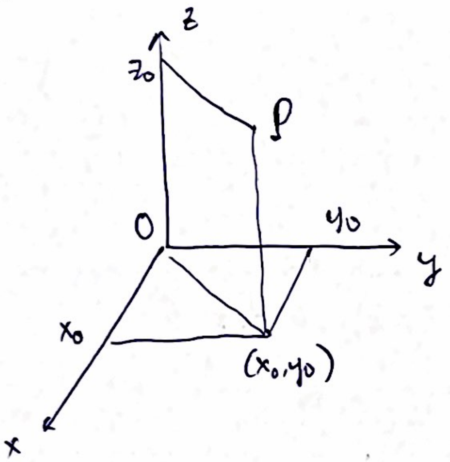
1 Three Dimensional Coordinate System

1.1 Space and Points in Space – Video Before Class

Objective(s):

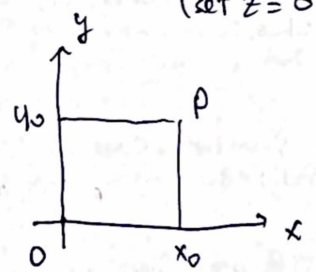
- Upgrade from two dimensional system to a three dimension system.
- Comprehend and be able to visualize our three dimensional system.
- Plot points in space.



a point $P(x_0, y_0, z_0) \to$ coordinates $\underbrace{(x_0, y_0, z_0)}$

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$
ordered tripple

the projection to xy-plane
(set $z = 0$)



Definition(s) 1.1.

(a) The Coordinate system consist of the x-axis, y-axis, and z-axis. Which all meet at a common point and are perpendicular to one another.

(b) The common meeting point of the coordinate axes is called the origin.

(c) The direction of the z-axis is determined by the right-hand-thumb rule

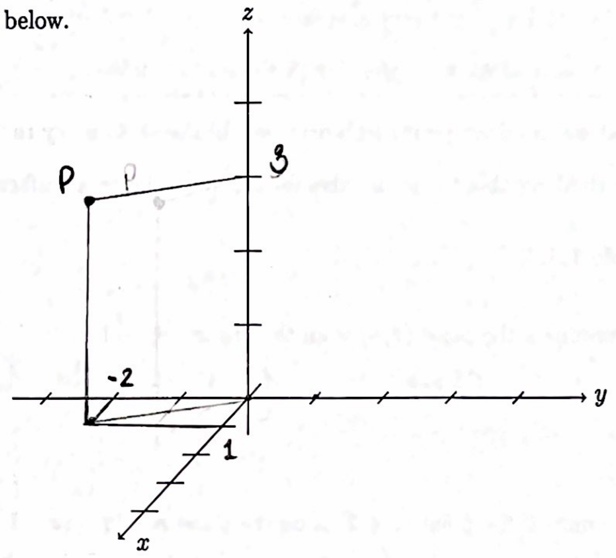
(d) The three coordinate axes determine three planes. The Oxy contains the x-axis, y-axis and so on.
($z = 0$)

(e) The coordinate planes divide space into 8 regions called octant.

(f) The Oxy is in the foreground, determined by the Ox, Oy axes.

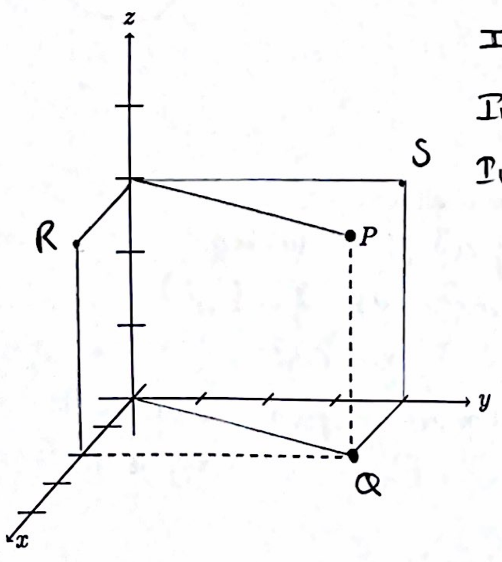
Just as points in a plane are determined by an (x, y) ordered pair, points in space are determined by an (x, y, z) ordered triple. If the point P is determined by $(1, 2, 3)$ then 1 is the x−coordinate, 2 is the y−coordinate, and 3 is the z−coordinate.

Example 1.2. Graph $P(1, -2, 3)$ on the coordinate axes below.



Although it is not obvious right now we will become very interested in projections.

Example 1.3. Below the point $P(2, 4, 3)$ is graphed. Graph the projections into the xy-plane (call it Q), the xz-plane (call it R), and the yz-plane (call it S) then write the coordinates for each of the projections.



In xy-plane: Q(2,4,0)
In xz-plane: R(2,0,3)
In yz-plane: S(0,4,3)

Definition(s) 1.4. The Cartesian product $\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. It is called a canonical coordinate system.

1.2 Surfaces in Space – During Class

Objective(s):

- Sketch simple surfaces in space.
- Determine when a point lies on a specified surface.

Now that we can draw points let's draw lots of them! So many that we start making some surfaces. To help us upgrade let's start by thinking about what it takes for a point to be on a surface... or a curve in $\mathbb{R}^2$.

Example 1.5.

(a) Determine if the point $(1, 4)$ is on the line $x - 4y = 1$

Check: $\underbrace{1 - 4 \cdot 4}_{-15} \neq 1$ No

(b) Determine if the point $(1, 4, 2)$ is on the plane $x - 4y + 8z = 1$

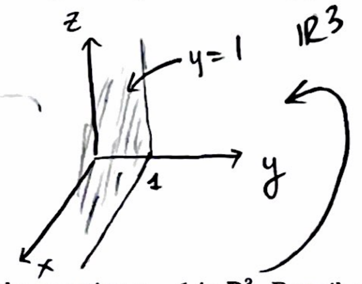
$\underbrace{1 - 4 \cdot 4 + 8 \cdot 2} = 1$ Yes

(c) Determine if the point $(1, -3, 0)$ is on the surface $xyz + x^2 = y$

$1 \cdot (-3) \cdot 0 + 1^2 = -3$ No

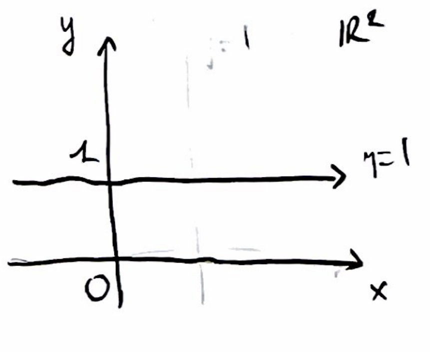
Example 1.6.

(a) Graph the equation $y = 1$ on the xy-plane. Describe it in words as best as possible.



the collection of all points having coordinates $(x, 1, z)$
$x, z \in \mathbb{R}$
the plane Oxz - shifted to the right by 1 unit

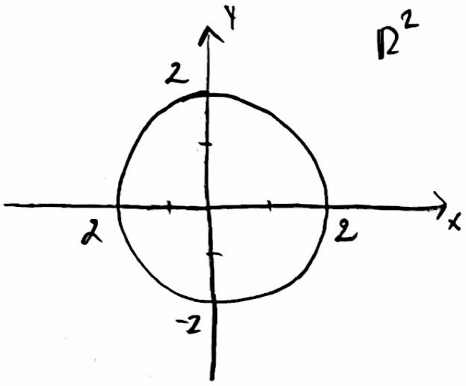
(b) Graph the equation $y = 1$ in $\mathbb{R}^3$. Describe it in words as best as possible.



the collection of all points $(x, 1)$, $x \in \mathbb{R}$
or the collection of all points having distance 1 to the x-axis

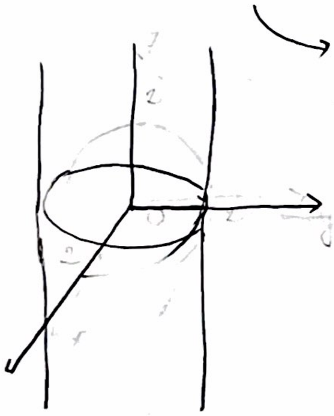
Example 1.7.

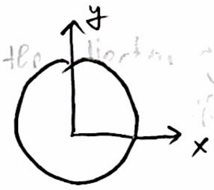
(a) Graph the equation $x^2 + y^2 = 4$ in the xy-plane. Describe it in words as best as possible.



the collection of all points having distance to the origin $= 2$ ($\mathbb{R}^2$)

(b) Graph the equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$. Describe it in words as best as possible.



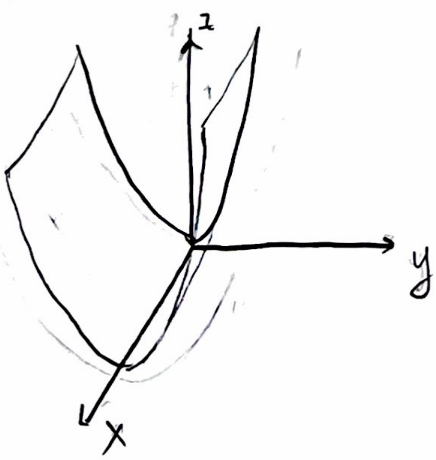


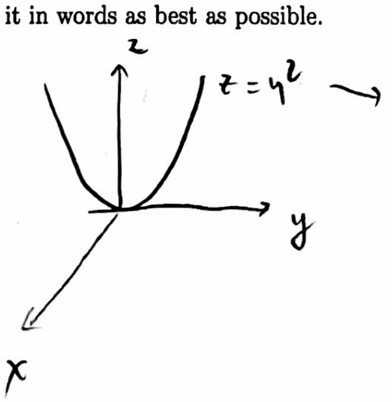
missing z-variable

↳ shift along the z-axis

(a cylinder)

(c) Graph the equation $z = y^2$ in $\mathbb{R}^3$. Describe it in words as best as possible.





→ shift along x-axis

Equations like these that are missing a variable are quite nice and will get a special name in 12.6. They will continue to come up throughout the course

1.3 Spheres in Space – During Class

Objective(s):

- Extend our well known distance equation from 2 variables to 3 variables.
- Draw a sphere in space.
- Be able to describe a sphere given its equation.

There will be several times in this course where we can upgrade from a well known 2 dimensional equation / system by "sprinkling in some z's". This is indeed one of those times

Definition(s) 1.8. The distance __________ between the points $P(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is given by:

$$|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

Example 1.9. Calculate the distance between $(1, 2, 3)$ and $(3, -1, 0)$.

$$\sqrt{(1-3)^2+(2-(-1))^2+(3-0)^2} = \sqrt{2^2+3^2+3^2}$$
$$= \sqrt{4+9+9} = \sqrt{22}$$

And with 3 dimensional distance we can define the set of all points equidistant from a center point (aka a __________)

Definition(s) 1.10. An equation of a sphere with center $C(h, k, l)$ and radius r is:

$$(x-h)^2+(y-k)^2+(z-l)^2 = r^2$$

Example 1.11 (Now you try! WW 12.1.3). Find an equation of a sphere with radius 2 centered at the point $(1, -2, 3)$.

$$(x-1)^2+(y+2)^2+(z-3)^2 = 4$$

This type of problem is relatively straight forward but we can turn it around to make a harder problem.

Example 1.12. Describe the surface $x^2 + y^2 + z^2 = 1$ in words as best as possible.

collection of all points in $\mathbb{R}^3$ having distance 1 to the origin

Example 1.13. Describe the surface $x^2 - 2x + y^2 + z^2 + 4z = 4$ in words as best as possible.

"Complete the square"

$a^2 + 2ab + b^2 = (a+b)^2$

$a^2 - 2ab + b^2 = (a-b)^2$

$$(x^2 - 2x + 1) + (y^2) + (z^2 + 4z + 4) = 1 + 4 + 4 = 9$$

$$(x-1)^2 + y^2 + (z+2)^2 = 3^2$$

center: $(1, 0, -2)$

radius: 3

the collection of all points in $\mathbb{R}^3$ having distance 3 to the point $(1,0,-2)$

We will always try to get as far as possible in class. Completed notes will be available on the course site for the "During Class" portions of the notes. The filled in video notes are available in the video!

2 Vectors

2.1 Basic Properties of Vectors – Video Before Class

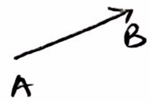
Objective(s):

- Comprehend what vectors are from their definitions and pictures.
- Define and determine the length of a vector.
- Define what it means for two vectors to be equal.

Definition(s) 2.1.

(a) A vector is used to indicate a quantity that has both length and direction (such as velocity or ?)

(b) Suppose a particle moves along a line segment form point A to point B. The corresponding vector has starting point A (the tail) and the ending point B (the tip). We indicate this vector by writing $\mathbf{v} = \overrightarrow{AB}$.



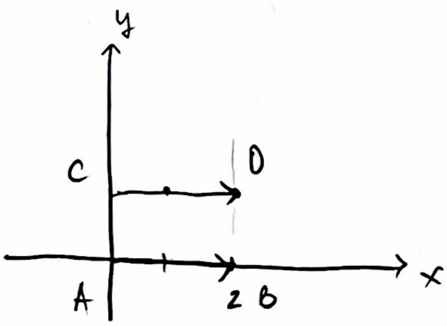
(c) Its length is denoted $|AB|$.

(a) If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ then: $|AB| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

(b) If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ then: $|AB| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

(d) Two vectors are considered equal if they have the same length and direction.

Picture:



$\overrightarrow{CD} = \overrightarrow{AB}$ since the have the same length and direction.

Now since vectors are considered equal so long as the same direction and length (or magnitude) we might as well move one of the points to be in a convenient location. In particular let's move the initial point to the ________.

Definition(s) 2.2.

(a) If **a** is a two-dimensional vector with initial point at the origin and terminal point $P(a_1, a_2)$ then

$\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2 \rangle$ is the coordinate of P.

(b) If **a** is a three-dimensional vector with initial point at the origin and terminal point $P(a_1, a_2, a_3)$ then

$\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the coordinate of P.

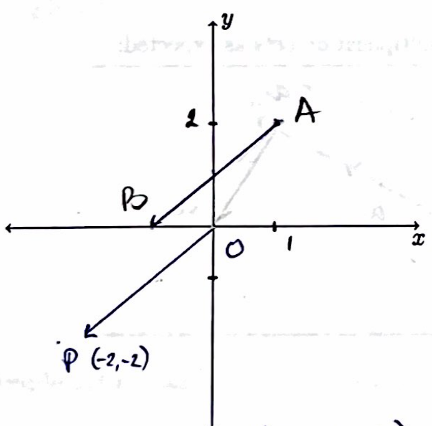
Example 2.3.

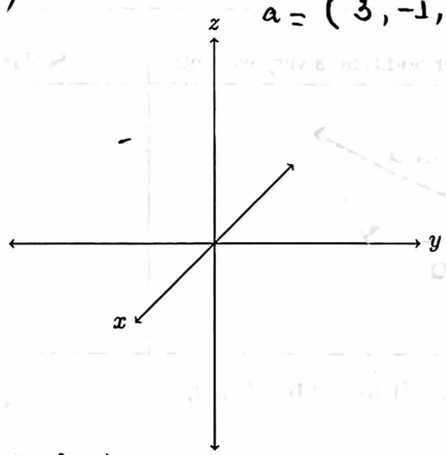
In each case find a position vector **a** so that $\mathbf{a} = \overrightarrow{AB}$. Graph both on the coordinate axes to verify that they are equal.

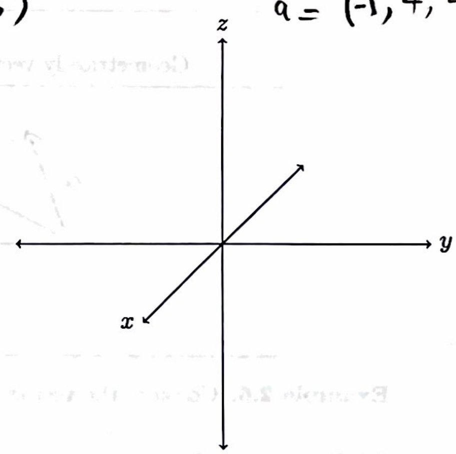
$A(1, 2),\ B(-1, 0)$ $a = (2,2)$

$A(2, 1, 0),\ B(-1, 2, -3)$ $a = (3,-1,3)$

$A(-1, 4, -5),\ B(0, 0, 0)$ $a = (-1,4,-5)$







$(O = (0,0))$ take $B - A$

$\overrightarrow{AB} = \overrightarrow{OP}$ what is $\vec{P}$

$\overrightarrow{AB} = (-1,0) - (1,2) = (-2,-2)$

, so we talk about $\overrightarrow{AB}$ like $(-2,-2)$ only

Definition(s) 2.4. The norm or magnitude of **a** is denoted by $|a|$ or $\|a\|$ is given by:

(a) If $\mathbf{a} = \langle a_1, a_2 \rangle$ then $|\mathbf{a}| = \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$

(b) If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ then $|\mathbf{a}| = \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

2.2 Basic Vector Operations – Video After Class

Objective(s):

- Define vector addition and scalar multiplication and be able to visualize their actions.
- Develop some properties of vector addition and scalar multiplication.
- Get exposure to types of problems that can be asked regarding vector addition and scalar multiplication.

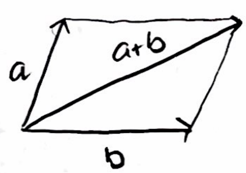
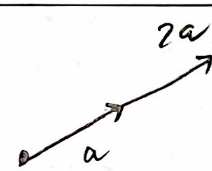
Now that we have these new mathematical objects we want to know how they interact with each other and real numbers (also called scalars).

Definition(s) 2.5. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors and c be a scalar. Then:

$$\mathbf{a} + \mathbf{b} = (a_1+b_1,\ a_2+b_2,\ a_3+b_3)$$

$$c\mathbf{a} = (ca_1, ca_2, ca_3)$$

Similarly for 2 dimensional vectors (or really any dimension we want).

Geometrically vector addition is very elegant:	Scalar multiplication acts as expected:
	

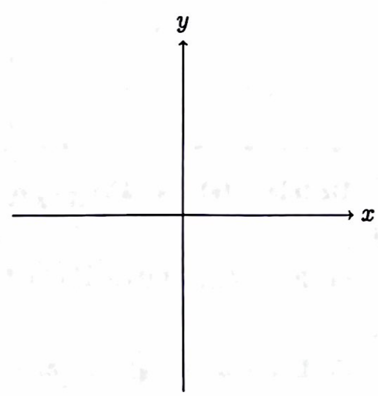
Example 2.6. Consider the vectors $\mathbf{a} = \langle 1, 3 \rangle$ and $\mathbf{b} = \langle -3, 2 \rangle$.

(a) Evaluate $\mathbf{a} + \mathbf{b}$

$a + b = (-2, 5)$

(b) Evaluate $-3\mathbf{a} + 2\mathbf{b}$

$-3(1,3) + 2(-3,2) = (-9, -5)$



(c) Sketch $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}+\mathbf{b}$, and $\mathbf{a}-\mathbf{b}$ on the graph to the right

Now lets see how magnitude is influenced by scalar multiplication:

Theorem 2.7. $|c \cdot \mathbf{a}| =$ $|c| \cdot \|a\|$

Proof:

$$\begin{aligned}
|c \cdot \mathbf{a}| &= |\langle ca_1, ca_2, ca_3\rangle| \\
&= \sqrt{c^2a_1^2 + c^2a_2^2 + c^2a_3^2} \\
&= \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)} \\
&= |c|\sqrt{a_1^2 + a_2^2 + a_3^2} \\
&= |c| \cdot \|\mathbf{a}\|
\end{aligned}$$

Definition(s) 2.8. A vector **a** is called a **unit vector** if it has magnitne 1.

Example 2.9. Find a unit vector in the same direction as $\mathbf{a} = \langle 1, 2, 3\rangle$.

$$\frac{a}{\|a\|}$$

$$\frac{\langle 1,2,3\rangle}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle$$

Example 2.10. Find a vector of length 3 in the direction opposite of $\mathbf{a} = \langle -2, 4, 1\rangle$.

$$3 \cdot \frac{a}{\|a\|}$$

$$\frac{3}{\sqrt{4+16+1}} \cdot \langle -2,4,1\rangle$$

Theorem 2.11. Suppose that $a \neq 0$, then:

Theorem 2.12 (Properties of Vector Operations).

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors and c, d be scalars:

(a) $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$

(b) $\mathbf{a}+\mathbf{0}=\mathbf{a}$

(c) $0\mathbf{a}=\mathbf{0}$

(d) $c(d\mathbf{a})=(cd)\mathbf{a}$

(e) $(c+d)\mathbf{a}=c\mathbf{a}+d\mathbf{a}$

(f) $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$

(g) $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$

(h) $1\mathbf{a}=\mathbf{a}$

(i) $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$

Definition(s) 2.13. The standard unit vectors are:

$\vec{i} = (1,0,0)$ $\quad\quad$ $\vec{j} = (0,1,0)$ $\quad\quad$ $\vec{k} = (0,0,1)$

Note: The standard base vectors are all unit vectors.

Any vector $\mathbf{a}=\langle a_1, a_2, a_3\rangle$ can be written as a linear combination of the standard unit vectors

$$\begin{aligned}
\mathbf{a} &= \langle a_1, a_2, a_3\rangle \\
&= \langle a_1, 0, 0\rangle + \langle 0, a_2, 0\rangle + \langle 0, 0, a_3\rangle \\
&= a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle \\
&= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}
\end{aligned}$$

Therefore we call a_1 the **i-component** of the vector **a**, a_2 the **j-component** of the vector **a**, and a_3 the **k-component** of the vector **a**.

Example 2.14. Write the following vectors as a linear combination of the standard unit vectors

(a) $\mathbf{a} = \langle 5, 2, -4\rangle$ $= 5i + 2j - 4k$

(b) $\mathbf{b} = \langle 0, \pi, 100\rangle$ $= 0i + \pi j + 100k$