The regularity with respect to domains of the additive eigenvalues of superquadratic Hamilton–Jacobi equation

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Presentation Overview

Optimal control and first-order Hamilton-Jacobi equation

Let U be a compact metric space. A *control* is a Borel measurable map $\alpha : [0,\infty) \mapsto U$. We are given:

$$\begin{cases} b = b(x,a) : \overline{\Omega} \times U \to \mathbb{R}^n & \text{velocity vector field} \\ f = f(x,a) : \overline{\Omega} \times U \to \mathbb{R} & \text{running cost.} \end{cases}$$

For $x \in \mathbb{R}^n$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves

$$\dot{y}(t) = b(y(t), \alpha(t)), \qquad t > 0, \qquad \text{and} \qquad y(0) = x$$

Question. Minimize the cost functional ($\lambda \geq 0$ - the discount factor)

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f\left(y^{x,\alpha}(s), \alpha(s)\right)\, ds.$$

Define $H(x,p) = \sup_{v\in U}\left(-b(x,v)\cdot p - f(x,v)\right)$ then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that $u \in C^\infty$ (using optimality or dynamic programming principle). However the *value function is usually not smooth!* $\longrightarrow$ *viscosity solution*.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

F is non-decreasing in u, non-increasing in D^2u (*degenerate elliptic*).
$\longrightarrow$ No integration by parts, only maximum principle.

Subsolution: $\varphi \in \mathrm{C}^2$, $u - \varphi$ *max* at x:

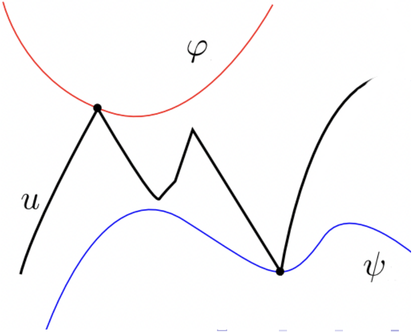
$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

Supersolution: $\psi \in \mathrm{C}^2$, $u - \psi$ *min* at x:

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

Viscosity solution is *both* subsolution and supersolution.

$\longrightarrow$ *physically correct solution*
$\longrightarrow$ *value function in optimal control theory*



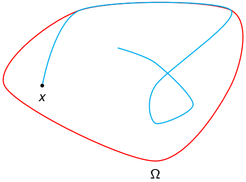
We consider

$$\begin{cases} \lambda u(x) + |Du|^p - f(x) \leq 0 & \text{in } \Omega, \\ \lambda u(x) + |Du|^p - f(x) \geq 0 & \text{on } \overline{\Omega} \end{cases} \qquad (PDE_0)$$

This is the state-constrain Hamilton-Jacobi equation Soner (1986), which describe the value function of a deterministic optimal control problem

$$u(x) = \inf_{\eta(0)=x} \left\{ \int_0^\infty e^{-\lambda s} L(\eta(s), -\dot{\eta}(s)) ds : \eta \in \mathrm{AC}, \eta([0,\infty)) \subset \overline{\Omega} \right\}.$$

Here $L(x,v) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is the running cost, Legendre's transform of $H(x,\xi) = |\xi|^p - f(x)$. Generally, if H is smooth and u is smooth



$$\begin{cases} \lambda u(x) + H(x, Du(x)) = 0 & \text{in } \Omega, \\ D_p H(x, Du(x)) \cdot \nu(x) \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Given stochastic control $\alpha(\cdot)$, we solve

$$\begin{cases} dX_t = \alpha(X_t)\, dt + \sqrt{2\varepsilon}\, d\mathbb{B}_t & \text{for } t > 0, \\ X_0 = x. \end{cases} \tag{1}$$

$\mathbb{B}_t \sim \mathcal{N}(0, t)$ is the Brownian motion, to constraint $X_t \in \Omega$, we define

$$\widehat{\mathcal{A}}_x = \left\{ \alpha(\cdot) \in \mathrm{C}(\Omega) : \mathbb{P}(X_t \in \Omega) = 1 \text{ for all } t \geq 0 \right\}$$

Minimize the cost function

$$u(x) = \inf_{\alpha \in \widehat{\mathcal{A}}_x} \mathbb{E}\left[\int_0^\infty e^{-\lambda t} L(X_t, \alpha(X_t))\, dt \right],$$

If $1 < p \leq 2$, $u \in \mathrm{C}^2(\Omega)$ Lasry and Lions (1989) is the solution to

$$\begin{cases} \lambda u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x, \partial\Omega) \to 0} u(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

If $p > 2$ then $u \in \mathrm{C}(\overline{\Omega})$. We focus on the case $p > 2$.

Using the stochastic, Lasry and Lions (1989) Dynamic Programming Principle, u solves

$$\begin{cases} \lambda u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x) \leq 0 & \text{in } \Omega, \\ \lambda u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x) \geq 0 & \text{on } \overline{\Omega}, \end{cases} \tag{2}$$

- u is a viscosity subsolution in Ω, that is if $x_0 \in \Omega$ and $\varphi \in \mathrm{C}^2(\Omega)$ with $u - \varphi$ has a maximum over Ω at x_0, then
$$\lambda u(x_0) + |D\varphi(x_0)|^p - f(x_0) - \varepsilon \Delta \varphi(x_0) \leq 0.$$
- u is a viscosity supersolution on $\overline{\Omega}$, that is that is if $x_0 \in \Omega$ and $\varphi \in \mathrm{C}^2(\overline{\Omega})$ with $u - \varphi$ has a maximum over $\overline{\Omega}$ at x_0, then
$$\lambda u(x_0) + |D\varphi(x_0)|^p - f(x_0) - \varepsilon \Delta \varphi(x_0) \geq 0.$$
- When $p > 2$, u is a unique viscosity solution, and
$$u \in \mathrm{C}^{0,\alpha}(\overline{\Omega}), \qquad \alpha = \frac{p-2}{p-1}.$$

Consider the problem:

$$\begin{cases} \lambda v_\lambda(x) + |Dv_\lambda(x)|^p - f(x) - \varepsilon \Delta v_\lambda(x) \leq 0 & \text{in } \Omega, \\ \lambda v_\lambda(x) + |Dv_\lambda(x)|^p - f(x) - \varepsilon \Delta v_\lambda(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases}$$

As $\lambda \to 0^+$,

- $\lambda v_\lambda \to -c(0)$
- $v_\lambda - v_\lambda(x_0) \to v$ (subsequence)

for a fixed $x_0 \in \Omega$ where v solves the ergodic problem

$$\begin{cases} |Dv(x)|^p - f(x) - \varepsilon \Delta v(x) \leq c(0) & \text{in } \Omega, \\ |Dv(x)|^p - f(x) - \varepsilon \Delta v(x) \geq c(0) & \text{on } \overline{\Omega}. \end{cases} \tag{3}$$

The additive eigenvalue denoted by $c(0)$ is defined as

$$c(0) = \min \left\{ c \in \mathbb{R} : |Du(x)|^p f(x) - \varepsilon \Delta u(x) \leq c \ \text{ in } \Omega \text{ has a solution} \right\}$$

and it is also the unique constant where (3) can be solved [Lasry and Lions (1989)].

Vanishing discount with changing domain

We consider $p > 2$, $\Omega_\lambda = (1 + r(\lambda))\Omega$ with

$$\lim_{\lambda\to 0} \frac{r(\lambda)}{\lambda} = \gamma \in (-\infty, +\infty),$$

and v_λ solves

$$\begin{cases} \lambda v_\lambda(x) + |Dv_\lambda(x)|^p - f(x) - \varepsilon\Delta v_\lambda(x) \leq 0 & \text{in } \Omega_\lambda, \\ \lambda v_\lambda(x) + |Dv_\lambda(x)|^p - f(x) - \varepsilon\Delta v_\lambda(x) \geq 0 & \text{on } \overline{\Omega}_\lambda, \end{cases} \qquad (\lambda, \Omega)$$

The corresponding ergodic problem is

$$\begin{cases} |Dv(x)|^p - f(x) - \varepsilon\Delta u(x) \leq c(\lambda) & \text{in } \Omega_\lambda, \\ |Dv(x)|^p - f(x) - \varepsilon\Delta u(x) \geq c(\lambda) & \text{on } \overline{\Omega}_\lambda. \end{cases} \qquad (0, \Omega_\lambda)$$

As $\lambda \to 0^+$, one expects that $v_\lambda \to v$ (under some normalization) and v solves the erogdic problem

$$\begin{cases} |Dv(x)|^p - f(x) - \varepsilon\Delta v(x) \leq c(0) & \text{in } \Omega, \\ |Dv(x)|^p - f(x) - \varepsilon\Delta v(x) \geq c(0) & \text{on } \overline{\Omega}. \end{cases} \qquad (0, \Omega)$$

Questions of interest and Literature

Motivation

1. In [Barles et al. (2010)], for $1 < p \leq 2$ then:
   - the map c_Ω is monotone with respect to Ω,
   - continuous with respsect to Hausdorff measure, under some appropriate perturbations.
2. For first-order equation ($\varepsilon = 0$), the map $\lambda \mapsto c(\lambda)$ has $c'_\pm(\cdot)$ exists and $c'(\cdot)$ exists a.e.
   - [Tu (2022)] for discount general $H(x,p)$,
   - [Tu and Zhang (2023)] for general contact Hamiltonians $H(x,p,u)$.

Questions: We want to study in more details the map $c(\lambda)$, in particular it leads to some associated questions:

1. Convergence of $v_\lambda \to v$?
2. Characterization of the limit v in terms of γ, i.e., $v = v^\gamma$ in some sense?
3. The regularity of the map $\lambda \mapsto c(\lambda)$.
4. Relations between the derivative $c'(\lambda)$ and the limiting solution v^γ.

Literature

State-constraint

1. Lasry and Lions (1989) (*PDEs approach* - 2nn-order equation)
2. Capuzzo-Dolcetta and Lions (1990) (*PDEs approach*)
3. Fabbri et al. (2017) (*stochastic control approach*)
4. Attouchi and Souplet (2020); Barles and Da Lio (2004); Barles et al. (2010); Tabet Tchamba (2010) for properties of solutions, time-dependent problem, large time behavior, ...

See also Porretta (2004); Porretta and Véron (2006)

The vanishing discount problem

1. Convergence of the vanishing discount is first established in [Davini et al. (2016)]
2. Subsequence works [Ishii et al. (2017a,b)] generalize the problem into many other settings (2nd-order, different BCs), $\longrightarrow$ duality method to construct Mather measures, (in contrast with using minimizing curves)
3. Contact Hamiltonians in Tu and Zhang (2023)

The main tool a representation of solutions using Mather measures.

Summary of main results 1

We write $\nabla L(x,v) = \big(D_x L(x,v), D_v L(x,v)\big)$ for $(x,v) \in \overline{\Omega} \times \mathbb{R}^n$. For a measure μ on $\overline{\Omega} \times \mathbb{R}^n$, we define

$$\langle \mu, \varphi \rangle_\Omega := \int_{\overline{\Omega}\times\mathbb{R}^n} \varphi(x,v)\, d\mu(x,v), \qquad \text{for } \varphi \in C(\overline{\Omega} \times \mathbb{R}^n) \cap L^1(\mu). \tag{4}$$

Theorem (Theorem 1 - Bozorgnia, Kwon and Tu, 2022)

For $p > 2$, the map $\lambda \mapsto c(\lambda)$ with respect to the scaling factor λ is one-sided differentiable:

$$c'_+(0) = \lim_{\substack{\lambda \to 0^+ \\ r(\lambda) > 0}} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) = \max_{\mu \in \mathcal{M}(\Omega)} \langle \mu, (-x, v) \cdot \nabla L(x,v) \rangle_\Omega \,, \tag{5}$$

$$c'_-(0) = \lim_{\substack{\lambda \to 0^+ \\ r(\lambda) < 0}} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) = \min_{\mu \in \mathcal{M}(\Omega)} \langle \mu, (-x, v) \cdot \nabla L(x,v) \rangle_\Omega \,. \tag{6}$$

Here, $L(x,v)$ is the Legendre transform of $H(x,\xi)$:

$$L(x,v) = C_p |v|^q + f(x), \qquad \textit{where} \qquad C_p = p^{-1/q}(p-1), \qquad p^{-1} + q^{-1} = 1. \tag{7}$$

Consider:

$$\begin{cases} \lambda u_\lambda(x) + H(x, Du_\lambda(x)) - \varepsilon \Delta u_\lambda(x) \le 0 & \text{in } \Omega_\lambda, \\ \lambda u_\lambda(x) + H(x, Du_\lambda(x)) - \varepsilon \Delta u_\lambda(x) \ge 0 & \text{on } \overline{\Omega}_\lambda, \end{cases} \tag{λ, Ω_λ}$$

Theorem (Theorem 2 - Bozorgnia, Kwon and Tu, 2022)

Let $u_\lambda \in C(\overline{\Omega}_\lambda)$ be the solution to $(\lambda, \Omega_\lambda)$.

(i) *We have $u_\lambda + \lambda^{-1} c(0) \to u^\gamma$ as $\lambda \to 0$ uniformly on $\overline{\Omega}$ and u^γ is a solution to (3).*

(ii) *Furthermore $u^\gamma = \max_{w \in \mathcal{E}^\gamma} w$ where $\mathcal{E}^\gamma$ denotes the family of subsolutions w to the ergodic problem (3) such that*

$$\gamma \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \langle \mu, w \rangle_\Omega \le 0 \qquad \text{for all } \mu \in \mathcal{M}(\Omega) \tag{8}$$

where $\gamma = \lim r(\lambda)/\lambda$.

Summary of main results 3

Key different with the 1st-order case: In the 2nd-order problem, solution to

$$\begin{cases} |Dv(x)|^p - f(x) - \varepsilon\Delta v(x) \leq c(0) & \text{in } \Omega, \\ |Dv(x)|^p - f(x) - \varepsilon\Delta v(x) \geq c(0) & \text{on } \overline{\Omega}. \end{cases}$$

is unique up to adding a constant. We can define $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{C}(\gamma) := u^\gamma(\cdot) - u^0(\cdot) \in \mathbb{R} \quad \longrightarrow \quad \mathcal{C}(\gamma) \text{is decreasing, concave, and } \mathcal{C}(0) = 0.$$

Theorem (Theorem 3 - Bozorgnia, Kwon and Tu, 2022)

We have $c'_+(0) = -\mathcal{C}'_+(0)$ *and* $c'_-(0) = -\mathcal{C}'_-(0)$. *Therefore*

$$c'(0) \text{ exists} \quad \Longleftrightarrow \quad \mathcal{C}'(0) \text{ exists}.$$

In which case

$$\mathcal{C}(\gamma) = -\gamma c'(0) \quad \text{for all } \gamma \in \mathbb{R}.$$

Special cases

1. If $f = \mathrm{const}$ then $\lambda \mapsto c(\lambda)$ is C^∞
2. If f is semiconcave then $\lambda \mapsto c(\lambda)$ is semiconvex.
3. If $p = 2$ then $\lambda \mapsto c(\lambda)$ is smooth (Hopf-Cole transform).

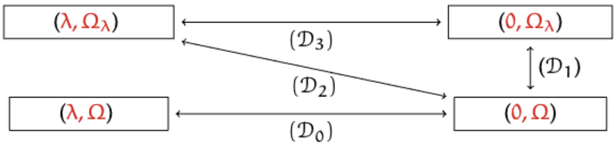
Difficulties

- The state-constraint boundary condition with 2nd-order equation is delicate, in terms of:
  - lack of finite-speed of propagation, technical problem related to comparison principle,
  - constructing Mather measures with 2nd-order structure using duality is very delicate, many technical problem arises

Contributions

(i) The technical generalization of Theorem 1 and 2 from the 1st-order case: lack of finite speed of propagation: tools in [Ishii et al. (2017a,b)] cannot be directly applied.

(ii) The new connection in Theorem between $\mathcal{C}'(\cdot)$ and $c'(\cdot)$.



- $(\mathcal{D}_0)$: the classical vanishing discount Ishii et al. (2017a,b).
- $(\mathcal{D}_1)$: the one-sided differentiability of $\lambda \mapsto c(\lambda)$
- $(\mathcal{D}_2)$: the vanishing discount coupled with changing domains
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Main tool: a duality representation

Let us define

$$\Phi^+(\overline{\Omega}\times\mathbb{R}^n) := \left\{\phi\in \mathrm{C}(\overline{\Omega}\times\mathbb{R}^n) : \phi(x,v) = tL(x,v)+\chi(x), t>0, \chi\in \mathrm{C}(\overline{\Omega})\right\}.$$

For each $\phi\in\Phi^+(\overline{\Omega}\times\mathbb{R}^n)$, define $H_\phi(x,\xi) = \sup_{v\in\mathbb{R}^n}(\xi\cdot v-\phi(x,v))$ for $(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$.
For $\delta\geq 0$ and $z\in\overline{\Omega}$ we define

$$\begin{aligned}\mathcal{F}_{\delta,\Omega} &= \Big\{(\phi,u)\in\Phi^+(\overline{\Omega}\times\mathbb{R}^n)\times \mathrm{C}(\overline{\Omega}) : \delta u + H_\phi(x,Du)-\varepsilon\Delta u\leq 0 \text{ in } \Omega\Big\},\\ \mathcal{G}_{z,\delta,\Omega} &= \Big\{\phi-\delta u(z) : (\phi,u)\in\mathcal{F}_{\delta,\Omega}\Big\},\\ \mathcal{G}'_{z,\delta,\Omega} &= \Big\{\mu\in\mathcal{R}(\overline{\Omega}\times\mathbb{R}^n) : \langle\mu,\varphi\rangle_\Omega\geq 0 \text{ for all } \varphi\in\mathcal{G}_{z,\delta,\Omega}\Big\}.\end{aligned}$$

We observe that $\Phi^+(\overline{\Omega}\times\mathbb{R}^n)$ is a convex cone in $\mathrm{C}(\overline{\Omega}\times\mathbb{R}^n)$ and $(x,\xi)\mapsto H_\phi(x,\xi)$ is well-defined and continuous for $\phi\in\Phi^+(\overline{\Omega}\times\mathbb{R}^n)$.

Theorem (Ishii et al. (2017a,b))

Let $(z,\lambda)\in\overline{\Omega}\times(0,\infty)$ and $u_\lambda\in \mathrm{C}(\overline{\Omega})$ be a solution of (λ,Ω). Then for $\lambda>0$ there holds

$$\lambda u_\lambda(z) = \min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{z,\lambda,\Omega}}\langle\mu,L\rangle_\Omega \quad \textit{and} \quad -c(0) = \min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{0,\Omega}}\langle\mu,L\rangle_\Omega. \tag{9}$$

We define the set of Mather measures on Ω_λ to be $\mathcal{M}(\Omega_\lambda)$. Consider $r(\lambda) > 0$

$$\begin{cases} H(x, Dw_\lambda(x)) - \varepsilon\Delta w_\lambda(x) \leq c(\lambda) & \text{in } \Omega_\lambda, \\ H(x, Dw_\lambda(x)) - \varepsilon\Delta w_\lambda(x) \geq c(\lambda) & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{10}$$

By scaling

$$H\big((1 + r(\lambda))\, x, (1 + r(\lambda))\, D\tilde{w}_\lambda(x)\big) - \varepsilon\Delta \tilde{w}_\lambda(x) \leq c(\lambda) \qquad \text{in } \Omega.$$

Using duality and definition of $\mathcal{M}_0$

$$\left\langle \mu, L\left((1 + r(\lambda))\, x, \frac{v}{1 + r(\lambda)} \right) - L(x, v) \right\rangle_\Omega + c(\lambda) - c(0) \geq 0. \tag{11}$$

for $\mu \in \mathcal{M}(\Omega)$, since $\langle \mu, L\rangle = -c(0)$.
$(0, \Omega_\lambda) \to (0, \Omega)$

$$\boxed{-\langle \mu, (-x, v) \cdot \nabla L(x, v)\rangle_\Omega + \liminf_{\lambda \to 0^+} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) \geq 0 \qquad \text{for all } \mu \in \mathcal{M}(\Omega)} \tag{12}$$

In the inverse direction, we start with $w \in \mathrm{C}(\overline{\Omega})$ solves

$$H(x, Dw(x)) - \varepsilon\Delta w(x) \leq c(0) \qquad \text{in } \Omega.$$

Scale to Ω_λ

$$H\left(\frac{x}{1+r(\lambda)}, \frac{1}{1+r(\lambda)} D\tilde{w}(x)\right) - \varepsilon\Delta\tilde{w}(x) \leq c(0) \qquad \text{in } \Omega_\lambda. \tag{13}$$

Take $\nu_\lambda \in \mathcal{M}(\Omega_\lambda)$, i.e., $v_\lambda \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega_\lambda}$ and $\langle \nu_\lambda, L\rangle_{\Omega_\lambda} = -c(\lambda)$, we obtain that

$$\left\langle \nu_\lambda, L\left(\frac{x}{1+r(\lambda)}, (1+r(\lambda))v\right) - L(x,v)\right\rangle_{\Omega_\lambda} - c(\lambda) + c(0) \geq 0.$$

As $\nu_\lambda \to \nu_0$ (after scaling, in measures sense and along the sequence lim sup) $(0, \Omega_\lambda) \to (0, \Omega)$

$$\boxed{\langle \nu_0, (-x, v)\cdot\nabla L(x,v)\rangle_\Omega \geq \limsup_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right)}. \tag{14}$$

From the previous inequalities:

$$\boxed{-\langle \mu, (-x, v)\cdot \nabla L(x,v)\rangle_{\Omega} + \liminf_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right) \geq 0 \qquad \text{for all } \mu \in \mathcal{M}(\Omega)}$$

and

$$\boxed{\langle \nu_0, (-x, v)\cdot \nabla L(x,v)\rangle_{\Omega} \geq \limsup_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right)}.$$

we obtain the result, as $\nu_0 \in \mathcal{M}(\Omega)$, and

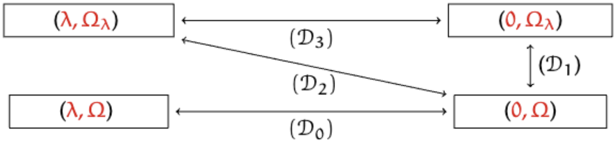
$$\lim_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right) = \langle \nu_0, (-x, v)\cdot \nabla L(x,v)\rangle_{\Omega} = \sup_{\mu\in\mathcal{M}} \langle \mu, (-x, v)\cdot \nabla L(x,v)\rangle_{\Omega}.$$

Similarly for $\liminf$.

Define

$$\mathcal{C}(\gamma) = u^{\gamma}(\cdot) - u^{0}(\cdot) \in \mathbb{R}.$$

This feature is only available in the 2nd-order case.



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1. The same method applies, but with $(\mathcal{D}_2)$ gives us the convergence of $u_\lambda + \lambda^{-1}c(\lambda) \to u^{\gamma}$.
2. Using $(\mathcal{D}_2)$ we obtain $\mathcal{C}'_{\pm}(0) = -c'_{\pm}(0)$.

We do not get useful information along other directions (yet)

If we can compute the Mather measures set, we can get more information.

1. If $f = \mathrm{const}$ then $\lambda \mapsto c(\lambda)$ is C^∞. This case $\langle \mu, L\rangle = \mathrm{const}$ for all $\mu \in \mathcal{M}$.
2. If f is semiconcave then $\lambda \mapsto c(\lambda)$ is semiconvex.
3. If $p = 2$ then $\lambda \mapsto c(\lambda)$ is smooth (Hopf-Cole transform).

$$\begin{cases} |Dv(x)|^2 - f(x) - \varepsilon \Delta v(x) = c(\lambda) & \text{in } \Omega_\lambda, \\ v(x) = +\infty & \text{on } \partial\Omega_\lambda. \end{cases} \tag{15}$$

Define $w_\lambda : \overline{\Omega} \to \mathbb{R}$ by $w_\lambda(x) = e^{-\hat{v}_\lambda(x)/\varepsilon}$ for $x \in \Omega_\lambda$ where $\hat{v}$ is chosen so that $\|w_\lambda\|_{L^2} = 1$. We obtain a linear problem

$$\begin{cases} -\varepsilon^2 \Delta w_\lambda(x) + f(x) w_\lambda(x) = c(\lambda) w_\lambda(x) & \text{in } \Omega_\lambda, \\ w_\lambda(x) = 0 & \text{on } \partial\Omega_\lambda. \end{cases} \tag{16}$$

Here $c(\lambda)$ is the normal eigenvalue of a linear problem.

$$c'(0) = -\varepsilon^2 \int_{\partial\Omega} \left| \frac{\partial w_0}{\partial \mathbf{n}}(x) \right|^2 (x \cdot \mathbf{n}) \; dS(x).$$

Open questions

1. Can we show that $\lambda \mapsto c(\lambda)$ is indeed differentiable everywhere? Or under what conditions do we have such a property?
2. The result for $1 < p \leq 2$? Such a duality representation is not available.
3. Contact structure?

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Thank you!

The End

Questions & Comments

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