Convergence rate for quasi periodic homogenization of Hamilton–Jacobi equations

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Overview

- How is particle's large-scale motion influenced by complex local patterns?
- At large scales, an effective operator with constant coefficients captures the coarse properties of a heterogeneous operator.
- Deriving explicit rates or next-order corrections remains a challenging task.
- Mathematical model: large-scale behavior of differential operators with highly oscillatory coefficients → effective operator with simpler coefficients.

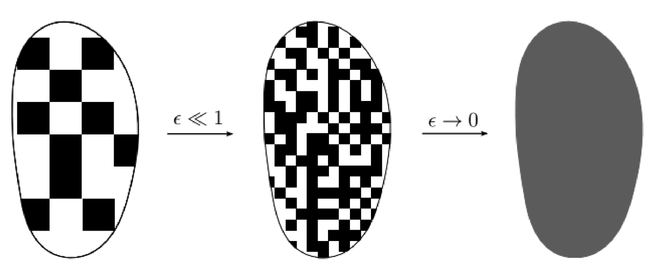

Figure 1. As $\varepsilon \to 0$, the medium is expected to *homogenize*. (Figure adapted from [7])

Background

For each $\varepsilon > 0$, let $u^\varepsilon \in C\big(\mathbb{R} \times [0,\infty)\big)$ be the viscosity solution to:

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{C$_\varepsilon$}$$

Initial data $u_0 \in \mathrm{BUC}(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, with Hamiltonian $H(x,p) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ continuous and satisfying:

- (almost periodicity): for $R > 0$, $\{H(\cdot + z, \cdot) : z \in \mathbb{R}\}$ is relatively compact in $\mathrm{BUC}(\mathbb{R}^n \times B_R(0))$;
- (coercivity): $H(x,p) \to \infty$ as $|p| \to \infty$ uniformly in x.

Homogenization: $u^\varepsilon \to u$ locally uniformly on $\mathbb{R} \times [0,\infty)$ as $\varepsilon \to 0^+$, which solves the effective equation [4]:

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{C}$$

Effective Hamiltonian $\overline{H}(p)$: the unique constant for which the *approximated cell problem* is solvable for all $\delta > 0$:

$$\overline{H}(p) - \delta \le H\big(x, p + Dw_\delta(x;p)\big) \le \overline{H}(p) + \delta \qquad \text{in } \mathbb{R}. \tag{CP$_\delta$}$$

First studied in the periodic setting in [5].

Rate of convergence

Heuristically, $u^\varepsilon(x,t) \approx u(x,t) + \varepsilon w_0\left(\frac{x}{\varepsilon}, Du(x,t)\right)$, implying $|u^\varepsilon - u| = \mathcal{O}(\varepsilon)$ if w_0 is bounded.
Periodic setting:

- PDE approach: $\mathcal{O}(\varepsilon^{1/3})$ result from [3].
- For convex H: The $\mathcal{O}(\varepsilon)$ rate was achieved in [8].

Almost-Periodic Setting: Noncompactness and other differences make extending methods from the periodic setting challenging.

- For strongly mixing media, a logarithmic-type rate was established in [2].
- A modulus-based rate was derived in [1].

We adopt an optimal control-based framework (inspired by [6]) for a new approach:

- New quantitative ergodic estimates for functions in the quasi-periodic setting.
- Connection between the regularity of $\overline{H}$ and the rate of convergence.
- Convergence rate for the Birkhoff average of **unbounded quasi-periodic functions**.

One-dimensional almost periodic functions

First studied by Bohr (1926). For $\varepsilon > 0$ and $f : \mathbb{R} \to \mathbb{R}$, we denote by

$$E(\varepsilon, f) = \{\tau \in \mathbb{R} : |f(x+\tau) - f(x)| < \varepsilon \text{ for all } x \in \mathbb{R}\}$$

the the set of all ε-periods. We say f is almost periodic, i.e., $f \in \mathrm{AP}(\mathbb{R})$ if for $\varepsilon > 0$, there exists $l_\varepsilon(f)$ such that any interval of length l_ε has an ε-period. In other words, for every $a \in \mathbb{R}$

$$[a, a + l_\varepsilon] \cap E(\varepsilon, f) \neq \emptyset.$$

Almost periodic functions have the **mean value property**:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)dx = \mathcal{M}(f).$$

If $f(x) = F(\xi x)$ is quasi-periodic, with $F \in C(\mathbb{T}^n)$ and $\xi \in \mathbb{R}^n$ a nonresonant frequency ($\xi \cdot \kappa \neq 0$ for all $\kappa \in \mathbb{Z}^n \setminus \{0\}$), then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx = \mathcal{M}(f) = \int_{\mathbb{T}^n} F(\mathbf{x})\,d\mathbf{x}.$$

Rate of convergence to the mean value

Almost periodic: For every $\varepsilon > 0$

$$\left|\frac{1}{T}\int_0^T f(x)\,dx - \mathcal{M}(f)\right| \le \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})}\frac{l_\varepsilon(f)}{T}.$$

Need an estimate of $l_\varepsilon(f)$ with respect to ε, but good as only L^∞ is needed.
Quasi periodic: $f(x) = \mathbf{F}(\xi x)$ for $x \in \mathbb{R}$.
If $\mathbf{F} \in C^{0,\alpha}(\mathbb{T}^n)$

- $n = 2$, for badly approximable ξ (a null set of $\mathbb{R}^2$): $l_\varepsilon(f) \le C\varepsilon^{\frac{-1}{\alpha}}$.
- for almost every n-frequencies ξ: $l_\varepsilon(f) \le C\varepsilon^{-\frac{n-1}{\alpha}}|\log(\varepsilon)|^{3(n-1)}$.

If $\mathbf{F} \in H^s(\mathbb{T}^n)$ for $s > \frac{n}{2} + \sigma_\xi$ then

$$\left|\frac{1}{T}\int_0^T \mathbf{F}(\xi x)\,dx - \int_{\mathbb{T}^n} \mathbf{F}(\mathbf{x})\,d\mathbf{x}\right| \le \frac{C(n,s)\|\mathbf{F}\|_{H^s(\mathbb{T}^n)}}{T}.$$

Here σ_ξ is a Diophantine condition of ξ: $\xi \cdot \kappa \ge \frac{C}{|\kappa|^\sigma}$ for all $\kappa \in \mathbb{Z}^n$.

Homogenization rate with quasi-periodic potentials

Theorem (Hu, Tu, Zhang '24): In $\mathbb{R}$, with H being convex and coercive (e.g., $\frac{1}{2}|p|^2$ for simplicity).

$$H(x,p) = \frac{|p|^2}{2} - V(x), \qquad V(x) = \mathbf{U}(\xi x), \mathbf{U} \in \mathrm{C}(\mathbb{T}^n), \mathbf{U} \ge 0.$$

There is $C(n,\alpha,\xi,V)$ such that

$$u^\varepsilon(x,t) - u(x,t) \ge \begin{cases} -C\varepsilon & \mathbf{U}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ -C\varepsilon^{\frac{\alpha}{\alpha+n-1}}|\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbf{U} \in C^\alpha(\mathbb{T}^n), \\ -C\varepsilon^{\frac{\alpha}{\alpha+1}} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

If $\overline{H} \in C^{1,\beta}(\mathbb{R})$ then

$$u^\varepsilon(x,t) - u(x,t) \le \begin{cases} C\varepsilon^{\frac{\beta}{\beta+1}} & \mathbf{U}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ C\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha}{\alpha+n-1}}|\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbf{U} \in C^\alpha(\mathbb{T}^n), \\ C\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha}{\alpha+1}} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

Example and Consequence

Example. Let $H(x,p) = \frac{1}{2}|p|^2 - \left(2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x)\right)^\gamma$ for $\gamma > 0$. Then

$$\left.\begin{matrix} -C\varepsilon \\ -C\varepsilon \\ -C\varepsilon^{1/2} \\ -C\varepsilon^{\frac{\gamma}{\gamma+1}} \end{matrix}\right\} \le u^\varepsilon - u \le \begin{cases} C\varepsilon^{\frac{\gamma-2}{3\gamma-2}} & \gamma > 2, \\ \frac{C}{|\log(\varepsilon)|} & \gamma = 2, \\ ? & \gamma \in (0,1), \\ ? & \gamma \in [1,2]. \end{cases}$$

Example of convergence rate for the (unbounded) Birkhoff average.

$$\left|\frac{1}{T}\int_0^T \frac{dx}{2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x)} - \int_{\mathbb{T}^2} \frac{dx_1 dx_2}{2 - \sin(2\pi x_1) - \sin(2\pi x_2)}\right| \le \frac{C}{T^{1/6}}.$$

Methodology and Future Directions

Optimal control formula:

$$u^\varepsilon(0,1) = \inf_{\eta\in\mathcal{T}} \underbrace{\left\{\varepsilon\int_0^{\varepsilon^{-1}} \left(\frac{|\dot\eta(s)|^2}{2} - V(\eta(s))\right) ds + u_0\left(\varepsilon\eta(\varepsilon^{-1})\right)\right\}}_{A^\varepsilon[\eta]}, \qquad \mathcal{T} = \{\eta(\cdot) \in \mathrm{AC}, \eta(0) = 0\}.$$

Conservation of energy: $\frac{1}{2}|\dot\eta(s)|^2 + V(\eta(s)) = r$ for $s \in (0, \varepsilon^{-1})$. Treat $r < 0$ and $r \ge 0$ separately:

$$u^\varepsilon(0,1) = \inf_r \left\{A^\varepsilon[\eta_\varepsilon] : \text{among minimizers } \eta_\varepsilon \text{ with energy } r\right\}.$$

Corrector w to $H(x, p + Dw(x)) = \overline{H}(p)$:

$$\frac{w(t)}{t} = \mathcal{O}\left(\frac{1}{t^\alpha}\right) \text{ as } t \to \infty \le u^\varepsilon - u \le \begin{cases} \text{shape and regularity of } \overline{H} \\ \text{averaging optimal path} : \left|\frac{\eta(t)}{t} - \overline{H}'(p)\right| \le \mathcal{O}\left(\frac{1}{t^\beta}\right). \end{cases}$$

- Lower bound: correctors, quantitative ergodic estimate, and Hopf-Lax formula.
- Upper bound: long-time average of characteristics, shape, and regularity of $\overline{H}$.

Outlook:

- If $\overline{H}$ is not smooth, there exists a periodic orbit whose period blows up as $r \to 0^-$, making the method inadequate for finding the upper bound.
- The quantitative ergodic estimate can be refined to achieve an optimal rate in certain cases, which will be explored in future work.

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