Rate of convergence for quasi-periodic homogenization of Hamilton–Jacobi equation and application

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Ergodic estimate

1. Given $\mathbb{F} \in C(\mathbb{T}^n)$ and $\xi = (\xi_1, \xi_2 \ldots, \xi_n)$ be a non-resonant vector, i.e., $\xi \cdot \kappa \neq 0$ for $\kappa \in \mathbb{Z}^n \backslash \{0\}$, then for $f(x) = \mathbb{F}(\xi x)$ in $\mathbb{R}$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{F}(\xi x)\, dx = \mathcal{M}(f) := \int_{\mathbb{T}^n} \mathbb{F}(\mathbf{x})\, d\mathbf{x}.$$

2. If $\mathbb{F}$ is unbounded, then what about

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dx}{\mathbb{F}(\xi x)} = \mathcal{M}(f^{-1}) := \int_{\mathbb{T}^n} \frac{d\mathbf{x}}{\mathbb{F}(\mathbf{x})}$$

given that $x \mapsto \frac{1}{F(\xi x)}$ is well-defined in $\mathbb{R}$?

3. Rate of convergence? Example (result from our work): $\mathbb{F}(x_1, x_2) = (2 - \sin(2\pi x_1) - \sin(2\pi x_2))^{1/2}$ for $\mathbf{x} = (x_1, x_2) \in \mathbb{T}^2$, then

$$\left| \frac{1}{T} \int_0^T \frac{dx}{\mathbb{F}(\xi x)} - \int_{\mathbb{T}^2} \frac{d\mathbf{x}}{\mathbb{F}(\mathbf{x})} \right| \leq \frac{C}{T^{1/6}} \qquad \text{if } \frac{\xi_2}{\xi_1} \text{ badly approximable.}$$

4. Consequence from homogenization of Hamilton–Jacobi equation

Viscosity solutions - Definition

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

F is non-decreasing in u, non-increasing in D^2u (*degenerate elliptic*).

$\longrightarrow$ No integration by parts, only maximum principle.

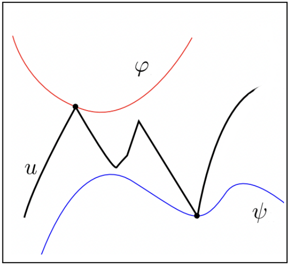
Subsolution: $\varphi \in C^2$, $u - \varphi$ *max* at x:

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

Supersolution: $\psi \in C^2$, $u - \psi$ *min* at x:

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

Viscosity solution is *both* subsolution and supersolution.



$\longrightarrow$ *physically correct solution*

$\longrightarrow$ *value function in optimal control theory*

Vanishing viscosity - Eikonal equation

The minimal amount of time required to travel from a point to the boundary with constant cost 1 is model by

$$|u'(x)| = 1 \qquad \text{in } (-1,1) \qquad \text{with } u(-1) = u(1) = 0.$$

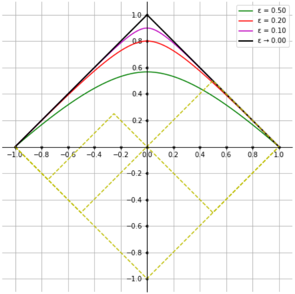
Infinitely many a.e. solutions, physically correct solution: $u(x) = 1 - |x|$.

Approximated equation with unique solution

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon (u^\varepsilon)'' & \text{in } (-1,1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

Vanishing viscosity

$$u^\varepsilon(x) = 1 - |x| + \varepsilon\left(e^{-1/\varepsilon} - e^{-|x|/\varepsilon}\right) \to u(x)$$



Optimal control theory - An infinite horizontal example

Let U be a compact metric space. A *control* is a Borel measurable map $\alpha : [0, \infty) \mapsto U$. We are given:

$$\begin{cases} b = b(x, a) : \overline{\Omega} \times U \to \mathbb{R}^n & \text{velocity vector field} \\ f = f(x, a) : \overline{\Omega} \times U \to \mathbb{R} & \text{running cost.} \end{cases}$$

For $x \in \mathbb{R}^n$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves

$$\dot{y}(t) = b(y(t), \alpha(t)), \qquad t > 0, \qquad \text{and} \qquad y(0) = x$$

Question. Minimize the cost functional ($\lambda \geq 0$)

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f\left(y^{x,\alpha}(s), \alpha(s)\right)\, ds.$$

Define $H(x, p) = \sup_{v \in U} \left(-b(x, v) \cdot p - f(x, v)\right)$ then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that $u \in \mathrm{C}^\infty$ (using optimality or dynamic programming principle). However the *value function is usually not smooth!*

Homogenization

In 1987, Lions, Papanicolaous and Varadhan [Lions-Papanicolaou-Varadhan'86] proved the homogenization result for a periodic, coercive Hamiltonian (possibly nonconvex)

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{T}^n \times \mathbb{R}^n \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

As $\varepsilon \to 0^+$, $u^\varepsilon \to u$ and u solves

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{T}^n \times \mathbb{R}^n \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

$\overline{H}(p)$ is the unique *constant* such that the ergodic (cell) problem can be solve

$$H(x, p + Dv(x)) = \overline{H}(p) \qquad \text{in } \mathbb{T}^n.$$

$\overline{H}(p)$ is called:

1. effective Hamiltonian
2. ergodic constant
3. additive eigenvalue of H
4. α-function in dynamical system
5. Máñẽ's critical value
6. ...

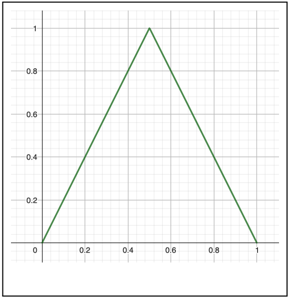
Homogenization - Example

In 1D, if

$$H(x,p) = \frac{|p|^2}{2} + V(x),$$

where

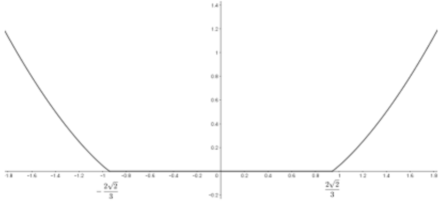
$$V(x) = \begin{cases} 2x & x \in \left[0, \frac{1}{2}\right], \\ -2x + 2 & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



Then

$$|p| = \frac{2\sqrt{2}}{3}\left[\left(\overline{H}(p)+1\right)^{\frac{3}{2}} - \overline{H}(p)^{\frac{3}{2}}\right].$$

Then $\overline{H}$ takes the form



Homogenization - Heuristic

- Introduce $y = \frac{x}{\varepsilon}$ as a fast variable, $x = \varepsilon y$ is a slow variable.
- *Ansatz*: $u^\varepsilon(x,t) = u^0(x,y,t) + \varepsilon u^1(x,y,t) + \varepsilon^2 u^2(x,y,t) + \dots$.
- Plug in the equation $u_t + H(\frac{x}{\varepsilon}, Du) = 0$

$$u_t^0(x,y,t) + H\left(y, D_x u^0(x,y,t) + \varepsilon^{-1} D_y u^0(x,y,t) + D_y u^1(x,y,t)\right) = 0.$$

- $D_y u^0 = 0$, i.e., $u^0 = u^0(x,t)$ independent of y

$$H\left(y, \boxed{D_x u^0(x,t)} + D_y u^1(x,y,t)\right) = \boxed{-u_t^0(x,t)}$$

- Ergodic or cell problem (fox a fixed (x,t))

$$H\left(y, \boxed{p} + D_y u^1(y)\right) = \boxed{\overline{H}(p)}$$

Homogenization

- The above ansatz gives

$$u^\varepsilon(x,t) \approx u^0(x,t) + \varepsilon u^1\left(\frac{x}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2).$$

- This means in homogenization as $\varepsilon \to 0$ then $u^\varepsilon \to u^0$.
- $v = u^1$ is a *corrector*

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon v\left(\frac{x}{\varepsilon}; Du(x,t)\right).$$

where

$$H(x, p + Dv(x;p)) = \overline{H}(p).$$

Solution v is not unique (up to adding a constant).

- If v is bounded then (the expected optimal rate)

$$|u^\varepsilon - u| = \mathcal{O}(\varepsilon).$$

- Via *doubling variable method*: can prove the convergence, but not the expansion.

Literature

This received quite a lot of attention in the past twenty years.
Assume: $x \mapsto H(x,p)$ **is Lipschitz locally in** p

- [Capuzzo-Dolcetta-Ishii'01]: $\mathcal{O}(\varepsilon^{1/3})$, PDE method, nonconvex and multi-scale $H(x, \frac{x}{\varepsilon}, Du^\varepsilon) \to \overline{H}(x, Du)$. : many works use this method
- $\mathcal{O}(\varepsilon^{1/2})$ if there is a Lipschitz selection $p \mapsto v(\cdot, p)$ of the cell problem

$$H(x, p + Dv(x;p)) = \overline{H}(p).$$

Convex Hamiltonian

- $\mathcal{O}(\varepsilon)$ in 1D [Mitake-Tran-Yu'19] and [Tu'18] for 1D multi-scale.
- Conditional $\mathcal{O}(\varepsilon)$ under smoothness assumption of $\overline{H}$ [Mitake-Tran-Yu'19]. first group utilized optimal control, optimal curve and metric distance
- Optimal rate $\mathcal{O}(\varepsilon)$ [Tran-Yu'21]. Burago Lemma and the metric distance.
- $\mathcal{O}(\varepsilon^{1/2})$ for multi-scale using Burago Lemma [Han-Jang'23].
- [Armstrong-Cardaliaguet-Souganidis'14]: followed [Capuzzo-Dolcetta-Ishii'01], $\mathcal{O}(\varepsilon^{1/8})$ for i.i.d, an abstract modulus $\omega(\varepsilon)$ for the almost periodic (PDE method).

Almost periodic homogenization

- For $f \in \mathrm{BUC}(\mathbb{R}^n)$, we way it is almost periodic if $\{f(\cdot + z) : z \in \mathbb{R}^n\}$ is relatively compact in $\mathrm{BUC}(\mathbb{R}^n)$.

$$\text{periodic} : x \mapsto H(x,p) \text{ is } \mathbb{Z}^n \text{ periodic}$$
$$\text{almost-periodic} : \{H(\cdot + z, \cdot) : z \in \mathbb{R}^n\} \text{ is relatively compact in } \mathrm{BUC}(\mathbb{R}^n \times B_R(0)).$$

- In one-dimensional case, for examle

$$H(x,p) = \frac{|p|^2}{2} - V(x), \qquad V(x) = 2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x).$$

- Quasi-periodic potential in 1D: $x \in \mathbb{R}$

$$V(x) = F(\xi x) \qquad \text{where } F \in C^k(\mathbb{T}^k),\ \xi \in \mathbb{R}^k \text{ is nonresonant.}$$

- The corrector is replaced by *almost corrector* [Ishii'00]

$$\overline{H}(p) - \delta \le H(y, p + Dv_\delta(y;p)) \le \overline{H}(p) + \delta.$$

Almost periodic function in 1D

First studied by Bohr (1926):

- For $\varepsilon > 0$, τ is an ε-period, if

$$|f(x+\tau) - f(x)| < \varepsilon \qquad \text{for all } x \in \mathbb{R}.$$

We say $E(\varepsilon, f) = \{\tau \in \mathbb{R} : |f(x+\tau) - f(x)| < \varepsilon\}$ the set of all ε-periods.

- $f \in \mathrm{AP}(\mathbb{R})$ if for $\varepsilon > 0$, there exists l_ε such that, for every $a \in \mathbb{R}$

$$[a, a + l_\varepsilon] \cap E(\varepsilon, f) \neq \emptyset$$

any interval of length l_ε has an ε-period.

- We say l_ε is an *inclusion interval length* of $E(\varepsilon, f)$.
- **Mean value property** If $f \in \mathrm{AP}(\mathbb{R})$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)dx = \mathcal{M}(f).$$

- If $f(x) = F(\xi x)$ is quasi-periodic, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)dx = \mathcal{M}(f) = \int_{\mathbb{T}^n} F(\mathbf{x})\, d\mathbf{x}.$$

Convergence to the mean value

If f is periodic of period 1, then $\mathcal{M}(f) = \int_0^1 f(x)dx$, and

$$\left| \frac{1}{T} \int_0^T f(x)\, dx - \mathcal{M}(f) \right| \leq \left(\int_0^1 f(x)dx \right) \frac{1}{T}.$$

Key ingredient for periodic homogenization rate $\mathcal{O}(\varepsilon)$ in 1D [Mitake-Tran-Yu'19, Tu'18].

- (Almost-periodic) For every $\varepsilon > 0$

$$\left| \frac{1}{T} \int_0^T f(x)\, dx - \mathcal{M}(f) \right| \leq \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})} \frac{l_\varepsilon(f)}{T}.$$

  Need an estimate of $l_\varepsilon(f)$ with respect to ε, but good as only L^∞ is needed.

- (Quasi-periodic) If $f(x) = \mathbf{F}(\xi x)$ and $\mathbf{F} \in H^s(\mathbb{T}^n)$ for $s > \frac{n}{2} + \sigma_\xi$ then

$$\left| \frac{1}{T} \int_0^T \mathbb{F}(\xi x)\, dx - \int_{\mathbb{T}^n} \mathbf{F}(\mathbf{x})\, d\mathbf{x} \right| \leq \frac{C(n,s)\|\mathbf{F}\|_{H^s(\mathbb{T}^n)}}{T}.$$

  Here σ_ξ is a Diophantine condition of ξ:

$$\xi \cdot \kappa \geq \frac{C}{|\kappa|^\sigma} \qquad \forall\, \kappa \in \mathbb{Z}^n.$$

  Need higher regularity, not applicable for some potentials.

Diophantine Approximations

For almost periodic f

$$\left|\frac{1}{T}\int_0^T f(x)\,dx - \mathcal{M}(f)\right| \leq \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})}\frac{l_\varepsilon(f)}{T}.$$

For quasi-periodic $f(x) = \mathbf{F}(\xi x)$ with $\mathbf{F} \in C^{0,\alpha}(\mathbb{T}^n)$

1. [Nai96] $n = 2$, badly approximable (null set)

$$l_\varepsilon(f) \leq C\varepsilon^{\frac{-1}{\alpha}}$$

2. [Ryn98] almost every n-frequencies

$$l_\varepsilon(f) \leq C\varepsilon^{-\frac{n-1}{\alpha}}|\log(\varepsilon)|^{3(n-1)}$$

Rate of convergence in 1D almost periodic

Theorem (Hu-Tu-Zhang '24): In 1D with H is convex, coercive ($\frac{1}{2}|p|^2$ for simplicity)

$$H(x,p) = \frac{|p|^2}{2} - V(x), \qquad V(x) = \mathbb{V}(\xi x), \mathbb{V} \in \mathrm{C}(\mathbb{T}^n), \mathbb{V} \geq 0.$$

There is $C(n,\alpha,\xi,V)$ such that

$$u^\varepsilon(x,t) - u(x,t) \geq \begin{cases} -C\varepsilon & \mathbb{V}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ -C\varepsilon^{\frac{\alpha}{\alpha+n-1}} |\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbb{F} \in C^\alpha(\mathbb{T}^n), \\ -C\varepsilon^{\frac{\alpha}{\alpha+1}} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

If $\overline{H} \in C^{1,\beta}(\mathbb{R})$ then

$$u^\varepsilon(x,t) - u(x,t) \leq \begin{cases} C\varepsilon^{\frac{\beta}{\beta+1}} & \mathbb{V}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ C\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha}{\alpha+n-1}} |\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbb{F} \in C^\alpha(\mathbb{T}^n), \\ C\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha}{\alpha+1}} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

Place in the literature

1. First algebraic rate for almost periodic setting (only abstract modulus rate, PDE method in the literature).
2. the relation between how irrational of ξ and the regularity of $\mathbb{V}$ is intricate.

Case study

Examples $\mathbb{V}(x,y) = (2 - \sin(2\pi x) - \sin(2\pi y))^\gamma$ and $\xi = (1, \sqrt{2})$.

$$H(x,p) = \frac{|p|^2}{2} - \left(2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x)\right)^\gamma, \qquad \gamma > 0.$$

Consider the homogenization problem in 1D

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \\ u^\varepsilon(x,0) = u_0(x) \end{cases} \longrightarrow \begin{cases} u_t + \overline{H}(Du) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Then

$\boxed{\gamma > 2}$ $$-C\varepsilon \le u^\varepsilon - u \le C\varepsilon^\tau, \quad \tau = \frac{\gamma - 2}{3\gamma - 2}$$

$\boxed{\gamma = 2}$ $$-C\varepsilon \le u^\varepsilon - u \le \frac{C}{|\log(\varepsilon)|}$$

$\boxed{\gamma < 2}$ $$u^\varepsilon - u \ge \begin{cases} -C\varepsilon^{\frac{\gamma}{\gamma+1}}, & \gamma \in (0,1), \\ -C\varepsilon^{1/2}, & \gamma \in [1,2]. \end{cases}$$

Idea of the proof

$$\boxed{\frac{v_p(t)}{t} = \mathcal{O}\left(\frac{1}{t^\alpha}\right) \text{ as } t \to \infty} \leq u^\varepsilon - u \leq \begin{cases} \text{shape and regularity of } \overline{H} \\ \text{averaging optimal path :} \\ \qquad \left|\frac{\eta(t)}{t} - \overline{H}'(p)\right| \leq \mathcal{O}\left(\frac{1}{t^\beta}\right). \end{cases}$$

1. Lower bound is easy: decay rate of correctors and Hopf-Lax formula

$$\mathcal{M}(f)$$

2. Upper bound is harder: long time average of characteristic (calibrated curve)

$$\mathcal{M}(f^{-1})$$

Shape of $\overline{H}$

To compute $\overline{H}(p)$, we look for a sublinear solution v_p to

$$H(x, p + Dv_p(x)) = \mu$$

Assume $\overline{H}(p) = \mu$, we look for p instead

$$\frac{|p + v'(x)|^2}{2} - \mathbb{V}(\xi x) = \mu \quad \Longrightarrow \quad v(x) = \int_0^x \sqrt{2(\mu + \mathbb{V}(x))}\, dx - px$$

Then

$$\frac{v(x)}{x} = \frac{1}{x} \int_0^x \sqrt{2(\mu + \mathbb{V}(x))}\, dx - p \to 0$$

With

$$p_\mu = \mathcal{M}(\sqrt{2(\mu + \mathbb{V})}) = \int_{\mathbb{T}^n} \sqrt{2(\mu + \mathbb{V}(\mathbf{x}))}\, d\mathbf{x}.$$

Sketch of the proof - 1

1. If $H(x,p) = \frac{|p|^2}{2} + V(x)$ then the Lagrangian $L(x,v) = \frac{|v|^2}{2} - V(x)$.
2. Let $(x,t) = (0,1)$, use optimal control formula (action minimizing)

$$A^\varepsilon[\eta] = \varepsilon \int_0^{\varepsilon^{-1}} L(\eta(s), -\dot{\eta}(s))\, ds + u_0\left(\varepsilon\eta(\varepsilon^{-1})\right)$$

   and

$$u^\varepsilon(0,1) = \inf_{\eta(0)=0} A^\varepsilon[\eta]$$

3. A minimizer has conservation of energy

$$\frac{|\dot{\eta}(s)|^2}{2} + V(\eta(s)) = r$$

4. Rewrite

$$u^\varepsilon(0,1) = \inf_r \left(\inf_{\eta_r} A^\varepsilon[\eta_r] \right)$$

5. For each energy r, averaging each terms of the action with rate

Sketch of the proof - 2

1. Lower bound is easy

$$A^\varepsilon[\eta_r] \geq u(0,1) + \inf_{|p|\geq p_0} \varepsilon v_p(\eta(\varepsilon^{-1}))$$

2. Lower bound correspond to decay rate of corrector $\frac{v_p(x)}{|x|}$ as $|x| \to \infty$, i.e., convergence rate to the mean value

$$\left| \frac{1}{T} \int_0^T \mathbb{V}^{1/2}(\xi x)\, d\mathbf{x} - \mathcal{M}(\mathbb{V}^{1/2}) \right| \leq \frac{C}{T^\theta}$$

3. For $|p| \geq p_0$

$$\left| \frac{v_p(t)}{t} \right| \leq \left| \frac{1}{t} \int_0^t \mathbb{F}_\mu(\xi x)\, dx - \mathcal{M}(\mathbb{F}_\mu) \right| \leq \begin{cases} C|t|^{-1} \\ C|t|^{-\frac{\alpha}{\alpha+n-1}} |\log(t)|^{3(n-1)} \end{cases}$$

- The first case happens for $\mathbb{F} \in H^s(\mathbb{T}^n)$ $(s > n/2 + \sigma_\xi)$
- The second case happens for a.e. $\xi \in \mathbb{R}^n$ with $\mathbb{F} \in C^{0,\alpha}(\mathbb{T}^n)$.

Sketch of the proof - 3

1. Upper bound is harder, obtainable when negative energy $r < 0$ does not play a role, i.e., $\overline{H} \in C^1$
2. Look at

$$A^\varepsilon[\eta_r] = (\varepsilon\eta_r(\varepsilon^{-1})) \underbrace{\left(\frac{1}{\eta_r(\varepsilon^{-1})} \int_0^{\eta_r(\varepsilon^{-1})} \sqrt{2(r - \mathbb{V}(\xi x))}\, dx\right)}_{p_r = \mathcal{M}(\sqrt{2(r-\mathbb{V})})} + u_0(\varepsilon\eta_r(\varepsilon^{-1})).$$

3. The difficult term is

$$\varepsilon\eta_r(\varepsilon^{-1}) \qquad \longleftrightarrow \qquad \frac{\eta(t)}{t} \to q \in \partial\overline{H}$$

This is the large time average of calibrated curve to a rotation vector.

4. Difficult to do directly in a uniform way as $r \to 0^+$, by Euler-Lagrange equation

$$\frac{1}{\varepsilon\eta(\varepsilon^{-1})} = \frac{1}{\eta(\varepsilon^{-1})} \int_0^{\eta(\varepsilon^{-1})} \frac{dx}{\sqrt{2(r - \mathbb{V}(\xi x))}} \to \mathcal{M}\left(\frac{1}{\sqrt{2(r - \mathbb{V})}}\right)$$

5. Using Hamilton–Jacobi equation: uniform in $r \to 0^+$

$$\overline{H} \in C^{1,\beta} \qquad \Longrightarrow \qquad \left|\frac{\eta_r(t)}{t} - \overline{H}'_+(p_r)\right| \le C\varepsilon^{\frac{\beta}{1+\beta}}.$$

Application to ergodic estimate

For $\mathbb{V}(x_1, x_2) = (2 - \sin(2\pi x_1) - \sin(2\pi x_2))^\gamma$ and $\xi = (\xi_1, \xi_2)$ with $\frac{\xi_2}{\xi_1}$ is badly approximable, $H(x,p) = \frac{|p|^2}{2} - \mathbb{V}(\xi x)$, then

$$\left|\frac{\eta(t)}{t} - \overline{H}'(p)\right| \le \begin{cases} C|t|^{-\frac{\gamma-2}{3\gamma-2}} & \gamma > 2 \\ C|t|^{-\frac{2-\gamma}{2(2+\gamma)}} & \gamma < 2 \\ C|\log(t)|^{-1} & \gamma = 2. \end{cases}$$

Consequently

$$\left|\frac{1}{T}\int_0^T \frac{dx}{\mathbb{V}^{1/2}(\xi x)} - \int_{\mathbb{T}^2} \frac{d\mathbf{x}}{\mathbb{V}(\mathbf{x})}\right| \le C\left(\frac{1}{T}\right)^{\frac{2-\gamma}{2(2+\gamma)}} \qquad \gamma < 2$$

while

$$\frac{1}{T}\int_0^T \frac{dx}{\mathbb{V}^{1/2}(\xi x)} \ge \begin{cases} C\left(\frac{1}{T}\right)^{\frac{\gamma-2}{3\gamma-2}} & \gamma > 2 \\ \frac{C}{|\log(T)|} & \gamma = 2. \end{cases}$$

Outlooks

1. For zero energy $r = 0$

$$\overline{H} \in C^{1,\alpha} \longrightarrow \varepsilon^{\frac{\alpha}{1+\alpha}} \longrightarrow \varepsilon^{\frac{\alpha(\alpha+1)}{\alpha(\alpha+1)+1}}$$

We have

$$\left|\frac{\eta_0(t)}{t}\right| \leq \left(\frac{1}{|t|}\right)^{\tau} \qquad \text{where } \tau = \frac{(\gamma-2)(3\gamma-2)}{(\gamma-2)(3\gamma-2)+4\gamma^2}.$$

If this holds uniformly for η_r as $r \to 0^*$ then we can improve the rate of homogenization

2. Nonsmooth $\overline{H}$?
3. Gaps in the quantitative estimate using two different methods?

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