Rate of convergence for periodic homogenization of convex Hamilton-Jacobi equations in one dimension

Son N.T. Tu
University of Wisconsin - Madison

Homogenization Theory of Hamilton-Jacobi Equation

Let $H(x,y,p) \in \mathrm{C}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ be uniformly coercive, locally uniformly bounded and Lipschitz in p and $\mathbb{Z}^n$-periodic in y.

For each $\varepsilon > 0$, let $u^\varepsilon \in \mathrm{C}(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to the Hamilton-Jacobi equation:

$$\begin{cases} u_t^\varepsilon(x,t) + H\left(x, \frac{x}{\varepsilon}, Du^\varepsilon(x,t)\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \qquad (\mathrm{C}_\varepsilon)$$

It is known (Lions-Papanicolaou-Varadhan, [4] for $H = H(y,p)$ and Evans [2, 3] for $H(x,y,p)$) that u^ε converges locally uniformly to u, the solution of the effective equation

$$\begin{cases} u_t(x,t) + \overline{H}(x, Du(x,t)) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \qquad (\mathrm{C})$$

$\overline{H}(x,p) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is called "effective Hamiltonian", a nonlinear averaging of the original H.

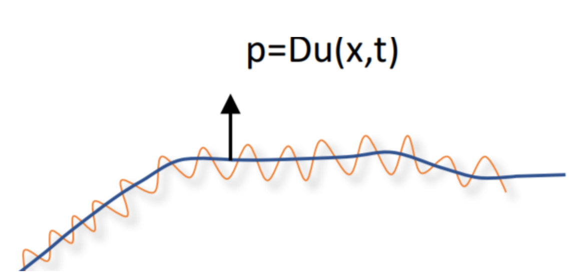
Cell problems and effective Hamiltonian

For each $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, there is a unique constant $\overline{H}(x,p) \in \mathbb{R}$ for which the following cell problem

$$H(x, y, p + D_y v(y)) = \lambda \qquad \text{in } \mathbb{T}^n \qquad (\mathrm{CP})$$

has a continuous solution $v(y) = v(y;x,p)$ (often named "corrector"). Heuristically, the two-scale asymptotic expansion (see [5]) says

$$u^\varepsilon(x,t) \approx u(x,t) + \varepsilon v\left(\frac{x}{\varepsilon}; x, Du(x,t)\right) + \mathcal{O}(\varepsilon^2),$$



The corrector $v(y;x,p)$ for $p = Du(x,t)$ basically captures the oscillation of Du^ε around (x,t).

How fast does u^ε converge to u as $\varepsilon \longrightarrow 0^+$?

According to the above formal expansion, it looks like

$$|u^\varepsilon - u| = \mathcal{O}(\varepsilon).$$

However, there is NO way to justify this expansion rigorously due to [5].

- In general, there does not even exists a continuous selection of $v(\cdot;x,p)$ with respect to p, let alone Lipschitz continuous selection.
- The solution $u(x,t)$ to (C) is only Lipschitz in (x,t), and is usually not C^1.

The best known result was due to I. Capuzzo-Dolcetta and H. Ishii [1] based on pure PDE approaches:

$$|u^\varepsilon - u| = \mathcal{O}(\varepsilon^{1/3}).$$

When $H(x,y,p) = H(y,p)$, H. Mitake, H. V. Tran and Y. Yu in [5] established an optimal rate $\mathcal{O}(\varepsilon)$ for the one dimensional case with convex Hamitonians along with other important results in higher dimensional spaces using tools from dynamical systems and weak KAM theory.

Main results

We consider the one dimensional case $n = 1$ and the convex Hamiltonian is of the form:

$$H(x,y,p) = H(p) + V(x,y)$$

for all $(x,y,p) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}$.

Theorem 1. Classical mechanics Hamiltonian
Assume $n = 1$ and $H(x,y,p) = \frac{1}{2}|p|^2 + V(x,y)$ where V is of the separable form

$$V(x,y) = a(x)b(y) + C_0$$

with C_0 is a constant and

(i) $a(x) \in \mathrm{C}^1(\mathbb{R})$ is bounded with $a(x) > 0$ for all $x \in \mathbb{R}$,
(ii) $b(y) \in \mathrm{C}(\mathbb{T})$ and $\max_{y\in\mathbb{T}} b(y) = 0$.

Assume $u_0 \in \mathrm{Lip}(\mathbb{R}) \cap \mathrm{BUC}(\mathbb{R})$, then for each $R, T > 0$ we have

$$\left\| u^\varepsilon - u \right\|_{L^\infty([-R,R]\times[0,T])} \leq C\varepsilon$$

where C is a constant depends on $R, T, \mathrm{Lip}(u_0)$, $a(x)$ and $\max|b(y)|$.

If $V(x,y) = V(y)$ does not depend on x, $u^\varepsilon \longrightarrow u$ uniformly in $\mathbb{R} \times [0,\infty)$ and the constant C above can be chosen explicitly as

$$C = 2\left(\|u_0'\|_{L^\infty(\mathbb{R})} + 4(\|V\|_{L^\infty})^{1/2} \right).$$

Theorem 2. A general class of convex Hamiltonians
If $H(x,y,p) = H(p) + V(x,y)$ where $H(p) \geq H(0) = 0$ such that:

- $H(p) \in \mathrm{C}^2(\mathbb{R})$ is strictly convex with $H''(0) > 0$, or $H(p) = |p|^\gamma$ where $\gamma \geq 2$.
- $\max_{\mathbb{R}\times\mathbb{T}} V(x,y) = 0$, there exists $y_0 \in \mathbb{T}$ such that $V(x,y_0) = 0$ for all $x \in \mathbb{R}$.
- For every compact interval $I \subset \mathbb{R}$ then $\alpha_I f_I(y) \leq |V(x,y)| \leq \beta_I f_I(y)$ for $\alpha_I, \beta_I > 0, f_I \in \mathrm{C}(\mathbb{R}, [0,\infty))$ and

$$\sup_{(x,y)\in I\times\mathbb{T}} \left| \frac{V_x(x,y)}{V(x,y)} \right| \leq C_I < \infty.$$

If $u_0 \in \mathrm{Lip}(\mathbb{R}) \cap \mathrm{BUC}(\mathbb{R})$ then for any $R, T > 0$ we have

$$\left\| u^\varepsilon - u \right\|_{L^\infty([-R,R]\times[0,T])} \leq C\varepsilon$$

where C is a constant depends only on $R, T, \mathrm{Lip}(u_0)$, $H(p)$ and $V(x,y)$.

In the case $V(x,y) = V(y)$, the method can be used to get the result for general convex Hamiltonians. We thus recover Theorem 1.3 in [5] and the convergence is uniform in this case. By Proposition 4.3 in [5], the rate $\mathcal{O}(\varepsilon)$ is optimal.

Sketch of the proof

Assume $V \in \mathrm{C}^2(\mathbb{R} \times \mathbb{T})$ and $C_0 = 0$. Now using optimal control formula:

$$u^\varepsilon(x_0,t_0) = \inf_{\eta\in\mathcal{T}} \underbrace{\left\{ \varepsilon \int_0^{\varepsilon^{-1}t_0} \left(\frac{|\dot\eta(s)|^2}{2} - V(\varepsilon\eta(s), \eta(s)) \right) ds + u_0\left(\varepsilon\eta(\varepsilon^{-1}t_0)\right) \right\}}_{A^\varepsilon[\eta]},$$

where $\mathcal{T} = \left\{ \eta(\cdot) \in \mathrm{AC}\left([0,\varepsilon^{-1}t_0]\right), \varepsilon\eta(0) = x_0 \right\}$. Minimizers satisfy the Euler-Lagrange equation

$$\begin{cases} \ddot\eta_\varepsilon(s) &= -\nabla V(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)) \cdot (\varepsilon, 1) \quad \text{on} \quad (0, \varepsilon^{-1}t_0), \\ \eta_\varepsilon(0) &= \varepsilon^{-1}x_0. \end{cases} \qquad (\text{E-L})$$

Sketch of the proof

Conservation of energy:

There exists a constant $r = r(\eta_\varepsilon) \in [V(0,0), +\infty)$ such that

$$\frac{|\dot\eta_\varepsilon(s)|^2}{2} + V(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)) = r \qquad \text{for all} \qquad s \in (0, \varepsilon^{-1}t_0).$$

The optimization problem is equivalent to

$$u^\varepsilon(x_0,t_0) = \inf_r \left\{ A^\varepsilon[\eta_\varepsilon] : \text{among all } \eta_\varepsilon(\cdot) \text{ solve (E-L) with energy } r \right\}.$$

- $r \leq 0$, by using structure of the potential V we have

$$\left| \inf_{r\leq 0} A^\varepsilon[\eta_\varepsilon] - u_0(x_0) \right| \leq \left(\sqrt{2\|V\|_{L^\infty}} + \|u_0'\|_{L^\infty} \right) \varepsilon.$$

- $r > 0$, by the conservation of energy the solutions η_ε can be determined by ODEs, and their averages can be determined by following fact

$$\left| \int_a^b F\left(x, \frac{x}{\varepsilon}\right) dx - \int_a^b \left(\int_0^1 F(x,y)\, dy \right) dx \right| \leq C\varepsilon$$

if $F(x,y) \in \mathrm{C}^1(\mathbb{R} \times \mathbb{T})$ and $a < b$ are real numbers.

Some remarks

1. For the one dimension case, the remaining question is to find the optimal rate for general coercive H (i.e. nonconvex H). It was conjectured by H. Mitake, H. V. Tran and Y. Yu that the general optimal rate is $\mathcal{O}(\sqrt{\varepsilon})$.
2. Although it is very reasonable to believe that the optimal convergence rate $\mathcal{O}(\varepsilon)$ is not achievable in general, an example with fractional convergence rate has not been found since this involves handling chaotic behaviors.

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References

See the full list of references in [6].

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