Remarks on the vanishing viscosity process of state-constraint Hamilton-Jacobi equations

Rate of convergence

Son Tu

Michigan State University

tuson@msu.edu

Academy of Mathematics and Systems Science
April 19, 2023

Presentation Overview

A model problem: escape of a light ray

- Let Ω be open with smooth boundary $\partial\Omega$ (the medium).
- A light ray starting from $x \in \Omega$ is a path $\gamma : [0, t] \to \Omega$ with $\gamma(0) = x$ for some $t > 0$.
- $c : \overline{\Omega} \to [0, +\infty)$ the medium constraint of the speed of light (inhomogeneity).
- $T_\gamma = \inf\{s \geq 0 : \gamma(s) \notin \Omega\}$: first time the light ray exists the medium and $T_\gamma = +\infty$ if $\gamma([0, \infty)) \subset \Omega$.

The light ray takes the path that exists the medium in the least amount of time with the speed constraint

$$|\dot{\gamma}(s)| \leq c(\gamma(s)), \qquad s \geq 0.$$

This leads to the introduction of the minimum time function

$$u(x) = \inf\left\{T_\gamma : \gamma(0) = x, |\dot{\gamma}(s)| \leq c(\gamma(s))\right\}$$

for $x \in \Omega$. *Assume* that $\nabla u(x)$ exists at all points, then using Bellman's optimality principle and a Taylor expansion:

$$\begin{cases} c(x)|Du(x)| = 1 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

This is *Eikonal equation*.

The minimal amount of time required to travel from a point to the boundary with constant cost 1 is model by

$$|u'(x)| = 1 \qquad \text{in } (-1,1) \qquad \text{with } u(-1) = u(1) = 0.$$

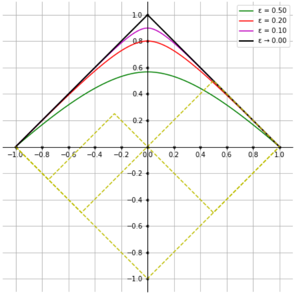
Infinitely many a.e. solutions, physically correct solution: $u(x) = 1 - |x|$.

Approximated equation with unique solution

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon (u^\varepsilon)'' & \text{in } (-1,1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

Vanishing viscosity

$$u^\varepsilon(x) = 1 - |x| + \varepsilon \left(e^{-1/\varepsilon} - e^{-|x|/\varepsilon} \right) \to u(x)$$



Let U be a compact metric space. A *control* is a Borel measurable map $\alpha : [0,\infty) \mapsto U$. We are given:

$$\begin{cases} b = b(x,a) : \overline{\Omega} \times U \to \mathbb{R}^n & \text{velocity vector field} \\ f = f(x,a) : \overline{\Omega} \times U \to \mathbb{R} & \text{running cost.} \end{cases}$$

For $x \in \mathbb{R}^n$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves

$$\dot{y}(t) = b(y(t), \alpha(t)), \qquad t > 0, \qquad \text{and} \qquad y(0) = x$$

Question. Minimize the cost functional ($\lambda \geq 0$ - the discount factor)

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f\left(y^{x,\alpha}(s), \alpha(s)\right)\, ds.$$

Define $H(x,p) = \sup_{v\in U}\left(-b(x,v)\cdot p - f(x,v)\right)$ then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that $u \in \mathrm{C}^\infty$ (using optimality or dynamic programming principle). However the *value function is usually not smooth!* $\longrightarrow$ *viscosity solution*.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

F is non-decreasing in u, non-increasing in D^2u (*degenerate elliptic*).

$\longrightarrow$ No integration by parts, only maximum principle.

Subsolution: $\varphi \in \mathrm{C}^2$, $u - \varphi$ *max* at x:

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

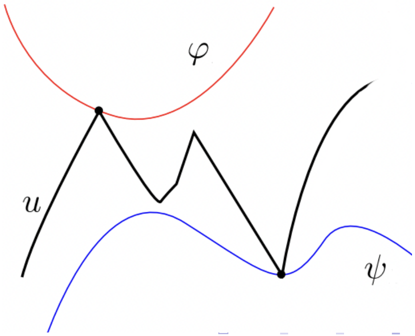
Supersolution: $\psi \in \mathrm{C}^2$, $u - \psi$ *min* at x:

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

Viscosity solution is *both* subsolution and supersolution.

$\longrightarrow$ *physically correct solution*

$\longrightarrow$ *value function in optimal control theory*



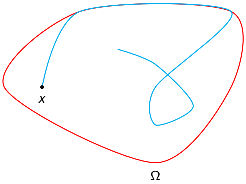
We consider

$$\begin{cases} u(x) + |Du|^p - f(x) \leq 0 & \text{in } \Omega, \\ u(x) + |Du|^p - f(x) \geq 0 & \text{on } \overline{\Omega} \end{cases} \tag{PDE_0}$$

This is the state-constrain Hamilton-Jacobi equation Soner (1986), which describe the value function of a deterministic optimal control problem

$$u(x) = \inf_{\eta(0)=x} \left\{ \int_0^\infty e^{-s} L(\eta(s), -\dot{\eta}(s)) ds : \eta \in \mathrm{AC}, \eta([0,\infty)) \subset \overline{\Omega} \right\}.$$

Here $L(x,v) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is the running cost, Legendre's transform of $H(x,\xi) = |\xi|^p - f(x)$. Generally, if H is smooth and u is smooth



$$\begin{cases} u(x) + H(x, Du(x)) = 0 & \text{in } \Omega, \\ D_p H(x, Du(x)) \cdot \nu(x) \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Given stochastic control $\alpha(\cdot)$, we solve

$$\begin{cases} dX_t = \alpha\,(X_t)\,dt + \sqrt{2\varepsilon}\,d\mathbb{B}_t & \text{for } t > 0, \\ X_0 = x. \end{cases} \tag{1}$$

$\mathbb{B}_t \sim \mathcal{N}(0,t)$ is the Brownian motion, to constraint $X_t \in \Omega$, we define

$$\widehat{\mathcal{A}}_x = \left\{\alpha(\cdot) \in \mathrm{C}(\Omega) : \mathbb{P}(X_t \in \Omega) = 1 \text{ for all } t \geq 0\right\}$$

Minimize the cost function

$$u^\varepsilon(x) = \inf_{\alpha \in \widehat{\mathcal{A}}_x} \mathbb{E}\left[\int_0^\infty e^{-t} L(X_t, \alpha(X_t))\,dt\right],$$

If $1 < p \leq 2$, $u^\varepsilon \in \mathrm{C}^2(\Omega)$ Lasry and Lions (1989) is the solution to

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

If $p > 2$ then $u^\varepsilon \in \mathrm{C}(\overline{\Omega})$. We focus on the subquadratic case $1 < p \leq 2$.

Using the stochastic, Lasry and Lions (1989) Dynamic Programming Principle, u^ε solves

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \leq 0 & \text{in } \Omega, \\ u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \geq 0 & \text{on } \overline{\Omega}, \end{cases} \tag{2}$$

- u^ε is a viscosity subsolution in Ω, that is if $x_0 \in \Omega$ and $\varphi \in \mathrm{C}^2(\Omega)$ with $u^\varepsilon - \varphi$ has a maximum over Ω at x_0, then
$$u^\varepsilon(x_0) + |D\varphi(x_0)|^p - f(x_0) - \varepsilon\Delta\varphi(x_0) \leq 0.$$
- u^ε is a viscosity supersolution on $\overline{\Omega}$, that is that is if $x_0 \in \Omega$ and $\varphi \in \mathrm{C}^2(\overline{\Omega})$ with $u^\varepsilon - \varphi$ has a maximum over $\overline{\Omega}$ at x_0, then
$$u^\varepsilon(x_0) + |D\varphi(x_0)|^p - f(x_0) - \varepsilon\Delta\varphi(x_0) \geq 0.$$

When $1 < p \leq 2$, u^ε is the unique solution with $u^\varepsilon(x) = +\infty$ on $\partial\Omega$.

Literature

Qualitative: As $\varepsilon \to 0$, $u^\varepsilon \to u$ in some sense, and in the limit, u is no longer blowing-up on the boundary:

1. Lasry and Lions (1989) (*PDEs approach*)
2. Capuzzo-Dolcetta and Lions (1990) (*PDEs approach*)
3. Fabbri et al. (2017) (*stochastic control approach*)

In the literature the solution is also called *large solutions*, and has been studied extensively. Blow-up rate of gradient is studied in Porretta (2004); Porretta and Véron (2006).
Quantitative: Rate of convergence: not yet done for state-constraint but for Dirichlet BC:

$$\begin{cases} u^\varepsilon(x) + H(x, Du^\varepsilon) - \varepsilon \Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = 0 & \text{on } \partial\Omega \end{cases} \longrightarrow \begin{cases} u(x) + H(x, Du) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The rate is $\mathcal{O}(\sqrt{\varepsilon})$, $\|u^\varepsilon - u\|_{L^\infty(\overline{\Omega})} \leq C\sqrt{\varepsilon}$ and the one-sided rate can be $\mathcal{O}(\varepsilon)$ for convex Hamiltonians

1. Fleming (1961)
2. Bardi and Capuzzo-Dolcetta (1997)
3. Crandall and Lions (1984)
4. Evans (2010), Tran (2011) (nonlinear adjoint method)

A blow up rate of u^ε near $\partial\Omega$

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

When $H(x,\xi) = |\xi|^p - f(x)$, we can compute the asymptotic expansion of u^ε near $\partial\Omega$. Assume

$$u^\varepsilon \sim \frac{C}{d(x)^\alpha}$$

we find

$$\boxed{u^\varepsilon(x) \sim \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}}, \qquad p < 2, \qquad \alpha = \frac{2-p}{p-1}, \quad C_\alpha = \frac{(\alpha+1)^{\alpha+1}}{\alpha}$$

$$\boxed{u^\varepsilon(x) \sim -\varepsilon \log(d(x))}, \qquad p = 2$$

Summary of main results

Theorem (Han and Tu (2022))

Without loss of generality, we can assume $f \geq 0$. Also assume f is Lipschitz.

1. *Assume $f = 0$ on $\partial\Omega$ then $|u^\varepsilon - u| \leq C\sqrt{\varepsilon}$ in the interior of Ω. More precisely,*

$$\begin{aligned} -C\sqrt{\varepsilon} \leq u^\varepsilon - u &\leq C\sqrt{\varepsilon} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}, & p < 2 \\ -C\sqrt{\varepsilon} \leq u^\varepsilon - u &\leq C\sqrt{\varepsilon} + C\varepsilon|\log(d(x))|, & p = 2 \end{aligned}$$

2. *If f is compactly supported in Ω then*

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\varepsilon + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}.$$

3. *If $f \in \mathrm{C}^2(\Omega)$ such that $Df = 0$ and $f = 0$ on $\partial\Omega$ then*

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\varepsilon^{1/p} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad 1 < p < 2.$$

Difficulties

- The blow-up behaviors at the boundary makes it a nontrivial task to apply conventional method: doubling variables.
  $\longrightarrow$ We construct a new blow-up solution near the boundary and glue things together
- A uniform bound for the Laplacian of u^ε is complicated with blow-up behavior.
  $\longrightarrow$ We avoid this by using a bound for the Laplacian of u instead. This is related to the semi-concavity of the solution u^ε and u.

Contributions

- The rate $\mathcal{O}(\varepsilon^{1/p})$ is new!
- Construct a new blow-up solution to deal with the blow-up behavior of u^ε (major difficulty)
- Specific (blow-up rate) of semi-concavity behavior of u of improve the one-sided rate.

A heuristic argument

Heuristic: Doubling variable method

$$\Phi(x,y) = u^{\varepsilon}(x) - u(y) - \frac{|x-y|^2}{\sigma}, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega}$$

- Viscosity solution $\sim$ *weak solution in* $L^{\infty} \Longrightarrow$ move the derivative to *test function without integration by parts* by maximum principle.
- Assume Φ has a maximum at x_σ, y_σ and $x_\sigma \in \Omega$ then $\Phi(x_\sigma, y_\sigma) \geq \Phi(x_\varepsilon, x_\varepsilon)$ implies that $|x_\sigma - y_\sigma| \leq C\sigma$
  - $\dfrac{|x-y_\sigma|^2}{\varepsilon}$ as a test function for $u^\varepsilon(x)$ in (PDE$_\varepsilon$) to get

$$u^{\varepsilon}(x_\sigma) + \left|\frac{2(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(x_\sigma) - \varepsilon\frac{2n}{\sigma} \leq 0$$

  - $-\dfrac{|x_\sigma - y|^2}{\varepsilon}$ as a test function for $u(y)$ in (PDE_0) to obtain

$$u(y_\sigma) + \left|\frac{2(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(y_\sigma) \geq 0$$

$$u^{\varepsilon}(x) - u(x) \leq u^{\varepsilon}(x_\sigma) - u(y_\sigma) \leq \frac{2n\varepsilon}{\sigma} + f(x_\sigma) - f(y_\sigma) \leq \frac{2n\varepsilon}{\sigma} + C\sigma$$

and the best choice here is $\sigma = \sqrt{\varepsilon}$.

- To overcome the difficulties in the argument, we instead use

$$\Phi(x,y) = \underbrace{u^\varepsilon(x) - \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}}_{\psi^\varepsilon(x)} - u(y) - \frac{C|x-y|^2}{\sigma} \tag{3}$$

- This forces the maximum happen at (x_σ, y_σ) where $x_\sigma \in \Omega$ (make C_α bigger). We also have

$$D\psi^\varepsilon(x) = Du^\varepsilon(x) + C_\alpha \alpha \left(\frac{\varepsilon}{d(x)}\right)^{\alpha+1} Dd(x). \tag{4}$$

- $|D\psi^\varepsilon(x)| \leq C$ if $d(x) \geq \varepsilon$ Armstrong and Tran (2015) $\Longrightarrow$ boundary layer is $\mathcal{O}(\varepsilon)$ from the boundary.
- However, we need $d(x_\sigma) \approx \varepsilon^\gamma$ for $\gamma \in (0,1)$. We need fine control of this after using some penalty to force the max happens.

Recall the equation

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

- Consider the case $f = 0$ first (then $u \equiv 0$), then ($\nu > 1$)

$$0 \leq u^\varepsilon \leq \underbrace{\frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\varepsilon^{\alpha+2}}_{\text{supersolution}}$$

- Compactly supported $\mathrm{supp}(f) \subset \Omega_\kappa = \{x \in \Omega : d(x) > \kappa\}$. If $\Phi(x,y)$ has max at (x_σ, y_σ)
  - (a) If $x_\sigma \in \Omega_\kappa$ then $d(x_\sigma) \geq C\kappa$, it is stronger than $d(x_\sigma) \approx \varepsilon^\gamma$.
  - (b) If $x_\sigma \in \Omega\backslash\Omega_\kappa$ we use a new *barrier*, bound solution by w that solves the PDE with $w = +\infty$ on $\partial\Omega_\kappa \cup \partial\Omega$.
- General case $f = 0$ on $\partial\Omega$: we do a cut-off $f_\kappa \to f$ as $\kappa \to 0$ and $\mathrm{supp}(f_\kappa) \subset \Omega_\kappa$. Since $f = 0$ on $\partial\Omega$, we can construct $\|f_\kappa - f\|_{L^\infty} \leq C\kappa$.

Heuristic

- To overcome $\kappa + \frac{C}{\kappa}$, which make the best rate is only $\mathcal{O}(\sqrt{\varepsilon})$ we use u^ε as a C^2 test function for u.
- Assume that $u^\varepsilon(x) - u(x)$ has a maximum over $\overline{\Omega}$ at some interior point $x_0 \in \Omega$, then

$$\max_{x\in\overline{\Omega}} \Big(u^\varepsilon(x) - u(x)\Big) \leq u^\varepsilon(x_0) - u(x_0) \leq \varepsilon \Delta u^\varepsilon(x_0).$$

- If u is uniformly semiconcave in $\overline{\Omega}$, then $\Delta u^\varepsilon(x_0) \leq \Delta u(x_0) \leq C$.

Difficulties

1. $u^\varepsilon = +\infty$ on $\partial\Omega$, we can subtract by $\frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}$ to make maximum happen in the interior (then we need the barrier to handle the case $d(x)$ is small $\leftarrow$ the barrier still plays a crucial role).
2. Unless $f \in \mathrm{C}^2_c(\Omega)$, in general, u is not uniformly semiconcave but only locally semiconcave. In fact

$$\Delta u(x) \leq \frac{C}{d(x)}$$

and this is enough to get $\mathcal{O}(\varepsilon)$ for compactly supported data.

The $\mathcal{O}(\varepsilon)$ rate

Compactly supported data - different cases

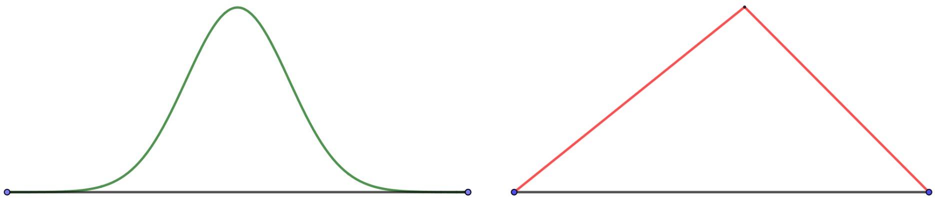


Figure: The different data that lead to different semiconcavities of u

- One the left: If f can be extended to a semiconcave function $\tilde{f}: \mathbb{R}^n \to \mathbb{R}$ by setting $f = 0$ on Ω^c, then u is uniformly semiconcave, i.e., $|Du| \leq C$, and hence an improvement on the rate happens.
- One the right: the best we can do is $|Du| \leq Cd(x)^{-1}$.

Semiconcavity of solutions to the first-order problem

We want to show that if $x, x-h, x+h \in \overline{\Omega}$ then

$$u(x+h) - 2u(x) + u(x-h) \leq C|h|^2$$

- If f can be extended to $f \in \mathbb{R}^n$ by setting $f = 0$ outside Ω and $\tilde{f}$ is semiconcave then u is the restriction of $\tilde{u}$ where
$$\tilde{u}(x) + |D\tilde{u}(x)|^p - \tilde{f}(x) = 0 \qquad \text{in } \mathbb{R}^n.$$
Equation in the whole space is easier to deal with, see Calder (2018).
- If $f = 0$ on $\partial\Omega$ but cannot be extended to semiconcave function globally by setting $f = 0$ outside Ω, we relies on *optimal control formula and* $p \leq 2 \Longrightarrow q = p^* > 2$. Take a minimizer η of x and let η hits $\partial\Omega$ at time T, then
$$u(x) = \int_0^T e^{-s}\left(c|\dot{\eta}(s)|^q + f(\eta(s))\right)ds.$$
- $\xi \mapsto |\xi|^q$ is C^2 if $q > 2$, thus locally semiconcave.
- Bounded velocity $|\dot{\eta}| \leq C$ implies $d(x) \leq CT$.

Discussion

Open questions

1. Can we remove the assumption $f = 0$ on $\partial\Omega$?
2. What is the optimal rate of convergence? (The semiconcavity of u in a more general setting was studied in a recent paper Han (2022)).
3. What is the rate of convergence for the super-quadratic case $p > 2$?
4. More general form of Hamiltonians?
5. A finer control of solution locally (which could leads to better rate) by using stochastic approach?

References

Armstrong, S. N. and Tran, H. V. (2015). Viscosity solutions of general viscous Hamilton–Jacobi equations. *Mathematische Annalen*, 361(3):647–687.

Bardi, M. and Capuzzo-Dolcetta, I. (1997). *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*. Modern Birkhäuser Classics.

Calder, J. (2018). Lecture notes on viscosity solutions.

Capuzzo-Dolcetta, I. and Lions, P.-L. (1990). Hamilton-Jacobi Equations with State Constraints. *Transactions of the American Mathematical Society*, 318(2):643–683.

Crandall, M. G. and Lions, P. L. (1984). Two Approximations of Solutions of Hamilton-Jacobi Equations.

Evans, L. C. (2010). Adjoint and Compensated Compactness Methods for Hamilton–Jacobi PDE. *Archive for Rational Mechanics and Analysis*, 197(3):1053–1088.

Fabbri, G., Gozzi, F., and Swiech, A. (2017). *Stochastic Optimal Control in Infinite Dimension: Dynamic Programming and HJB Equations*. Probability Theory and Stochastic Modelling.

Fleming, W. H. (1961). The convergence problem for differential games. *Journal of Mathematical Analysis and Applications*, 3(1):102–116.

Han, Y. (2022). Global semiconcavity of solutions to first-order Hamilton-Jacobi equations with state constraints. (arXiv:2205.01615). arXiv:2205.01615 [math] type: article.

Han, Y. and Tu, S. N. T. (2022). Remarks on the Vanishing Viscosity Process of State-Constraint Hamilton–Jacobi Equations. *Applied Mathematics & Optimization*, 86(1):3.

Lasry, J. M. and Lions, P.-L. (1989). Nonlinear Elliptic Equations with Singular Boundary Conditions and Stochastic Control with State Constraints. I. The Model Problem. *Mathematische Annalen*, 283(4):583–630.

Porretta, A. (2004). Local estimates and large solutions for some elliptic equations with absorption. *Advances in Differential Equations*, 9(3-4):329–351. Publisher: Khayyam Publishing, Inc.

Porretta, A. and Véron, L. (2006). Asymptotic Behaviour of the Gradient of Large Solutions to Some Nonlinear Elliptic Equations. *Advanced Nonlinear Studies*, 6(3):351–378. Publisher: Advanced Nonlinear Studies, Inc. Section: Advanced Nonlinear Studies.

Soner, H. (1986). Optimal Control with State-Space Constraint I. *SIAM Journal on Control and Optimization*, 24(3):552–561.

Tran, H. V. (2011). Adjoint methods for static Hamilton–Jacobi equations. *Calculus of Variations and Partial Differential Equations*, 41(3):301–319.

The End

Questions & Comments

Thank you

- Co-author: Yuxi Han
- This work was done at University of Wisconsin - Madison
- The work was supported in part by the GSSC Fellowship (UW-Madison), NSF grant DMS1664424 and NSF CAREER grant DMS-1843320