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On the regularity of stochastic effective Hamiltonian

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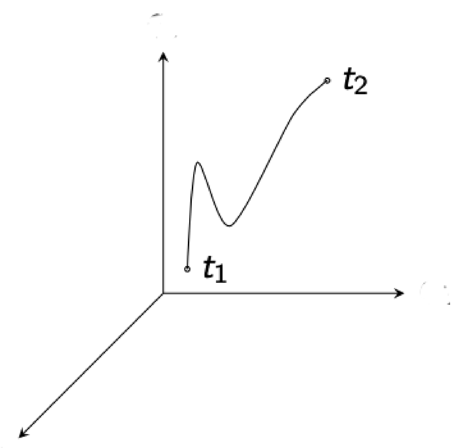
Introduction

The system's state at time t is defined by the values of the generalized coordinates. This is represented by a point in the "configuration space". The motion is characterized by the path traced out by this point in configuration space.



- **Lagrangian mechanics**. $L = L(x, v)$ (x-position, v-velocity) is the Lagrangian. $\mathbf{x}$ solves the Euler-Lagrange equations
$$-\frac{d}{dt}\left(D_v L(\mathbf{x}, \dot{\mathbf{x}})\right) = D_x L(\mathbf{x}, \dot{\mathbf{x}}). \tag{1}$$
By *principle of least action*, $\mathbf{x}$ minimizes the action $\int_0^T L(\mathbf{x}(t), \dot{\mathbf{x}}(t))dt$.
- **Hamiltonian mechanics**. $p = D_v L(x, v)$ (momentum), and $H = L^*$. Let $\mathbf{p}(t) = D_v L(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ then $(\mathbf{x}, \mathbf{p})$ solves
$$\begin{cases} \dot{\mathbf{x}} = D_p H(\mathbf{x}, \mathbf{p}) \\ \dot{\mathbf{p}} = -D_x H(\mathbf{x}, \mathbf{p}). \end{cases} \quad \text{and} \quad \frac{d}{dt} H(\mathbf{x}(t), \mathbf{p}(t)) = 0. \tag{2}$$
These are the *Hamilton's equation* and *conservation of energy*.
- **Hamilton-Jacobi PDE**. Let $u(P, x)$ solves
$$H(x, D_x u) = \overline{H}(P) \qquad \text{then} \qquad \begin{cases} \dot{\mathbf{X}} = D\overline{H}(\mathbf{P}) \\ \dot{\mathbf{P}} = 0. \end{cases} \tag{3}$$
via the canonical change of variable $(p, x) \mapsto (P, X)$: $\overline{H}$ is the *effective Hamiltonian*, uniquely defined [2] (via homogenization).

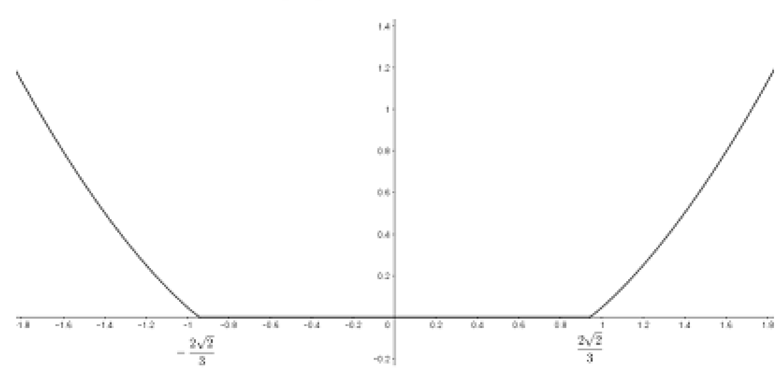


Figure 1: $\overline{H}$ where $H(x,p) = \frac{1}{2}|p|^2 - (1 + |2x-1|)$ on $\mathbb{T}^1 \times \mathbb{R}$

- **Stochastic effective Hamiltonian** In PDE language (motivated from Homogenization [2]), we concern the ergodic problem
$$H(x, p + Du(x)) = \overline{H}(p) \qquad \text{in } \mathbb{T}^n \tag{4}$$
and its stochastic version [1]
$$H(x, p + Du^\varepsilon(x)) - \varepsilon \Delta u^\varepsilon = \overline{H}^\varepsilon(p) \qquad \text{in } \mathbb{T}^n. \tag{5}$$

Contributions

1. Representation and regularity of $\overline{H}^\varepsilon, \overline{H}$.
2. On the semi-classical limit $\overline{H}^\varepsilon \to \overline{H}$ as $\varepsilon \to 0^+$:
$$|\overline{H}^\varepsilon - \overline{H}| = \mathcal{O}(\varepsilon)$$
for *convex* Hamiltonians.

1 Mather measures

Stochastic minimizing measures [1] extends classical Mather measures [3, 4]. A probability measure $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ is called a *holonomic measure* if
$$\int_{\mathbb{T}^n \times \mathbb{R}^n} v \, d\mu < \infty, \quad \int_{\mathbb{T}^n \times \mathbb{R}^n} \big(v \cdot D\varphi(x) - \varepsilon \Delta \varphi(x)\big) \, d\mu = 0 \quad \forall\, \varphi \in C^2(\mathbb{T}^n).$$
Let $\mathcal{C}(\varepsilon)$ be the set of holonomic measures, then
$$-\overline{H}^\varepsilon(0) = \inf_{\mu \in \mathcal{C}(\varepsilon)} \int_{\mathbb{T}^n \times \mathbb{R}^n} L \, d\mu. \tag{6}$$
Stochastic Mather measures are mesures that minimize (6).

2 Regularity

Theorem 1 ([5]). *Assume convexity, then $\varepsilon \mapsto H^\varepsilon(0)$ is C^1-smooth in $\varepsilon > 0$*
$$\frac{d}{d\varepsilon} \overline{H}^\varepsilon(0) = -\int_{\mathbb{T}^n \times \mathbb{R}^n} \Delta u^\varepsilon(x) \, d\mu \qquad \textit{for all } \mu \in \mathcal{M}_0(\varepsilon). \tag{7}$$
$\mathcal{M}_0(\varepsilon)$ is the set of Mather measures associated with (5) with $p = 0$, and u^ε solves (5).

Theorem 2 ([5]). *Assume convexity, then for $\varepsilon = 0$, $p \mapsto \overline{H}(p)$ has one-sided directional derivatives in any direction $\zeta \in \mathbb{R}^n$ and*
$$\begin{aligned} D_{\xi+}\overline{H}(p) &= \max_{\mu \in \mathcal{M}_p(0)} \int_{\mathbb{T}^n \times \mathbb{R}^n} v \cdot \xi \, d\mu(x, v) \\ D_{\xi-}\overline{H}(p) &= \min_{\mu \in \mathcal{M}_p(0)} \int_{\mathbb{T}^n \times \mathbb{R}^n} v \cdot \xi \, d\mu(x, v). \end{aligned} \tag{8}$$

- $\mathcal{M}_p(0)$: Mather measures associated to $(x, \xi) \mapsto H(x, p + \xi)$.
- If $\mathcal{M}_p(0) = \{\mu\}$ is a singleton, then $p \mapsto \overline{H}(p)$ is differentiable at p with $D\overline{H}(p) = \int_{\mathbb{T}^n \times \mathbb{R}^n} v \, d\mu(x, v)$.
- Theorem 2 relies on rather weak assumptions: $H \in C^0(\mathbb{T}^n \times \mathbb{R}^n)$ is both coercive and convex.

3 Rate of convergence

Theorem 3. *Assume convexity, then $\varepsilon \mapsto \overline{H}^\varepsilon(0)$ is uniformly Lipschitz for $\varepsilon \in [0, 1]$. Consequently*
$$|\overline{H}^\varepsilon(0) - \overline{H}(0)| = \mathcal{O}(\varepsilon). \tag{9}$$

- For general nonconvex setting: $|\overline{H}^\varepsilon(0) - \overline{H}(0)| = \mathcal{O}(\varepsilon^{1/2})$.
- For special $H(x, p) = \frac{1}{2}|p|^2 + V(x)$, the rate (9) is known. We generalized this to all convex Hamiltonians.

4 Conclusions || Discussion

The proof technique relies on the scaling structure of Mather measures and has been applied to various domain perturbation problems, starting with [6].

References

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