)$$

Theorem

(a) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(b) $(\alpha\vec{a}) \times (\beta\vec{b}) = (\alpha\beta)(\vec{a} \times \vec{b})$

(c) $\vec{0} \times \vec{a} = \vec{0}$

(d) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(e) $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

Example 4.8. Given that $\underset{\vec{a}}{\langle 1,1,0\rangle} \times \underset{\vec{b}}{\langle 3,4,-2\rangle} = \langle -2,2,1\rangle$ quickly calculate the following:

(a) $\langle 3,4,-2\rangle \times \langle 1,1,0\rangle$

(b) $\langle 4,4,0\rangle \times \langle 3,4,-2\rangle$

a) $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} = -(-2,2,1) = (2,-2,1)$

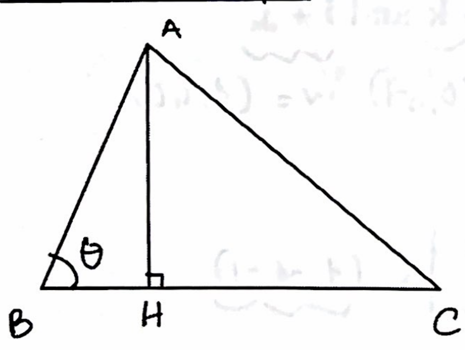
b) $4\vec{a} \times \vec{b} = (-8,8,4)$

Area of triangle

$S_{ABC} = \frac{1}{2} AB \cdot BC \cdot \sin\theta$

why?

$AH = AB \sin\theta$



$S_{ABC} = \frac{1}{2} AH \cdot BC$

$= \frac{1}{2} AB \cdot BC \sin\theta$.

thus

$S_{ABC} = |\overrightarrow{BA} \times \overrightarrow{BC}|$