

**HOMOGENIZATION OF HAMILTON-JACOBI EQUATIONS
NOTES AND REMARKS**

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ABSTRACT. These notes contain the essential materials and some further remarks of the summer course "Homogenization of Hamilton-Jacobi equations" taught by Prof. Hung V. Tran¹ at University of Science, Ho Chi Minh City, Viet Nam from July 6th to July 14th, 2015.

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Notations:

- $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is the n -dimensional donut (torus).
- w.r.t: with respect to

LECTURE 1

Clear contest : $\varepsilon > 0$: scale of the equation, $\varepsilon \ll 1$: ε is very small.

Equation of interest. Following the Hamilton-Jacobi equation

$$(I) \quad u_t^\varepsilon(x, t) + H\left(\frac{x}{\varepsilon}, Du^\varepsilon(x, t)\right) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

where

- $u^\varepsilon(x, t) : \mathbb{R}^n \times [0, \infty) \longrightarrow \mathbb{R}$,
 x is the location variable (spatial) and t is the time variable.
- $u_t^\varepsilon(x, t) = \frac{\partial}{\partial t} u^\varepsilon(x, t)$.
- $Du^\varepsilon(x, t) = \nabla_x(u^\varepsilon)(x, t) = \left(\frac{\partial u^\varepsilon}{\partial x_1}, \dots, \frac{\partial u^\varepsilon}{\partial x_n}\right)$.
- H is the Hamiltonian (total energy)

$$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(y, p) \longmapsto H(y, p) \in \mathbb{R}.$$

Example 1 (Classical mechanics).

$$H(y, p) = \frac{1}{2}|p|^2 + V(y)$$

where the first term is the kinetic energy and the second one is the potential energy.

Question. We want to understand (I) as $\varepsilon \longrightarrow 0$. Is there some u such that $u_\varepsilon \longrightarrow u$ and is u solve something simpler? This is one of the key questions in homogenization theory.

Assumptions. The following assumptions are extremely important:

(H1) (coercivity)

$$\lim_{|p| \rightarrow \infty} H(y, p) = \infty \quad \text{uniformly in } y,$$

(H2) (periodicity)

$$H(y + k, p) = H(y, p) \quad \forall k \in \mathbb{Z}^n.$$

Formal Analysis. (not rigorous) Consider the equation

$$(I) \quad u_t^\varepsilon(x, t) + H\left(\frac{x}{\varepsilon}, Du^\varepsilon(x, t)\right) = 0$$

where t is time variable, x is spatial variable (slow variable) and $y = \frac{x}{\varepsilon}$ is the fast oscillatory variable. There are several ways to begin our analysis:

- Numerical analysis: plot u^ε .
- Guessing: some asymptotic expansion w.r.t ε .

We consider some ansatz²

$$(1) \quad u^\varepsilon(x, t) = u(x, t) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}, t\right) + \varepsilon^2 u^2\left(x, \frac{x}{\varepsilon}, t\right) + \dots$$

$$(2) \quad u^\varepsilon(x, t) = u\left(x, \frac{x}{\varepsilon}, t\right) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}, t\right) + \dots$$

$$(3) \quad u^\varepsilon(x, t) = u(x, t) + \varepsilon^5 u^1\left(\frac{x}{\varepsilon}, t\right) + \dots$$

$$(4) \quad u^\varepsilon(x, t) = u(x, t) + \varepsilon u^1\left(\frac{x}{\varepsilon}, t\right) + \varepsilon^2 u^2\left(\frac{x}{\varepsilon}, t\right) + \dots$$

Now using (4), we have

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon u^1\left(\frac{x}{\varepsilon}, t\right) + \varepsilon^2 u^2\left(\frac{x}{\varepsilon}, t\right) + \dots$$

$$Du^\varepsilon(x, t) = Du(x, t) + Du^1\left(\frac{x}{\varepsilon}, t\right) + \varepsilon Du^2\left(\frac{x}{\varepsilon}, t\right) + \dots$$

Plugging these into (I) we have

$$\left[u_t^\varepsilon(x, t) + \varepsilon u_t^1\left(\frac{x}{\varepsilon}, t\right) + \dots \right] + H\left(\frac{x}{\varepsilon}, Du(x, t) + Du^1\left(\frac{x}{\varepsilon}, t\right) + \varepsilon Du^2\left(\frac{x}{\varepsilon}, t\right) + \dots\right) = 0.$$

Denote $y = \frac{x}{\varepsilon}$, we obtain

$$\left[u_t^\varepsilon(x, t) + \varepsilon u_t^1(y, t) + \dots \right] + H\left(y, Du(x, t) + Du^1(y, t) + \varepsilon Du^2(y, t) + \dots\right) = 0.$$

We will assume a "big lie": x and y are independent. Matching asymptotic expansion zero-order term $O(1)$, we get

$$u_t(x, t) + H\left(y, Du(x, t) + Du^1(y, t)\right) = 0.$$

Now, fix $(x, t) \in \mathbb{R}^n \times (0, \infty)$, since $u_t(x, t)$ is a constant w.r.t y ,

$$H(y, Du(x, t) + Du^1(y, t))$$

is expected to be a constant w.r.t y , i.e.,

$$H(y, Du(x, t) + Du^1(y, t)) = C(p, t).$$

²The word *Ansatz* in German is some thing like the word *formulation* in English.

Let $p = Du(x, t) \in \mathbb{R}^n$, then

$$H(y, p + Du^1(y, t)) = C(p, t).$$

Assume one more reduction $u^1(y, t) = u^1(y)$, we then have the following PDE

$$H(y, p + Du^1(y)) = C(p).$$

This is called the cell (or ergodic) problem.

Theorem 1 (Lions-Papanicolau-Varadhan). *Fix $p \in \mathbb{R}^n$, there exists a unique constant $C \in \mathbb{R}^n$ such that the cell problem*

$$H(y, p + Du^1(y)) = C(p) \quad \text{in } \mathbb{R}^n$$

has a periodic solution u^1 .

Define the effective Hamiltonian $\bar{H}(p) = C(p)$. We are back to our original question: consider

$$(C_\varepsilon) \quad \begin{cases} u_t^\varepsilon + H(\frac{x}{\varepsilon}, Du^\varepsilon) = 0 \\ u^\varepsilon(x, 0) = u_0(x) \end{cases} .$$

Do the solutions u^ε of (C_ε) converges to a solution u of

$$(HJ) \quad \begin{cases} u_t + \bar{H}(Du) = 0 \\ u(x, 0) = u_0(x) \end{cases} ?$$

Remark 1. *This is a natural question since from the Hamiltonian $H(\frac{x}{\varepsilon}, Du^\varepsilon)$ we derived the effective Hamiltonian \bar{H} that is independent of ε so as we pass $\varepsilon \rightarrow 0$, (C_ε) should become (HJ) . The answer to this problem is “yes” and will be elaborated in lecture 3.*

LECTURE 2

Before we prove theorem 1, let's talk about viscosity solutions.

Introduction to viscosity solutions. Consider

- The static problem

$$(S) \quad u(y) + H(y, Du(y)) = 0.$$

- The Cauchy problem

$$(C) \quad \begin{cases} u_t(x, t) + H(y, Du(x, t)) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(y, 0) = u_0(y) & \text{on } \mathbb{R}^n \end{cases}.$$

We then look into the static problem in \mathbb{R}^n . The idea here is that the problem

$$(S_\varepsilon) \quad u^\varepsilon + H(Du^\varepsilon(y), y) = \varepsilon \Delta u^\varepsilon \text{ in } \mathbb{R}^n$$

has a smooth solution u^ε . Assume further that $u^\varepsilon \rightarrow u$ locally uniformly in \mathbb{R}^n . Take a smooth function ϕ such that

$$\begin{cases} (u - \phi)(x_0) = 0 \\ (u - \phi)(x) < 0 & \text{else where} \end{cases}.$$

That is, $u - \phi$ has a strict max at x_0 .

Exercise 1. For $\varepsilon > 0$ small enough, $u^\varepsilon - \phi$ has a max at x_ε near by x_0 and there is a subsequence $\varepsilon_j \rightarrow 0$ such that $x_{\varepsilon_j} \rightarrow x_0$.³

Play with u^ε . Since $u^\varepsilon - \phi$ has max at x_ε , we then have, of course by second derivative test,

$$(1) \quad \begin{cases} Du^\varepsilon(x_\varepsilon) = D\phi(x_\varepsilon) & \implies Du^\varepsilon(x_\varepsilon) = D\phi(x_\varepsilon) \\ \Delta(u^\varepsilon - \phi)(x_\varepsilon) \leq 0 & \iff \Delta u^\varepsilon(x_\varepsilon) \leq \Delta\phi(x_\varepsilon) \end{cases}.$$

Reconsider the PDE,

$$u^\varepsilon(x_\varepsilon) + H(x_\varepsilon, Du^\varepsilon(x_\varepsilon)) = \varepsilon \Delta u^\varepsilon(x_\varepsilon).$$

From (1), we can see that

$$u^\varepsilon(x_\varepsilon) + H(x_\varepsilon, D\phi(x_\varepsilon)) \leq \varepsilon \Delta\phi(x_\varepsilon).$$

³Moreover, if u_m is continuous and ϕ is smooth such that $u_m \rightarrow u$ uniformly on a compact set $\bar{\Omega} \subset \mathbb{R}^n$, and assume $u - \phi$ has a strict (local) maximum over $\bar{\Omega}$ at x_0 , $u_m - \phi$ has (local) maximum over $\bar{\Omega}$ at x_m . Then we must have $x_m \rightarrow x_0$ as $m \rightarrow \infty$. We have the convergence of whole sequence here, while in case $u^\varepsilon \rightarrow u$ uniformly as $\varepsilon \rightarrow 0$ we only have the convergence of sub-sequence.

Let $\varepsilon_j \rightarrow 0$, we then have

$$u(x_0) + H(x_0, D\phi(x_0)) \leq 0.$$

Inspired by this derivation, we have the definition of viscosity solutions.

Definition 1 (Definition of viscosity solution for the static equation). Assume u is continuous on its domain

- (Subsolution) u is called a viscosity subsolution if for all $\phi \in C^\infty(\mathbb{R}^n)$ such that $(u - \phi)(x_0)$ is a strict max then

$$u(x_0) + H(x_0, D\phi(x_0)) \leq 0.$$

- (Supersolution) u is called a viscosity supersolution if for all $\phi \in C^\infty(\mathbb{R}^n)$ such that $(u - \phi)(x_0)$ is a strict min then

$$u(x_0) + H(x_0, D\phi(x_0)) \geq 0.$$

- u is called a viscosity solution if it is both a subsolution and a supersolution.

Geometric descriptions. Using the notions of sub-differential and super-differential, we will give a geometric descriptions for viscosity solution as the following

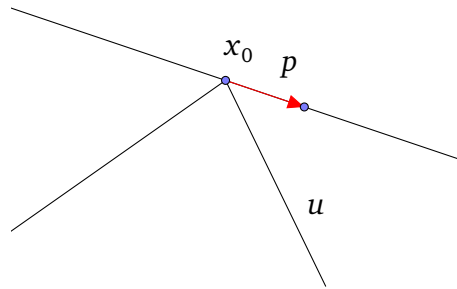
Definition 2. Let u be a real valued function defined on the open set $\Omega \subset \mathbb{R}^n$. For any $x \in \Omega$, the sets

$$D^-u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

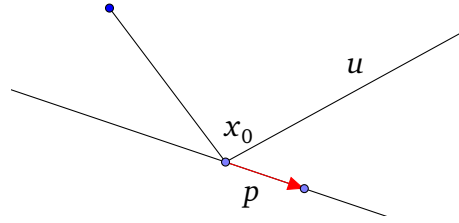
$$D^+u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{\|y - x\|} \leq 0 \right\}$$

are called, respectively the **subdifferential** and **superdifferential** of u at x .

We can see that $p \in D^+u(x_0)$ if $u(x) \leq u(x_0) + p \cdot (x - x_0)$ for all $x \in B(x_0, r)$.



Similarly, $p \in D^-u(x)$ if $u(x) \geq u(x_0) + p \cdot (x - x_0)$ for all $x \in B(x_0, r)$.



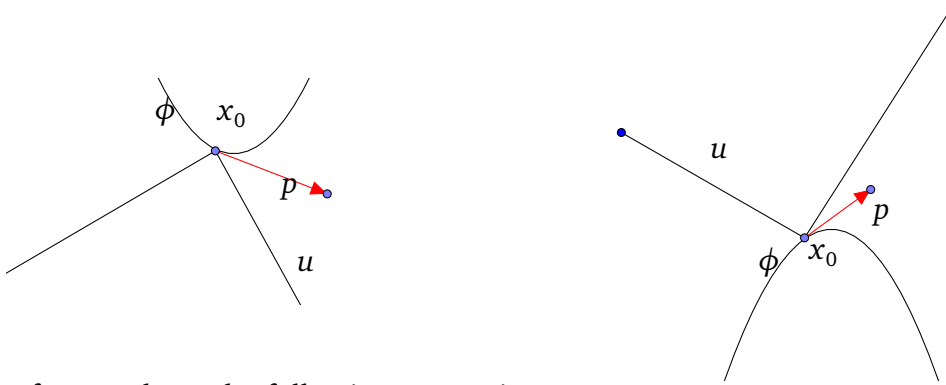
From this, we can see that, up to a constant,

$u - \phi$ has a strict max at $x_0 \iff u$ is touched from above by ϕ at x_0

$\iff D\phi(x_0) \in D^+u(x_0),$

$u - \phi$ has a strict min at $x_0 \iff u$ is touched from below by ϕ at x_0

$\iff D\phi(x_0) \in D^-u(x_0).$



In fact, we have the following properties:

Proposition 1 (Properties of sub-differentials and super-differentials).

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $x \in \Omega$ where Ω is open, then the following properties hold

- $D^+f(x) = -D^-(-f)(x).$
- $D^+f(x)$ and $D^-f(x)$ are convex (possibly empty).
- If $f \in C(\Omega)$, then $p \in D^+f(x)$ if and only if there is a function $\varphi \in C^1(\Omega)$ such that $\nabla\varphi(x) = p$ and $f - \varphi$ has a local maximum at x .
- If $f \in C(\Omega)$, then $p \in D^-f(x)$ if and only if there is a function $\varphi \in C^1(\Omega)$ such that $\nabla\varphi(x) = p$ and $f - \varphi$ has a local minimum at x .
- $D^+f(x)$ and $D^-f(x)$ are both nonempty if and only if f is differentiable at x . In this case we have that $D^+f(x) = D^-f(x) = \{\nabla f(x)\}.$
- If $f \in C(\Omega)$, the sets of points where a one-sided differential exists

$$\Omega^+ = \{x \in \Omega : D^+f(x) \neq \emptyset\} \quad \Omega^- = \{x \in \Omega : D^-f(x) \neq \emptyset\}$$

are both non-empty. Indeed, they are dense in Ω .

Proof.

(a) It's clear since if $a_n \rightarrow a$ then

$$\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$$

(b) It's also clear from the definitions.

(c) Assume that $p \in D^+f(x)$, by definition, we can find $\delta > 0$ and a continuous increasing function $\sigma : [0, \infty) \rightarrow \mathbb{R}$ with $\sigma(0) = 0$ such that

$$(2) \quad f(y) \leq f(x) + \langle p, y - x \rangle + \|y - x\| \sigma(\|y - x\|)$$

for $\|y - x\| < \delta$. Define

$$\rho(r) = \int_0^r \sigma(t) dt$$

then

$$\rho(0) = \rho'(0) = 0 \quad \text{and} \quad r\sigma(r) \leq \rho(2r) \leq r\sigma(2r)$$

Now for $y \in B(x, \delta)$ we set

$$\varphi(y) = f(x) + \langle p, y - x \rangle + \rho(2\|y - x\|).$$

Since $f(x) = \varphi(x)$, clearly φ is differentiable. Furthermore, because

$$\sigma(r) \leq \frac{\rho(2\|y - x\|)}{\|y - x\|} \leq \sigma(2\|y - x\|)$$

and $\sigma(r) \rightarrow 0$ as $r \rightarrow 0$, we conclude that $\nabla\varphi(x) = p$. Now for $y \in B(x, \delta)$, from (2) we have

$$\begin{aligned} f(y) - f(x) &\leq \langle p, y - x \rangle + \|y - x\| \sigma(\|y - x\|) \\ &\leq \langle p, y - x \rangle + \rho(2\|y - x\|) = \varphi(y) - \varphi(x). \end{aligned}$$

Therefore, $(f - \varphi)(y) \leq (f - \varphi)(x)$ for $y \in B(x, \delta)$, i.e, $f - \varphi$ has a local maximum at x .

For the converse, if $\varphi \in C^1(\Omega)$ such that $f - \varphi$ has a local maximum at x and $f(x) = \varphi(x)$, $\nabla\varphi(x) = p$. Then, since $f(y) - f(x) \leq \varphi(y) - \varphi(x)$ in a neighborhood of x , we have

$$\limsup_{y \rightarrow x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{\|y - x\|} \leq \limsup_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - \langle p, y - x \rangle}{\|y - x\|} = 0.$$

Therefore, $p \in D^+f(x)$.

(d) This completely similar to (c).

- (e) If f is differentiable at x , then clearly $\nabla f(x) \in D^+f(x) \cap D^-f(x)$. Furthermore, if $p \in D^+f(x)$, then there exists $\varphi \in C^1(\Omega)$ such that

$$\varphi(x) = f(x) \quad \nabla\varphi(x) = p$$

and $f - \varphi$ has a local maximum at x , clearly $p = \nabla\varphi(x) = \nabla f(x)$. Doing similarly for $D^-f(x)$ we have $D^+f(x) = D^-f(x) = \{\nabla f(x)\}$.

For the converse, assume that $D^+f(x)$ and $D^-f(x)$ are both nonempty. Assume $a \in D^+f(x)$ and $b \in D^-f(x)$, then there exists $\varphi, \psi \in C^1(\Omega)$ such that

$$\varphi(x) = \psi(x) = f(x) \text{ and } \begin{cases} \nabla\varphi(x) = p & f - \varphi \text{ has local maximum at } x \\ \nabla\psi(x) = q & f - \psi \text{ has local minimum at } x \end{cases}.$$

Therefore, in neighborhood $B(x, \delta)$ we have

$$\psi(y) \leq f(y) \leq \varphi(y) \quad \forall y \text{ s.t. } \|y - x\| < \delta.$$

Since $\psi, \varphi \in C^1(\Omega)$, it's easy to see that f is also differentiable at x , and so we also have the formula $D^+f(x) = D^-f(x) = \{\nabla f(x)\}$.

- (f) Let $x_0 \in \Omega$ and $\varepsilon > 0$ be given. We will show that there exists a function $\varphi \in C^1(\Omega)$ such that $f - \varphi$ has local maximum in $B(x_0, \varepsilon)$ at some point y in $B(x_0, \varepsilon)$. Consider the smooth function in $C^1(B(x_0, \varepsilon))$ given by

$$\varphi(x) = \frac{1}{\varepsilon^2 - \|x - x_0\|^2} \quad x \in B(x_0, \varepsilon).$$

It's easy to extend φ into a function in $C^1(\Omega)$. Also observe that

$$\varphi(x) \longrightarrow +\infty \quad \text{as} \quad \|x - x_0\| \longrightarrow \varepsilon.$$

Since f is continuous, we have $f - \varphi$ has a local maximum in $B(x_0, \varepsilon)$ at some point y . By (c), $p = \nabla\varphi(y) \in D^+f(x)$ and thus $D^+f(x) \neq \emptyset$, i.e $y \in \Omega^+$. Furthermore, for every $x_0 \in \Omega$ and $\varepsilon > 0$ so small enough, the set Ω^+ contains a point $y \in B(x_0, \varepsilon)$. This shows that Ω^+ is dense in Ω .

Similarly, if we consider the $C^1(B(x_0, \varepsilon))$ function given by

$$\varphi(x) = \frac{-1}{\varepsilon^2 - \|x - x_0\|^2} \quad x \in B(x_0, \varepsilon).$$

The case Ω^- is dense in Ω by a similar argument.

□

From this, we have the equivalent definition for viscosity solution of static equation as following

Definition 3 (Equivalent definition of viscosity solution for the static equation). Assume u is continuous on its domain

- (Subsolution) u is called a viscosity subsolution if

$$\forall x_0 \in \mathbb{R}^n \quad u(x_0) + H(x_0, p) \leq 0 \quad \forall p \in D^+u(x_0).$$

- (Supersolution) u is called a viscosity supersolution if

$$\forall x_0 \in \mathbb{R}^n \quad u(x_0) + H(x_0, p) \geq 0 \quad \forall p \in D^-u(x_0).$$

- u is called a viscosity solution if it is both a subsolution and a supersolution.

Thus, in general we have the following definition for viscosity solution:

Definition 4 (General definition for viscosity solutions). Consider the first order partial differential equation

$$(3) \quad F(x, u(x), \nabla u(x)) = 0$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function from an open set $\Omega \subset \mathbb{R}^n$. A function $u \in C(\Omega)$ is a **viscosity subsolution** of (3) if

$$\text{for every } x \in \Omega \quad F(x, u(x), p) \leq 0 \quad \forall p \in D^+u(x).$$

Similarly, $u \in C(\Omega)$ is **viscosity supersolution** of (3) if

$$\text{for every } x \in \Omega \quad F(x, u(x), p) \geq 0 \quad \forall p \in D^-u(x).$$

We say u is a viscosity solution of (3) if it is both a supersolution and a subsolution in the viscosity sense.

Remark 2. The basic questions about well-posedness of PDE in sense of viscosity solution, the existence, uniqueness and stability were solved by Evans, Crandall-Lions in 1970 for the equation

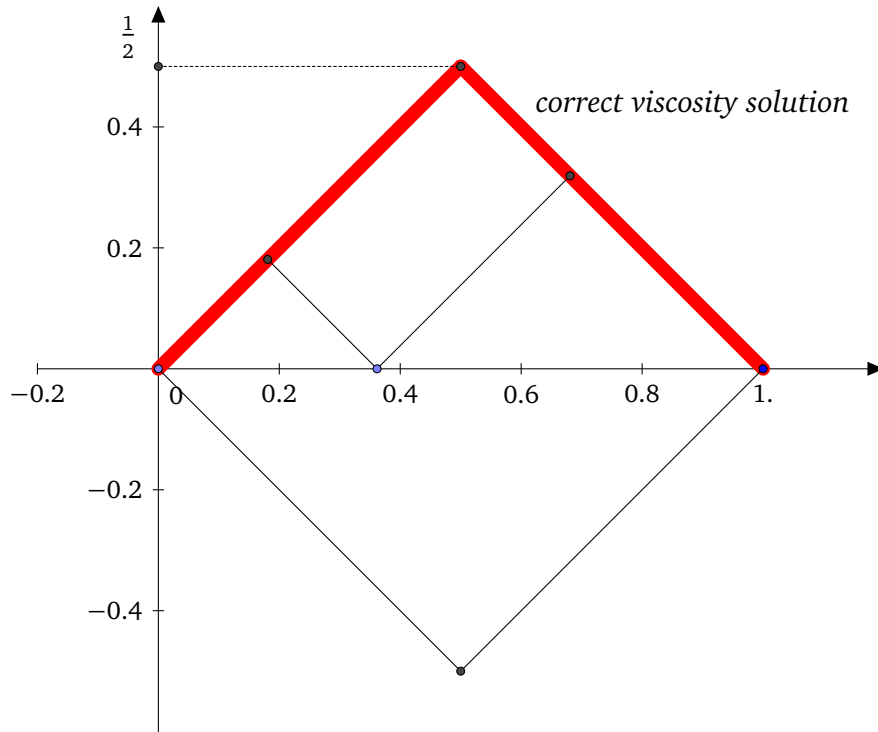
$$H(y, Du^\varepsilon(y)) = \varepsilon \Delta u^\varepsilon(y).$$

by the method adding small viscosity.

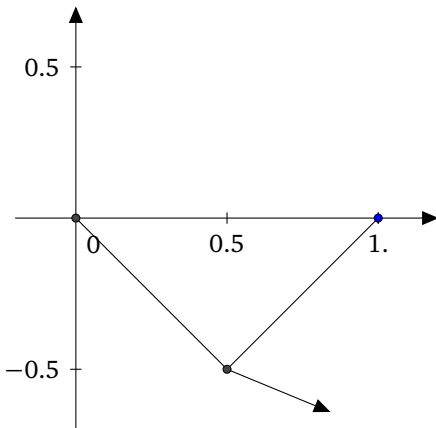
Example 2 (Eikonal's equation). Let H denote the Hamiltonian, $H(u, p) = |p|$ in \mathbb{R} , consider the Eikonal's equations

$$\begin{cases} H(y, Du(y)) = 1 & \text{in } (0, 1) \\ u(0) = u(1) = 0 \end{cases} \implies \begin{cases} |u'(y)| = 1 & \text{in } (0, 1) \\ u(0) = u(1) = 0 \end{cases}.$$

We can see that there are infinite many solutions (in weak sense) of this problem, but there is only one correct viscosity solution.



Finding a viscosity solution. We already know this equation has infinitely many solutions. Consider a solution u that its graph contains a ∇ shape, for example



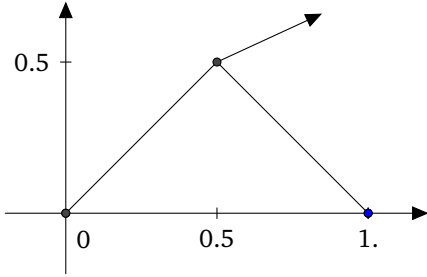
$$D^+u\left(\frac{1}{2}\right) = \emptyset \quad ; \quad D^-u\left(\frac{1}{2}\right) = [-1, 1].$$

Thus, if u is a viscosity supersolution then

$$\forall p \in D^-u\left(\frac{1}{2}\right) : |p| - 1 \geq 0.$$

It's contradiction.

The only u which is not contain this shape is



$$D^+u\left(\frac{1}{2}\right) = [-1, 1] \quad ; \quad D^-u\left(\frac{1}{2}\right) = \emptyset.$$

Thus, if u is a viscosity subsolution then

$$\forall p \in D^+u\left(\frac{1}{2}\right) : |p| - 1 \leq 0.$$

Thus, this is the only viscosity solution to the problem. □

Remark 3. In general, Hamilton-Jacobi equations have infinitely a.e solutions. Viscosity solutions give 2-further additional requirement to select one good solution in form these infinitely many.

Now we are ready to prove theorem 1. Recall

The ergodic (cell) problem. Fix $p \in \mathbb{R}^n$, there exists a unique $c \in \mathbb{R}$ such that

$$(E_p) \quad H(y, p + Dv(y)) = c \quad \text{in } \mathbb{R}^n.$$

has a periodic solution v .

Proof of theorem 1. We prove through some steps.

Discounted approximation. For $\varepsilon > 0$, consider the following problem

$$\varepsilon v^\varepsilon(y) + H(y, p + Dv^\varepsilon(y)) = 0 \text{ in } \mathbb{R}^n$$

Claim without proof: There exists a unique viscosity solution $v^\varepsilon(y)$ to the above problem.

1st observation: if $v^\varepsilon(y)$ is a solution then $v^\varepsilon(y + k)$ for $k \in \mathbb{Z}^n$ is also a solution by periodicity of H . Thus $v^\varepsilon(y)$ is periodic, i.e $v^\varepsilon(y) = v^\varepsilon(y + k)$ by the uniqueness of the solution v^ε .

2nd observation: there are apriori estimates. We try to bound v^ε . First of all, take $\phi = c_1$, and observe $H(y, p)$ is periodic, this is bounded, assume $|H(y, p)| \leq C$. If we choose $\phi = c_1 = \frac{-C}{\varepsilon}$ then

$$\varepsilon \phi + H(y, p + D\phi(y)) = \varepsilon c_1 + H(y, p) \leq \varepsilon c_1 + C = 0.$$

If we choose $\phi = c_1 = \frac{C}{\varepsilon}$ then

$$\varepsilon \phi + H(y, p + D\phi(y)) = \varepsilon c_1 + H(y, p) \geq \varepsilon c_1 - C = 0.$$

Thus we obtain

$$\begin{aligned} \phi = -\frac{C}{\varepsilon} \quad & \text{for } C > 0 \text{ large is a subsolution,} \\ \phi = \frac{C}{\varepsilon} \quad & \text{for } C > 0 \text{ large is a supersolution.} \end{aligned}$$

Therefore since v^ε is a solution, we get

$$-\frac{C}{\varepsilon} \leq v^\varepsilon \leq \frac{C}{\varepsilon} \quad \implies \quad |\varepsilon v^\varepsilon| \leq C,$$

which implies

$$|H(y, p + Dv^\varepsilon(y))| \leq C \quad \forall y.$$

Now by coercivity of H , we must have $|Dv^\varepsilon(y)| \leq C'$ for some constant $C' > 0$.

3rd observation: $\{v^\varepsilon\}$ is equi-Lipschitz and v^ε is periodic. Since

$$|v^\varepsilon(x) - v^\varepsilon(y)| \leq |Dv^\varepsilon(\zeta)| \cdot |x - y| \leq C'|x - y|$$

for some $\zeta \in [x, y] = \{tx + (1-t)y : t \in [0, 1]\}$ and clearly C' is independent to ε . In other words, for any $\varepsilon > 0$

$$v^\varepsilon \in \text{Lip}(\mathbb{T}^n) \quad \text{and} \quad \|Dv^\varepsilon\|_{L^\infty(\mathbb{T}^n)} \leq C'.$$

Remark 4. *At this point we want to use Arzela-Ascoli theorem. However, v^ε aren't bounded as $\varepsilon \rightarrow 0$ so we can't get the uniform bound for every ε thus Arzela-Ascoli cannot be applied here. Luckily, we have something lurking underneath so Arzela-Ascoli theorem could be applied.*

Define

$$w^\varepsilon(y) = v^\varepsilon(y) - v^\varepsilon(0).$$

We have $|Dw^\varepsilon(y)| = |Dv^\varepsilon(y)| \leq C'$ and so

$$|w^\varepsilon(y)| = |v^\varepsilon(y) - v^\varepsilon(0)| \leq C'|y| \leq C'\sqrt{n}$$

since \mathbb{T}^n is bounded. Thus w^ε is bounded and we can apply the Arzela-Ascoli theorem to $\{w^\varepsilon\}$, there exists $\varepsilon_j \rightarrow 0$ such that as $j \rightarrow \infty$,

$$(4) \quad w^{\varepsilon_j}(y) \rightarrow w(y) \quad \text{uniformly in } \mathbb{T}^n.$$

Since $|\varepsilon v^\varepsilon(0)| \leq C$, by Bozalno-Weitrass's principle, we can assume (passing to subsequence if necessary) that $\varepsilon_j v^{\varepsilon_j}(0) \rightarrow -c$ for some constant $c \in \mathbb{R}$. We claim furthermore that

$$\varepsilon_j v^{\varepsilon_j}(y) \rightarrow -c \quad \text{as} \quad j \rightarrow \infty.$$

Indeed, we have

$$\varepsilon_j |v^{\varepsilon_j}(y) - v^{\varepsilon_j}(0)| \leq \varepsilon_j \|Dv^{\varepsilon_j}\|_{L^\infty(\mathbb{T}^n)} \leq C' \varepsilon_j \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

Now look at

$$\varepsilon v^\varepsilon(y) + H(y, p + Dv^\varepsilon(y)) = 0 \quad \text{in } \mathbb{R}^n.$$

It becomes

$$(5) \quad \varepsilon_j w^{\varepsilon_j}(y) + \varepsilon_j v^{\varepsilon_j}(0) + H(y, p + Dw^{\varepsilon_j}(y)) = 0.$$

4th observation: Now we claim that

$$(6) \quad H(y, p + Dw(y)) = c$$

in the viscosity sense. (We don't have $Dw^{\varepsilon_j} \rightarrow Dw$.) Indeed,

- **Sub-test:** Fix $x \in \mathbb{R}^n$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} (w - \varphi)(x) = 0 \\ w - \varphi < 0 \text{ else where} \end{cases} \quad \text{i.e, } w - \varphi \text{ has a strict max at } x.$$

Since $w^{\varepsilon_j} \rightarrow w$ uniformly on some compact neighborhood of x , there exists a subsequence $\{j_k\}$ such that

$$\begin{cases} w^{\varepsilon_{j_k}} - \varphi \text{ has a max at } x_{\varepsilon_{j_k}} \text{ near } x \\ x_{\varepsilon_{j_k}} \rightarrow x \text{ as } k \rightarrow \infty \\ w^{\varepsilon_{j_k}}(x_{\varepsilon_{j_k}}) \rightarrow w(x) \text{ as } k \rightarrow \infty \end{cases}.$$

Look at (5), in viscosity sense, we must have

$$\varepsilon_{j_k} w^{\varepsilon_{j_k}}(x_{\varepsilon_{j_k}}) + \varepsilon_{j_k} v^{\varepsilon_{j_k}}(0) + H(x_{\varepsilon_{j_k}}, p + D\varphi^{\varepsilon_{j_k}}(x_{\varepsilon_{j_k}})) \leq 0.$$

Now let $k \rightarrow \infty$ we obtain

$$-c + H(x, p + D\varphi(x)) \leq 0 \iff H(x, p + D\varphi(x)) \leq c.$$

Thus, w is a viscosity subsolution of (6).

- **Super-test:** Fix $x \in \mathbb{R}^n$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} (w - \varphi)(x) = 0 \\ w - \varphi > 0 \text{ else where} \end{cases} \quad \text{i.e, } w - \varphi \text{ has a strict min at } x.$$

Since $w^{\varepsilon_j} \rightarrow w$ uniformly on some compact neighborhood of x , there exists a subsequence $\{j_k\}$ such that

$$\begin{cases} w^{\varepsilon_{j_k}} - \varphi \text{ has a min at } x_{\varepsilon_{j_k}} \text{ near } x \\ x_{\varepsilon_{j_k}} \rightarrow x \text{ as } k \rightarrow \infty \\ w^{\varepsilon_{j_k}}(x_{\varepsilon_{j_k}}) \rightarrow w(x) \text{ as } k \rightarrow \infty \end{cases} .$$

Look at (5), in viscosity sense, we must have

$$\varepsilon_{j_k} w^{\varepsilon_{j_k}}(x_{\varepsilon_{j_k}}) + \varepsilon_{j_k} v^{\varepsilon_{j_k}}(0) + H(x_{\varepsilon_{j_k}}, p + D\varphi^{\varepsilon_{j_k}}(x_{\varepsilon_{j_k}})) \geq 0.$$

Now let $k \rightarrow \infty$ we obtain

$$-c + H(x, p + D\varphi(x)) \geq 0 \iff H(x, p + D\varphi(x)) \geq c.$$

Thus, w is a viscosity supersolution of (6).

Thus, w is a viscosity solution of (6). It's clearly periodic.

The uniqueness of c : Assume that there exists (v_1, c_1) and (v_2, c_2) where $c_1 < c_2$

$$H(y, p + Dv_1(y)) = c_1 < c_2 = H(y, p + Dv_2(y)) \quad \text{in } \mathbb{R}^n.$$

Since v_1 and v_2 are periodic, these are bounded, and thus we can take $\varepsilon > 0$ small enough so that

$$\varepsilon v_1(y) + H(y, p + Dv_1(y)) \leq \frac{c_1 + c_2}{2} \leq \varepsilon v_2(y) + H(y, p + Dv_2(y)).$$

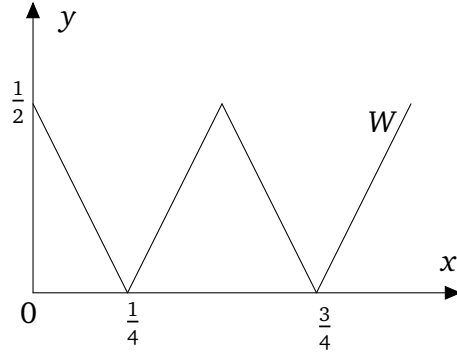
This means that v_1 and v_2 are sub- and super-solutions, respectively, to the problem

$$\varepsilon v(y) + H(y, p + Dv(y)) = \frac{c_1 + c_2}{2}.$$

Therefore, by comparison principle, $v_1 \leq v_2$. However, we have that both v_1 and v_2 are bounded; and if v_1 is a subsolution then so is $v_1 + C$. So, for large enough C we have $v_1 + C > v_2$, which is a contradiction. \square

Remark 5. *One should note that although c is unique, the solution to cell problem is not unique in general. One of the homework is an example this claim.*

Exercise 2. *Let $n = 1$, $H(x, p) = |p| - W(x)$ where W is defined by its graph*



That is $W : \mathbb{R} \rightarrow \mathbb{R}$ is periodic. Consider the ergodic cell problem (E_0)

$$(E_0) \quad H(y, 0 + Dv(y)) = |Dv(y)| - W(y) = c \quad \text{in } \mathbb{T}.$$

- (a) Find the unique constant c such that (E_0) has solution.
- (b) Find an infinite number of solutions for (E_0) for all $y \in \mathbb{T}^n$.

LECTURE 3

In this lecture, our task is to prove the following homogenization theorem, answering the question in lecture 1.

Theorem 2 (Lions-Papanicolau-Varadhan, Evans). *Assume u_0 is bounded, uniformly continuous on \mathbb{R}^n and for any $\varepsilon > 0$, u^ε solves the 1st P.D.E*

$$(C_\varepsilon) \quad \begin{cases} u_t^\varepsilon(x, t) + H\left(\frac{x}{\varepsilon}, Du^\varepsilon(x, t)\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u^\varepsilon(x, 0) = u_0(x) & \text{on } \mathbb{R}^n \end{cases}.$$

Then, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly⁴ to u in $\mathbb{R}^n \times [0, \infty)$, and u solves

$$(C) \quad \begin{cases} u_t(x, t) + \bar{H}(Du(x, t)) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^n \end{cases}.$$

The idea of the proof of this theorem comes from Evans's perturbed test function method. The following is a heuristic proof.

Lemma 1. *There is a priori estimate*

$$|u^\varepsilon| + |Du^\varepsilon| \leq C.$$

Proof. Notes from last year summer school. □

Proof of theorem 2. Using above priori estimate, we deduce by Arzela-Ascoli theorem there exists a sequence $\{\varepsilon_j\} \searrow 0$ such that $u^{\varepsilon_j} \rightarrow u$ locally uniformly. We are left to show u solves (C).

Sub-test. One might be tempted to take the following approach.

Failed attempt. Let $\phi \in C^\infty$ and $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ is a strict maximum of $u - \phi$. We want to show that

$$\phi_t(x_0, t_0) + \bar{H}(D\phi(x_0, t_0)) \leq 0.$$

By exercise 1 we have $u^{\varepsilon_j} - \phi$ has a max at (x_j, t_j) and $(x_j, t_j) \rightarrow (x_0, t_0)$ as $j \rightarrow \infty$. Since u^{ε_j} is a viscosity solution of (C_{ε_j}) , we obtain

$$\phi_t(x_j, t_j) + H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_j, t_j)\right) \leq 0.$$

As $j \rightarrow \infty$, we have $\phi_t(x_j, t_j) \rightarrow \phi_t(x_0, t_0)$. However, the catch is that we're not sure about the term

⁴That means the convergence is uniformly on any compact subset of $\mathbb{R}^n \times [0, \infty)$

$$H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_j, t_j)\right)$$

converges to

$$\overline{H}(D\phi(x_0, t_0))$$

in any sense. The following method, called the perturbed test function method was introduced by Evans in 1988 - 1989 deal with this problem.

Heuristic proof - Assume everything is smooth. Let $p_0 = D\phi(x_0, t_0) \in \mathbb{R}^n$. Take v_0 to be a (Lipschitz, periodic) solution of

$$(7) \quad H(y, p_0 + Dv(y)) = \overline{H}(p_0) = \overline{H}(D\phi(x_0, t_0)) \quad \forall y \in \mathbb{R}^n.$$

A very important observation is that v_0 is bounded by its periodicity and Dv_0 is bounded the coercivity assumption of H . From this, we also have the function

$$(x, t) \longmapsto u^\varepsilon(x, t) - \varepsilon v_0\left(\frac{x}{\varepsilon}\right)$$

converges uniformly to u as $\varepsilon \rightarrow 0$. Thus by exercise 1 we obtain there exists a sequence $\{\varepsilon_j\} \searrow 0$ such that

$$(8) \quad u^{\varepsilon_j}(x, t) - \left[\phi(x, t) + \varepsilon_j v_0\left(\frac{x}{\varepsilon_j}\right) \right] \quad \text{has a max at } (x_j, t_j) \text{ near } (x_0, t_0)$$

and $(x_j, t_j) \rightarrow (x_0, t_0)$ as $j \rightarrow \infty$. The function u^{ε_j} is called the perturbed test function. So, by the definition of viscosity subsolutions, we have⁵

$$\phi_t(x_j, t_j) + H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_j, t_j) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) \leq 0.$$

Also observe that

$$\begin{aligned} \phi_t(x_j, t_j) &\longrightarrow \phi_t(x_0, t_0), \\ D\phi(x_j, t_j) &\longrightarrow D\phi(x_0, t_0). \end{aligned}$$

Furthermore,⁶

$$\begin{aligned} &\left| H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_j, t_j) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) - H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_0, t_0) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) \right| \\ &\leq C |D\phi(x_j, t_j) - D\phi(x_0, t_0)| \longrightarrow 0 \end{aligned}$$

⁵Here is the point where the proof goes wrong as v_0 may not be differentiable at $\frac{x_j}{\varepsilon_j}$

⁶Also, (I think we need to) assume that H is Lipschitz in the p -coordinate to apply here.

as $j \rightarrow \infty$. So, combine the 2 facts we obtain

$$\begin{aligned} \phi_t(x_j, t_j) + H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_j, t_j) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) &= \phi_t(x_j, t_j) + H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_0, t_0) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) \\ &+ \left[H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_j, t_j) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) - H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_0, t_0) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) \right]. \end{aligned}$$

From (7) we have

$$H\left(\frac{x_j}{\varepsilon_j}, D\phi(x_0, t_0) + Dv_0\left(\frac{x_j}{\varepsilon_j}\right)\right) = \bar{H}(D\phi(x_0, t_0)) = \bar{H}(p_0).$$

Thus let $j \rightarrow 0$ in the former formula we thus obtain

$$\phi_t(x_0, t_0) + \bar{H}(p_0) \leq 0.$$

So, u is a viscosity sub-solution of (C). \square

Based on the idea above plus another technique called doubling of variables, we have a real proof.

Rigorous proof. Recall $u^{\varepsilon_j} \rightarrow u$ locally uniformly on $\mathbb{R}^n \times (0, \infty)$. We perform the subsolution test, the supersolution case is similar and left as an exercise. Suppose $u - \phi$ has a strict max at (x_0, t_0) . We want to show that

$$\phi_t(x_0, t_0) + H(D\phi(x_0, t_0)) \leq 0.$$

Denote $p_0 = D\phi(x_0, t_0)$, consider an open neighborhood $\Omega \times (0, T) \subset \mathbb{R}^n \times (0, \infty)$ with compact closure which contains (x_0, t_0) such that

$$u(x, t) - \phi(x, t) < u(x_0, t_0) - \phi(x_0, t_0) \quad \forall (x, t) \in \bar{\Omega} \times [0, T].$$

We have $u^{\varepsilon_j} \rightarrow u$ uniformly on $\bar{\Omega}$. Let v_0 be the periodic Lipschitz viscosity solution of the cell problem

$$H(y, p_0 + Dv_0(y)) = \bar{H}(p_0).$$

Then v_0 is bounded by its periodicity, now let $\eta > 0$ small consider

$$\Phi^{\varepsilon_j, \eta} : \bar{\Omega} \times \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$$

$$(x, y, t) \quad \longmapsto u^{\varepsilon_j}(x, t) - \phi(x, t) - \varepsilon_j v_0(y) - \frac{\left| \frac{x}{\varepsilon_j} - y \right|^2}{\eta}.$$

Since v_0 is bounded, the function $\Phi^{\varepsilon_j, \eta}$ is continuous and bounded above.

Step 0. There exists $(x_{j\eta}, y_{j\eta}, t_{j\eta}) \in \bar{\Omega} \times \mathbb{R}^n \times [0, T]$ such that $\Phi^{\varepsilon_j, \eta}$ attains its maximum there. Indeed, let

$$\alpha_{\varepsilon_j, \eta} = \sup_{\bar{\Omega} \times \mathbb{R}^n \times [0, T]} \Phi^{\varepsilon_j, \eta}(x, y, t).$$

Then there exists some sequence $(x_m, y_m, t_m) \in \bar{\Omega} \times \mathbb{R}^n \times [0, T]$ such that

$$\Phi^{\varepsilon_j, \eta}(x_m, y_m, t_m) \longrightarrow \alpha_{\varepsilon_j, \eta}.$$

Since $\bar{\Omega} \times [0, T]$ is compact, we can assume there exists a sub-sequence $\{m_k\}$ such that

$$(x_{m_k}, t_{m_k}) \longrightarrow (x', t') \in \bar{\Omega} \times [0, T] \quad \text{as} \quad k \longrightarrow \infty.$$

We claim that $\{y_{m_k}\}$ is bounded. Assume the contrary, then we can assume further that (passing a sub-sequence) that $|y_{m_k}| > 2^k$ for all $k \in \mathbb{N}$. We have

$$\begin{aligned} \Phi^{\varepsilon_j, \eta}(x_{m_k}, y_{m_k}, t_{m_k}) &= u^{\varepsilon_j, \eta}(x_{m_k}, t_{m_k}) - \phi(x_{m_k}, t_{m_k}) - \varepsilon_j v_0(y_{m_k}) - \frac{\left| \frac{x_{m_k}}{\varepsilon_j} - y_{m_k} \right|^2}{\eta} \\ &\leq u^{\varepsilon_j, \eta}(x_{m_k}, t_{m_k}) - \phi(x_{m_k}, t_{m_k}) - \varepsilon_j v_0(y_{m_k}) - \frac{\left(|y_{m_k}| - \left| \frac{x_{m_k}}{\varepsilon_j} \right| \right)^2}{\eta} \\ &\leq u^{\varepsilon_j, \eta}(x_{m_k}, t_{m_k}) - \phi(x_{m_k}, t_{m_k}) - \varepsilon_j v_0(y_{m_k}) - \frac{\left(2^k - \left| \frac{x_{m_k}}{\varepsilon_j} \right| \right)^2}{\eta}. \end{aligned}$$

Let $k \longrightarrow \infty$, we obtain

$$\alpha_{\varepsilon_j, \eta} \leq -\infty.$$

It's contradiction, so $\{y_{m_k}\}$ must be bounded. Thus, we can also assume that (passing a sub-sequence)

$$y_{m_k} \longrightarrow y' \quad \text{for some } y' \in \mathbb{R}^n.$$

In conclusion, $\Phi^{\varepsilon_j, \eta}$ attains its maximum at $(x', y', t') \in \bar{\Omega} \times \mathbb{R}^n \times [0, T]$. We reindex this point into $(x_{j\eta}, y_{j\eta}, t_{j\eta}) \in \bar{\Omega} \times \mathbb{R}^n \times [0, T]$.

Note, also, that $y_{j\eta} \approx \frac{x_{j\eta}}{\varepsilon_j}$ because of the penalty effect as η gets small.

Step 1. Since $\bar{\Omega} \times [0, T]$ is compact, by passing to sub-sequence, we can assume that there exists some point $(x_j, t_j) \in \bar{\Omega} \times [0, T]$ such that

$$(x_{j\eta}, t_{j\eta}) \longrightarrow (x_j, t_j) \quad \text{as} \quad \eta \longrightarrow 0.$$

We claim that as $\eta \longrightarrow 0$, then

$$(9) \quad (x_{j\eta}, y_{j\eta}, t_{j\eta}) \longrightarrow (x_j, y_j, t_j) \quad \text{and} \quad y_j = \frac{x_j}{\varepsilon_j}.$$

To see that, we observe that

$$\Phi^{\varepsilon_{j\eta}}(x_{j\eta}, y_{j\eta}, t_{j\eta}) \geq \Phi^{\varepsilon_{j\eta}}(x_{j\eta}, y, t_{j\eta}) \quad \forall y \in \mathbb{R}^n.$$

This is equivalent to

$$\varepsilon_j v_0(y) + \frac{\left| \frac{x_{j\eta}}{\varepsilon_j} - y \right|}{\eta} \geq \varepsilon_j v_0(y_{j\eta}) + \frac{\left| \frac{x_{j\eta}}{\varepsilon_j} - y_{j\eta} \right|}{\eta}.$$

Thus if we choose $y = \frac{x_{j\eta}}{\varepsilon_j}$, then it becomes

$$\varepsilon_j \left[v_0\left(\frac{y_{j\eta}}{\varepsilon_j}\right) - v_0(y_{j\eta}) \right] \geq \frac{\left| \frac{x_{j\eta}}{\varepsilon_j} - y_{j\eta} \right|}{\eta}.$$

Since v_0 is bounded, the LHS of the above inequality is bounded. Thus, at the limit as $\eta \rightarrow 0$, we must have $\frac{x_j}{\varepsilon_j} = y_j$.⁷ Thus (9) is true.

Step 2. Also, for any $(x, t) \in \bar{\Omega} \times [0, T]$ we have

$$\Phi^{\varepsilon_{j\eta}}(x_{j\eta}, y_{j\eta}, t_{j\eta}) \geq \Phi^{\varepsilon_{j\eta}}\left(x, \frac{x}{\varepsilon_j}, t\right) = u^{\varepsilon_j}(x, t) - \phi(x, t) - \varepsilon_j v_0\left(\frac{x}{\varepsilon_j}\right).$$

Since v_0 is bounded, let $\eta \rightarrow 0$ and using the fact that $(x_{j\eta}, y_{j\eta}, t_{j\eta}) \rightarrow (x_j, \frac{x_j}{\varepsilon_j}, t_j)$, we obtain

$$u^{\varepsilon_j}(x_j, t_j) - \phi(x_j, t_j) - \varepsilon_j v_0\left(\frac{x_j}{\varepsilon_j}\right) \geq u^{\varepsilon_j}(x, t) - \phi(x, t) - \varepsilon_j v_0\left(\frac{x}{\varepsilon_j}\right)$$

for all $(x, t) \in \bar{\Omega} \times [0, T]$. Thus the function

$$\psi^{\varepsilon_j}(x, t) = u^{\varepsilon_j}(x, t) - \varepsilon_j v_0\left(\frac{x}{\varepsilon_j}\right)$$

satisfies $\psi^{\varepsilon_j} - \phi$ has a maximum over $\bar{\Omega} \times [0, T]$ at (x_j, t_j) . Also, since v_0 is bounded, $\psi^{\varepsilon_j} \rightarrow u$ uniformly on $\bar{\Omega} \times [0, T]$. Thus by exercise 1 we must have (maybe some subsequence) $(x_j, t_j) \rightarrow (x_0, t_0)$ as $j \rightarrow \infty$ since $u - \phi$ has strict maximum over $\bar{\Omega} \times [0, T]$ at (x_0, t_0) .

⁷Note that

$$\Phi\left(x_0, \frac{x_0}{\varepsilon_j}, t_0\right) = u^{\varepsilon_j}(x_0, t_0) - \phi(x_0, t_0) - \varepsilon_j v_0(y_0) \geq -c$$

for some constant c , but I honestly don't see the usefulness of this fact.

Step 3. Fix $y = y_{j_\eta}$ then the function

$$(x, t) \mapsto \Phi^{\varepsilon_{j_\eta}}(x, y_{j_\eta}, t) \quad \text{has a max at} \quad (x_{j_\eta}, t_{j_\eta}).$$

So,

$$(x, t) \mapsto u^{\varepsilon_j}(x, t) - \phi(x, t) - \frac{\left| \frac{x}{\varepsilon_j} - y_{j_\eta} \right|^2}{\eta} \quad \text{has a max at} \quad (x_{j_\eta}, t_{j_\eta}).$$

Observe that the mapping

$$(x, t) \mapsto \phi(x, t) + \frac{\left| \frac{x}{\varepsilon_j} - y_{j_\eta} \right|^2}{\eta}$$

is smooth, thus by definition of viscosity solution for the equation $u_t^{\varepsilon_j} + H\left(\frac{x}{\varepsilon_j}, Du^{\varepsilon_j}\right) = 0$, we have

$$(10) \quad \phi_t(x_{j_\eta}, t_{j_\eta}) + H\left(\frac{x_{j_\eta}}{\varepsilon_j}, D\phi(x_{j_\eta}, t_{j_\eta}) + \frac{\frac{2}{\varepsilon_j} \left(\frac{x_{j_\eta}}{\varepsilon_j} - y_{j_\eta}\right)}{\eta}\right) \leq 0.$$

Step 4. Observe also that as $\eta \rightarrow 0$, the following term is bounded, that is there exists $C > 0$ such that

$$(11) \quad \frac{\frac{2}{\varepsilon_j} \left(\frac{x_{j_\eta}}{\varepsilon_j} - y_{j_\eta}\right)}{\eta} < C.$$

Assume the converse, this term is tend to $+\infty$ as $\eta \rightarrow 0$, then from (10) and by the coercivity of H we obtain a contradiction since $(x_{j_\eta}, y_{j_\eta}, t_{j_\eta}) \rightarrow (x_j, \frac{x_j}{\varepsilon_j}, t_j)$. Thus (11) is true, therefore we can assume (maybe some subsequence) that there exists some $q \in \mathbb{R}^n$ such that

$$(12) \quad \lim_{\eta \rightarrow \infty} \frac{\frac{2}{\varepsilon_j} \left(\frac{x_{j_\eta}}{\varepsilon_j} - y_{j_\eta}\right)}{\eta} = q.$$

Using (12) in (10), let $\eta \rightarrow 0$ we obtain

$$(13) \quad \phi_t(x_j, t_j) + H\left(y_j, D\phi(x_j, t_j) + q\right) \leq 0.$$

Step 5. Now, fix (x_{j_η}, t_{j_η}) , we have the map

$$y \longmapsto \Phi(x_{j_\eta}, y, t_{j_\eta}) \quad \text{has a max at} \quad y_{j_\eta}.$$

So,

$$y \longmapsto -\varepsilon_j v_0(y) - \frac{\left| \frac{x}{\varepsilon_j} - y \right|^2}{\eta} \quad \text{has a max at} \quad y_{j_\eta}.$$

Thus

$$y \longmapsto v_0(y) - \left[\frac{-\frac{1}{\varepsilon_j} \left| \frac{x_{j_\eta}}{\varepsilon_j} - y \right|^2}{\eta} \right] \quad \text{has a min at} \quad y_{j_\eta}.$$

Therefore, by mean of viscosity solution of $H(y, p_0 + Dv_0(y)) = \bar{H}(p_0)$ we have

$$H \left(y_{j_\eta}, p_0 + D \left(\frac{-\frac{1}{\varepsilon_j} \left| \frac{x_{j_\eta}}{\varepsilon_j} - y \right|^2}{\eta} \right) \Big|_{y=y_{j_\eta}} \right) \geq \bar{H}(p_0),$$

which is the same as

$$H \left(y_{j_\eta}, p_0 + \frac{\frac{2}{\varepsilon_j} \left(\frac{x_{j_\eta}}{\varepsilon_j} - y_{j_\eta} \right)}{\eta} \right) \geq \bar{H}(p_0).$$

Pass $\eta \longrightarrow 0$ and using (12), we obtain from the continuity of H that

$$(14) \quad H(y_j, p_0 + q) \geq \bar{H}(p_0).$$

Using (14) we obtain

$$(15) \quad \phi_t(x_j, t_j) + \bar{H}(p_0) \leq \phi_t(x_j, t_j) + H(y_j, p_0 + q).$$

Observe that from (13) and the continuity of H we have⁸

$$\begin{aligned} \phi_t(x_j, t_j) + H(y_j, D\phi(x_j, t_j) + q) &\leq 0 \quad \forall j \in \mathbb{N}, \\ H(y_j, p_0 + q) - H(y_j, D\phi(x_j, t_j) + q) &\longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \end{aligned}$$

Using this fact in (15) and letting $j \longrightarrow \infty$, we obtain

$$\phi_t(x_0, t_0) + \bar{H}(p_0) \leq 0.$$

Therefore, u is a viscosity sub-solution of (C). □

⁸Note that since $y_j = \frac{x_j}{\varepsilon_j}$, so that as $j \longrightarrow \infty$ and $x_j \longrightarrow x_0$, we have $\|y_j\| \longrightarrow \infty$. And thus I think we need H is Lipschitz in p -coordinate to apply here.

Exercise 3. *Give a formal proof for case super-solution.*

Remark 6. *The deep part of this proof is that it passes the indifferenciability of v_0 to a test smooth function ϕ . This part is achieved by introducing the variable y .*

LECTURE 4

In this lecture, we study the properties of the effective Hamiltonian, \bar{H} . Recall that there exists a Lipschitz viscosity solution v^ε to the following equation

$$\varepsilon v^\varepsilon(y) + H(y, p + Dv^\varepsilon(y)) = 0 \text{ in } \mathbb{R}^n.$$

We also know that $\bar{H}(p)$ can be defined as the limit of $-\varepsilon_j v^{\varepsilon_j}(0)$ as $j \rightarrow \infty$ in view of theorem 1. We want to understand \bar{H} qualitatively, quantitatively and numerically.

Proposition 2. *Consider the following Cauchy problem*

$$\begin{cases} w_t(y, t) + H(y, p + Dw(y)) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ w(y, 0) = 0 & \text{on } \mathbb{R}^n \end{cases}.$$

Then

$$\lim_{t \rightarrow \infty} \frac{w(y, t)}{t} = -\bar{H}(p).$$

Proof. We try to bound the solution,

$$\text{subsolution} \leq w(y, t) \leq \text{supersolution}$$

Let v be the solution to the cell problem

$$H(y, p + Dv(y)) = \bar{H}(p).$$

Denote

$$\phi^+(y, t) = v(y) + c - \bar{H}(p)t \quad \text{and} \quad \phi^-(y, t) = v(y) - c - \bar{H}(p)t.$$

We have

$$\begin{aligned} \phi_t^+(y, t) + H(y, p + D\phi^+(y, t)) &= -\bar{H}(p) + H(y, p + Dv(y)) = 0, \\ \phi_t^-(y, t) + H(y, p + D\phi^-(y, t)) &= -\bar{H}(p) + H(y, p + Dv(y)) = 0. \end{aligned}$$

For $t = 0$, $\phi^+(y, 0) = v(y) + c \geq 0$ and $\phi^-(y, 0) \leq 0$ for c large enough. So, they are super-solution and sub-solutions. Thus

$$v(y) - c - \bar{H}(p)t = \phi^-(y, t) \leq w(y, t) \leq \phi^+(y, t) = v(y) + c - \bar{H}(p)t.$$

Therefore,

$$\frac{v(y) - c}{t} - \bar{H}(p) \leq \frac{w(y, t)}{t} \leq \frac{v(y) + c}{t} - \bar{H}(p).$$

Let $t \rightarrow \infty$, we obtain the final result. □

Remark 7. *Finding \bar{H} numerically is extremely hard.*

We also have some further more properties of the effective Hamiltonian

Exercise 4. Suppose $H \in \text{Lip}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies (H1)-(H2), i.e. coercivity and periodicity in y -coordinate. Then \bar{H} is Lipschitz and coercive.

We now switch gear to find some representations for \bar{H} . We already knew 2, one is from the definition, another one is right above as the negative of large time average of the solution of proposition 2. The third one is the following,

Proposition 3 (Representation formula of $\bar{H}(p)$).

$$\bar{H}(p) = \inf\{c \in \mathbb{R} : \exists v \in C(\mathbb{T}^n) : H(y, p + Dv) \leq c \text{ in } \mathbb{T}^n \text{ in viscous sense}\}.$$

Proof. There exists a viscosity solution $v \in \text{Lip}(\mathbb{T}^n)$ such that

$$H(y, p + Dv(y)) = \bar{H}(p) \quad \text{in } \mathbb{T}^n.$$

Therefore, $\bar{H}(p) \geq RHS$. We then want to show that $\bar{H}(p) \leq RHS$. By way of contradiction, assume that $RHS < \bar{H}(p)$. That implies there exists $c \leq \bar{H}(p)$ such that there exists $w \in C(\mathbb{T}^n)$ such that

$$H(y, p + Dw) \leq c < \bar{H}(p) = H(y, p + Dv) \quad \text{in } \mathbb{T}^n.$$

Since both v and w are bounded, we can take $\varepsilon > 0$ small such that

$$\varepsilon w + H(y, p + Dw) \leq \frac{c + \bar{H}(p)}{2} \leq \varepsilon v + H(y, p + Dv).$$

Therefore, $w \leq v$ in \mathbb{T}^n , a contradiction since we can always add a constant to make $w > v$. \square

In case $p \mapsto H(\cdot, p)$ is convex, we also have a further presentation of \bar{H} .

Proposition 4 (inf – sup formula). Assume $p \mapsto H(y, p)$ is convex. Then

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \sup_{y \in \mathbb{T}^n} H(y, p + D\phi(y)).$$

Proof. Denote the formula in the previous proposition to be A , i.e

$$A = \inf\{c \in \mathbb{R} : \exists v \in C(\mathbb{T}^n) : H(y, p + Dv) \leq c \text{ in } \mathbb{T}^n \text{ in viscous sense}\}$$

and

$$B = \left\{ \sup_{y \in \mathbb{T}^n} H(y, p + D\phi(y)) : \phi \in C^1(\mathbb{T}^n) \right\}.$$

- Take $\phi \in C^1(\mathbb{T}^n)$, let

$$C = \max_{y \in \mathbb{T}^n} H(y, p + D\phi(y)).$$

So, $H(y, p + D\phi(y)) \leq C$ in \mathbb{T}^n and C is admissible in the formula of A . Therefore, $B \subseteq A$ and thus $RHS = \inf B \geq \inf A = \bar{H}(p)$.

- We show that $RHS = \inf B \leq \inf A = \bar{H}(p) = H(y, p + Dv(y))$ in \mathbb{T}^n . Note that v is only Lipschitz so to make it smooth we convolute it. Take the standard mollifier

$$\phi \in C_c^\infty(\mathbb{R}^n) \quad \phi \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \phi(x) dx = 1.$$

For any $\varepsilon > 0$ let

$$\phi^\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon}\right) \quad \implies \quad \int_{\mathbb{R}^n} \phi^\varepsilon(x) dx = 1.$$

Recall that $\text{supp}(\phi^\varepsilon)(x - \cdot) \subseteq B(x, C\varepsilon)$. Now take

$$v^\varepsilon(x) = (v^\varepsilon \star v)(x) = \int_{\mathbb{R}^n} \phi^\varepsilon(x - y)v(y) dy.$$

Then we have (in viscous sense)

$$\begin{aligned} \bar{H}(p) &= H(y, p + Dv(y)) \\ &= \int_{\mathbb{R}^n} H(y, p + Dv(y)) \phi^\varepsilon(x - y) dy \\ &= \int_{B(x, C\varepsilon)} H(x, p + Dv(y)) \phi^\varepsilon(x - y) dy - C'\varepsilon. \end{aligned}$$

By Jensen's inequality for convex Hamiltonian H in p , we obtain

$$\begin{aligned} \bar{H}(p) &\geq H\left(x, \int_{B(x, C\varepsilon)} (p + Dv(y)) \phi^\varepsilon(x - y) dy\right) - C'\varepsilon \\ &= H(x, p + Dv^\varepsilon(y)) - C'\varepsilon \geq H(y, p + Dv^\varepsilon(y)) - 2C'\varepsilon. \end{aligned}$$

Thus there exists some constant $C' > 0$ such that

$$\bar{H}(p) + 2C'\varepsilon \geq H(y, p + Dv^\varepsilon(y)).$$

Since v^ε is smooth and $\max_{y \in \mathbb{T}^n} H(y, p + Dv^\varepsilon(y)) \in B$, we obtain

$$\bar{H}(p) + 2C'\varepsilon \geq \max_{y \in \mathbb{T}^n} H(y, p + Dv^\varepsilon(y)) \geq \inf B.$$

Pass $\varepsilon \rightarrow 0$, we get

$$\bar{H}(p) \geq \inf B = RHS.$$

□

Exercise 5. *If $p \mapsto H(x, p)$ is convex then $p \mapsto \bar{H}(p)$ is also convex.*

APPENDIX. SOLUTIONS TO PROBLEMS

Notes.

- Proof 1 will be of Son Tu and proof 2 will be of Son Van.
- The proof of exercise 3 is completely similar to case sub-solution in the lecture and so it will be omitted.

Exercise 1. Let $\phi \in C^1(\mathbb{R}^n)$ such that $(u - \phi)(x_0) = 0$ and $u - \phi$ has a strict max at x_0 . Assume u^ε converges to u locally uniformly on \mathbb{R}^n . Prove that for $\varepsilon > 0$ small enough, $u^\varepsilon - \phi$ has a max at x_ε near by x_0 and there is a subsequence $\varepsilon_j \rightarrow 0$ such that $x_{\varepsilon_j} \rightarrow x_0$.

Proof 1. In this exercise, we will understand the notion of maximum (or strict maximum) as local maximum (or strict maximum).

Assume $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict (local) maximum at x_0 , so there exists an open neighborhood of x_0 with compact closure Ω such that

$$u(y) - \phi(y) < u(x_0) - \phi(x_0) = 0 \quad \forall y \in \overline{\Omega} \setminus \{x_0\}.$$

Clearly u is continuous on $\overline{\Omega}$ since $u^\varepsilon \rightarrow u$ uniformly on $\overline{\Omega}$. Let $r > 0$ small such that $B'(x_0, 2r) \subset \Omega$. Clearly

$$u(y) - \phi(y) < 0 \quad \forall y \in S(x_0, r) = \{y \in \mathbb{R}^n : \|y - x_0\| = r\}.$$

Claim. There exists $\delta_r > 0$ such that for every $\varepsilon < \delta_r$, $u^\varepsilon - \phi$ only attains its maximum over $\overline{\Omega}$ at some points $x_\varepsilon \in B'(x, r)$.

Proof of claim. Assume the converse, then for δ arbitrarily small, there exists $\varepsilon < \delta$ such that $u^\varepsilon - \phi$ attains its maximum over $\overline{\Omega}$ outside $B'(x, r)$.

Take $\delta = \frac{1}{n}$, we can construct a sequence $\varepsilon_n \rightarrow 0$ such that $u^{\varepsilon_n} - \phi$ attains its maximum over $\overline{\Omega}$ at $x_{\varepsilon_n} \in \overline{\Omega} \setminus B'(x_0, r)$, i.e. $|x_{\varepsilon_n} - x_0| \geq r$ for all $n \in \mathbb{N}$ and

$$(16) \quad u^{\varepsilon_n}(x_{\varepsilon_n}) - \phi(x_{\varepsilon_n}) \geq u^{\varepsilon_n}(x) - \phi(x) \quad \forall x \in \overline{\Omega}.$$

Since $\overline{\Omega}$ is compact, there exists some point $x' \in \overline{\Omega}$ such that $x_{\varepsilon_n} \rightarrow x'$. Clearly we also have $x' \in \overline{\Omega} \setminus B'(x, r)$. Observe that since $u^{\varepsilon_n} \rightarrow u$ uniformly on $\overline{\Omega}$ as $n \rightarrow \infty$, we have

$$\left| u^{\varepsilon_n}(x_{\varepsilon_n}) - u(x') \right| \leq \sup_{x \in \overline{\Omega}} |u^{\varepsilon_n}(x) - u(x)| + \left| u(x_{\varepsilon_n}) - u(x') \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

So that if we let $n \rightarrow \infty$ in (16) we obtain

$$u(x') - \phi(x') \geq u(x) - \phi(x) \quad \forall x \in \overline{\Omega}.$$

Therefore $u - \phi$ attain a maximum over $\bar{\Omega}$ at $x' \neq x_0$. It's a contradiction since $u - \phi$ has strict max over $\bar{\Omega}$ at x_0 .

Now return to our exercise, denote $\alpha_r = \max\{u(y) - \phi(y) : y \in S(x_0, r)\} < 0$, the maximum is archived since $u - \phi$ is continuous on compact set $S(x_0, r)$. Now since $u^\varepsilon \rightarrow u$ uniformly on $\bar{\Omega}$, there exists $\zeta_r > 0$ such that for any $\varepsilon < \zeta_r$, then

$$|u^\varepsilon(y) - u(y)| < \frac{-\alpha_r}{2} \quad \forall y \in \bar{\Omega}.$$

Now for any $y \in S(x_0, r)$ we have

$$(17) \quad u^\varepsilon(y) - \phi(y) = u^\varepsilon(y) - u(y) + u(y) - \phi(y) < \frac{\alpha_r}{2}.$$

Choose $\lambda_r = \min\{\zeta_r, \delta_r\}$, then for any $\varepsilon < \lambda_r$, $u^\varepsilon - \phi$ is continuous on $\bar{\Omega}$, thus it obtains a maximum over $\bar{\Omega}$ at some point $x_\varepsilon \in B'(x_0, r)$, denote its maximum by $\alpha = u^\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$, then clearly

$$(18) \quad \alpha = \max_{\bar{\Omega}}\{u^\varepsilon(y) - \phi(y)\} \geq u^\varepsilon(x_0) - \phi(x_0) = u^\varepsilon(x_0) - u(x_0) > \frac{\alpha_r}{2}.$$

Combine (17) and (18) we obtain x_ε must lie in $B(x_0, r)$. Therefore for any $\varepsilon > 0$ small enough, $u^\varepsilon - \phi$ only has maximum over $\bar{\Omega}$ at some points $x_\varepsilon \in B(x_0, r)$. The rest is pretty simple, let $r = \frac{1}{n}$, then for each n , choose

$$\varepsilon_n = \min\left\{\lambda_{\frac{1}{n}}, \frac{1}{n}\right\} \implies \varepsilon_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

and $u^{\varepsilon_n} - \phi$ has a max at x_{ε_n} inside $B(x_0, \frac{1}{n})$, clearly $x_{\varepsilon_n} \rightarrow x_0$ and furthermore

$$|u^{\varepsilon_n}(x_{\varepsilon_n}) - u(x_0)| \leq |u^{\varepsilon_n}(x_{\varepsilon_n}) - u(x_{\varepsilon_n})| + |u(x_{\varepsilon_n}) - u(x_0)| \longrightarrow 0$$

as $n \rightarrow \infty$. □

Proof 2. Observe that this theorem is wrong if $u^\varepsilon - \phi$ and $u - \phi$ are not continuous. One can construct a counterexample using a modified version of the topologist's sine curve (remove the largest points and give them different smaller values).

So assume that u^ε and u are continuous. The problem now is equivalent to the following problem:

“Suppose f is a continuous function with global max at x_0 and that f^ε are continuous function that converges uniformly to f as $\varepsilon \rightarrow 0$. Then for small enough ε , f^ε has a max x_ε near x_0 .”

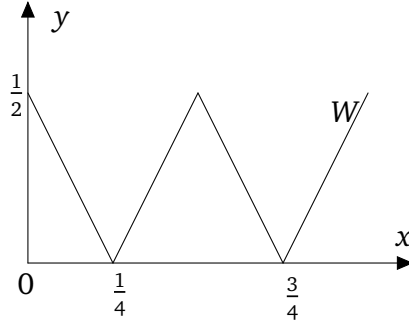
Pick $r_1 > r_2 > 0$, we have that $f|_{\{x: r_2 \leq |x-x_0| \leq r_1\}}$ has a max at x' . Furthermore, $f(x_0) > f(x')$. Define $h := f(x_0) - f(x')$. For small enough ε' , we have $|f^\varepsilon(x) - f(x)| < \frac{h}{3}$ for $\varepsilon < \varepsilon'$. Also, for small enough r (we can assume $r < r_1$), we have

$|f(x_0) - f(x)| < \frac{h}{3}$ for $x \in B(x_0, r)$. Therefore, pick $x \in B(x_0, r)$ and $\varepsilon < \varepsilon'$, we have $|f(x_0) - f^\varepsilon(x)| < \frac{2h}{3}$. Pick $y \in \{x : r_2 \leq |x - x_0| \leq r_1\}$. We have $f^\varepsilon(x) > f(x') + \frac{h}{3} > f(y) + \frac{h}{3} > f^\varepsilon(y)$. This means that, $\max_{\overline{B(x_0, r_1 - \delta)}} f^\varepsilon(t) \geq f^\varepsilon(x) > f^\varepsilon(y)$ for arbitrary small $\delta > 0$. This means that $\max_{\overline{B(x_0, r_2)}} f^\varepsilon(t)$ must be at $t_0 \in \overline{B(x_0, r_1)} \subseteq B(x_0, \frac{r_1+r_2}{2})$. Therefore, $f^\varepsilon(t_0) \geq f^\varepsilon(t)$ for all $t \in B(x_0, \frac{r_1+r_2}{2})$. Thus, t_0 is our desired x_ε .

(Not the cleanest proof but it works...)

□

Exercise 2. Let $n = 1$, $H(x, p) = |p| - W(x)$ where W is defined by its graph



i.e, $W : \mathbb{R} \rightarrow \mathbb{R}$ is periodic. Consider the ergodic cell problem (E_0)

$$(E_0) \quad H(y, 0 + Dv(y)) = |Dv(y)| - W(y) = c \quad \text{in } \mathbb{T}.$$

- Find the unique constant c such that (E_0) has solution.
- Find infinitely solutions for (E_0) for all $y \in \mathbb{T}^n$.

Proof 1.

- Assume v is continuous a viscosity solution of (E_0) , since \mathbb{T} is compact we have v is bounded and archived its minimum, called

$$m = v(x_m) = \min\{v(x) : x \in \mathbb{T}\} \leq v(x) \quad \forall x \in \mathbb{T}$$

- Super-test. At x_m , we can see that $q = 0$ belong to $Dv^-(x_m)$. Thus

$$0 = |q| \geq W(x_m) + c \geq \min_{x \in \mathbb{T}} W(x) + c = c \implies 0 \geq c.$$

- Sub-test. Setting $\mathbb{T}^+ = \{x \in \mathbb{T} : D^+v(x) \neq \emptyset\}$, let $p \in D^+v(x)$ then

$$0 = |p| \leq W(x) + c \quad \forall x \in \mathbb{T}^+.$$

But \mathbb{T}^+ is dense in \mathbb{T} , thus there exists a sequence $\{x_n\} \in \mathbb{T}^+$ such that⁹ $x_n \rightarrow \frac{1}{4}$, combine with the fact that W is continuous we obtain

$$c \geq -W(x_n) \quad \forall n \in \mathbb{N} \implies c \geq -W(1/4) = 0 \implies c \geq 0.$$

⁹We write $1/4$ mean the element $1/4$ in \mathbb{T}

Thus $c = 0$ is the only possible value such that (E_0) has a viscosity solution.

(b) For each $c \in \left[\frac{1}{2}, \frac{3}{4}\right]$, consider the following function

$$u_c(x) = \left(x^2 - \frac{x}{2}\right) \chi_{[0, \frac{1}{2}]} + \left(-x^2 + \frac{3}{2}x - \frac{1}{2}\right) \chi_{[\frac{1}{2}, c] \cup [\frac{3}{2}-c, 1]} \\ + \left(x^2 - \frac{3}{2}x - 2c^2 + 3c - \frac{1}{2}\right) \chi_{[c, \frac{3}{2}-c]}.$$

For example as $c = 0.6$ and $c = 0.7$ we have

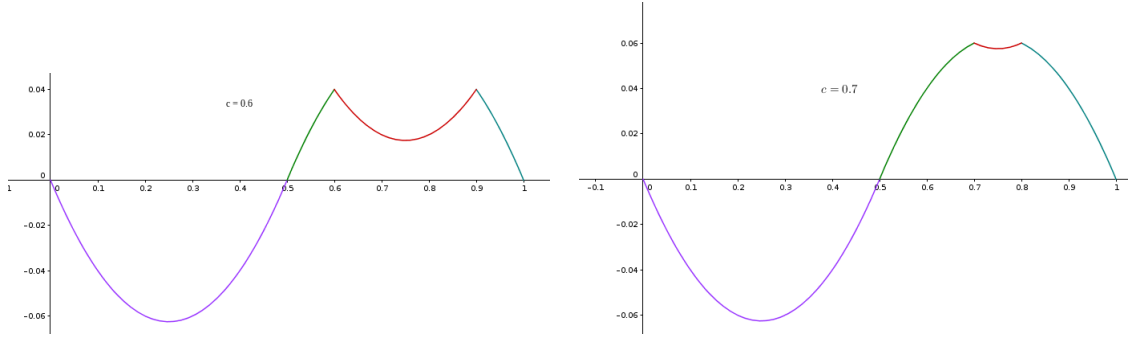


FIGURE 1. The function u_c as $c = 0.6$ and $c = 0.7$

Now we will show that u_c is a viscosity solution of $|Du_c(x)| = W(x)$ for any $c \in \left[\frac{1}{2}, \frac{3}{4}\right]$. To do this, we only need to consider points in which u_c is not smooth, that is $x = c$ and $x = \frac{3}{2} - c$. Also, we only need to consider $D^+u_c(x)$ since the set $D^-u_c(x)$ is always empty at non-smooth points of u_c . To begin with, we will give explicit formulas for $D^+u_c(c)$ and $D^+u_c\left(\frac{3}{2} - c\right)$.

Case $x = c$. Recall that $p \in D^+u(c)$ iff $u(x) - u(c) \leq p(x - c)$

– Consider $x < c$, then it becomes

$$(x - c) \left(\frac{3}{2} - x - c \right) \leq p(x - c) \implies \frac{3}{2} - x - c \geq p \implies \frac{3}{2} - 2c \geq p.$$

– Consider $x > c$, then it becomes

$$(x - c) \left(x + c - \frac{3}{2} \right) \leq p(x - c) \implies x + c - \frac{3}{2} \leq p \implies 2c - \frac{3}{2} \leq p.$$

Thus in this case we have

$$p \in D^+u_c(c) \iff |p| \leq \frac{3}{2} - 2c.$$

Now we need to check that $|p| \leq W(c)$ for all $p \in D^+u_c(c)$, we have

$$p \in D^+u_c(c) \implies |p| \leq \frac{3}{2} - 2c = \frac{3-4c}{2} = W(c).$$

Case $x = \frac{3}{2} - c$. Since u_c and W are symmetric through the line $x = \frac{3}{4}$, this case is completely similar to case $x = c$.

□

Proof 2. Consider the following equation

$$(HJ) \quad \varepsilon u^\varepsilon(y) + H(y, Du^\varepsilon) = 0.$$

Plug the formula of H into (HJ) we have

$$\varepsilon u^\varepsilon(y) + |Du^\varepsilon(y)| - W(y) = 0.$$

Observe that for each $\varepsilon > 0$, $u^\varepsilon = 0$ is a subsolution and $u^\varepsilon = \frac{1}{\varepsilon}$ is a supersolution. To see this, for $u^\varepsilon = 0$, let $\phi \in C^\infty$ s.t. $(\phi - u^\varepsilon)(y_0) = 0$ is a max. We have

$$D\phi(y_0) = Du^\varepsilon(y_0) = 0.$$

So

$$\varepsilon \phi(y_0) + H(y_0, D\phi(y_0)) = 0 - W(y_0) \leq 0.$$

Similarly, for $u^\varepsilon = \frac{1}{\varepsilon}$, let $\phi \in C^\infty$ s.t. $(\phi - u^\varepsilon)(y_0) = 0$ is a min. We have

$$\varepsilon \phi(y_0) + H(y_0, D\phi(y_0)) = 1 - W(y_0) \geq 0.$$

Therefore, if v^ε is a viscosity solution to (HJ), we have

$$1 \geq \varepsilon v^\varepsilon \geq 0.$$

Thus, we have

$$(19) \quad 1 \geq W(y) \geq |Dv^\varepsilon(y)|.$$

From here we conclude that v^ε is equi-Lipschitz with modulus 1. This is not enough to conclude anything about the convergence of $\varepsilon v^\varepsilon$ except of the result obtained in class. Luckily, we have more– observe that

$$\begin{aligned} \left| Dv^\varepsilon \left(\frac{1}{4} \right) \right| &\leq W \left(\frac{1}{4} \right) = 0, \\ \left| Dv^\varepsilon \left(\frac{3}{4} \right) \right| &\leq W \left(\frac{3}{4} \right) = 0. \end{aligned}$$

So,

$$Dv^\varepsilon \left(\frac{1}{4} \right) = Dv^\varepsilon \left(\frac{3}{4} \right) = 0.$$

That means, for all $\varepsilon > 0$, by plugging the above to our equation, $\varepsilon v^\varepsilon \left(\frac{1}{4} \right) = 0$, which means $v^\varepsilon \left(\frac{1}{4} \right) = 0$. Similarly, $v^\varepsilon \left(\frac{3}{4} \right) = 0$ for all $\varepsilon > 0$. Combine with (19), we deduce that for all $\varepsilon > 0$,

$$|v^\varepsilon(y)| = \left| \int_{\frac{1}{4}}^y v^{\varepsilon'}(t) dt \right| \leq \int_{\frac{1}{4}}^y |v^{\varepsilon'}(t)| dt \leq \int_{\frac{1}{4}}^y 1 dt = \left| y - \frac{1}{4} \right| \leq 1.$$

So, $\varepsilon v^\varepsilon(y) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and, therefore, $c = 0$. The solutions to the problem are obtained similarly to proof 1. \square

Exercise 4. Let $H \in \text{Lip}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$(H1) \quad \lim_{|p| \rightarrow \infty} H(y, p) = +\infty \text{ uniformly in } y,$$

$$(H2) \quad H(y + k, p) = H(y, p) \text{ for all } k \in \mathbb{Z}^n.$$

Then $\bar{H}(p) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lipschitz and coercive function.

Proof 1. For $u \in C(\mathbb{T}^n)$ and $F : \mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, we define

$$F(x, u(x), Du(x)) \leq C \text{ in the viscous sense} \iff F(x, u(x), p) \leq C \quad \forall p \in D^+u(x)$$

$$\iff u \text{ is sub-solution of } F(x, u(x), Du(x)) = C,$$

$$F(x, u(x), Du(x)) \geq C \text{ in the viscous sense} \iff F(x, u(x), p) \geq C \quad \forall p \in D^-u(x)$$

$$\iff u \text{ is super-solution of } F(x, u(x), Du(x)) = C.$$

By assumption, there exists a constant $C > 0$ such that

$$|H(x, p) - H(x, q)| \leq C \|p - q\| \quad \forall x, p, q \in \mathbb{R}^n.$$

We will prove the Lipschitz property of \bar{H} by using the representation

$$(20) \quad \bar{H}(q) = \inf \{ c \in \mathbb{R} : \exists v \in C(\mathbb{T}^n) : H(x, q + Dv(x)) \leq c \text{ in } \mathbb{T}^n \text{ in viscous sense} \}.$$

Step 1. First of all, we prove that

$$(21) \quad \bar{H}(p) - \bar{H}(q) \leq C \|p - q\|.$$

Let $u \in \text{Lip}(\mathbb{T}^n)$ is a periodic viscosity solution of the problem

$$(E_q) \quad H(x, q + Du(x)) = \bar{H}(q).$$

By definition

$$\zeta \in D^+u(x) \implies H(x, q + \zeta) \leq \bar{H}(q).$$

We will show that

$$(22) \quad H(x, p + Du(x)) \leq C\|p - q\| + \bar{H}(q)$$

in the viscous sense. Indeed, for $\zeta \in D^+u(x)$, by the Lipschitz's property of H , we have

$$\begin{aligned} |H(x, p + \zeta) - H(x, q + \zeta)| &\leq C\|p - q\| \implies H(x, p + \zeta) \leq C\|p - q\| + H(x, q + \zeta) \\ &\implies H(x, p + \zeta) \leq C\|p - q\| + \bar{H}(q) \quad \forall \zeta \in D^+u(x) \\ &\implies H(x, p + Du(x)) \leq C\|p - q\| + \bar{H}(q) \quad (\text{vis-sense}). \end{aligned}$$

Thus (22) is true, from the representation of \bar{H} in (20), we obtain (21) is true since

$$\bar{H}(p) \leq C\|p - q\| + \bar{H}(q) \iff \bar{H}(p) - \bar{H}(q) \leq C\|p - q\|.$$

Step 2. Similarly, doing quite similar to step 1, we obtain

$$(23) \quad \bar{H}(q) - \bar{H}(p) \leq C\|p - q\|.$$

From (21) and (23) we obtain the Lipschitz's property of \bar{H} ,

$$|\bar{H}(p) - \bar{H}(q)| \leq C\|p - q\|.$$

Step 3. We now prove the coercivity of \bar{H} . First restate the coercivity of H as

$$(24) \quad \lim_{|p| \rightarrow \infty} \left(\min_{x \in \mathbb{T}^n} H(x, p) \right) = +\infty.$$

We start with a similar result as representation formula of \bar{H} in (20), that is

$$(25) \quad \bar{H}(q) = \sup \{c \in \mathbb{R} : \exists v \in C(\mathbb{T}^n) : H(x, q + Dv(x)) \geq c \text{ in } \mathbb{T}^n \text{ in viscous sense}\}.$$

Indeed, define

$$\alpha = \sup \{c \in \mathbb{R} : \exists v \in C(\mathbb{T}^n) : H(x, q + Dv(x)) \geq c \text{ in } \mathbb{T}^n \text{ in viscous sense}\}.$$

Let $u \in C(\mathbb{T}^n)$ be the viscosity solution of the cell problem

$$H(x, p + Du(x)) = \bar{H}(p) \quad \text{in } \mathbb{T}^n.$$

Clearly, $\bar{H}(p)$ is admissible in the right hand of formula (25), thus $\bar{H}(p) \leq \alpha$. Now we want to prove $\alpha \leq \bar{H}(p)$, assume the contradiction, that is $\bar{H}(p) < \alpha$. Then there exists $(c, w) \in \mathbb{R} \times C(\mathbb{T}^n)$, $\bar{H}(p) < c \leq \alpha$ such that

$$H(x, p + Dw(x)) \geq c > \bar{H}(p) = H(x, p + Du(x)).$$

Since u, w are bounded, we can take $\varepsilon > 0$ such that

$$\varepsilon u(x) + H(x, p + Du(x)) < \frac{c + \bar{H}(p)}{2} < \varepsilon w(x) + H(x, p + Dw(x)).$$

By comparison principle, we have $u \leq w$ in \mathbb{T}^n , it's a contradiction since we can always add a constant to make $w < u$.

Now using this representation, let $\varphi = C \in C^1(\mathbb{T}^n)$, and define

$$c = \min_{x \in \mathbb{T}^n} H(x, p + D\varphi(x)) = \min_{x \in \mathbb{T}^n} H(x, p).$$

Then, c is admissible in the formula of the representation (25), so that $\bar{H}(p) \geq c$, i.e

$$(26) \quad \bar{H}(p) \geq \min_{x \in \mathbb{T}^n} H(x, p).$$

Now combine (26) with (24) we obtain

$$\lim_{|p| \rightarrow \infty} \bar{H}(p) \geq \lim_{|p| \rightarrow \infty} \left(\min_{x \in \mathbb{T}^n} H(x, p) \right) = +\infty.$$

Thus $p \mapsto \bar{H}(p)$ is coercive and the proof is complete. \square

Proof 2. Lipschitz part.

Let $p, q \in \mathbb{R}^n$, we want to show that $|\bar{H}(p) - \bar{H}(q)| \leq C|p - q|$. Let u^ε and v^ε be solutions to the following problems

$$(27) \quad \varepsilon u^\varepsilon + H(x, Du^\varepsilon + p) = 0$$

and

$$(28) \quad \varepsilon v^\varepsilon + H(x, Dv^\varepsilon + q) = 0.$$

We have, by ‘‘Lipschitzivity’’,

$$|H(x, Du^\varepsilon + p) - H(x, Du^\varepsilon + q)| \leq C|p - q|.$$

Thus,

$$|-\varepsilon u^\varepsilon - H(x, Du^\varepsilon + q)| \leq C|p - q|,$$

which implies

$$(29) \quad \varepsilon u^\varepsilon + C|p - q| + H(x, Du^\varepsilon + q) \geq 0.$$

Thus, $u^\varepsilon + \frac{C|p-q|}{\varepsilon}$ is a supersolution to (29) and

$$u^\varepsilon + \frac{C|p - q|}{\varepsilon} \geq v^\varepsilon,$$

meaning

$$\varepsilon v^\varepsilon - \varepsilon u^\varepsilon \leq C|p - q|.$$

Similarly, we obtain

$$-C|p - q| \leq \varepsilon v^\varepsilon - \varepsilon u^\varepsilon.$$

Thus, $|\varepsilon u^\varepsilon - \varepsilon v^\varepsilon| \leq C|p - q|$. Let $\{\varepsilon_j\}$ be a sequence such that $\varepsilon_j u^{\varepsilon_j} \rightarrow \bar{H}(p)$ and $\varepsilon_j v^{\varepsilon_j} \rightarrow \bar{H}(q)$, we then have

$$|\bar{H}(p) - \bar{H}(q)| \leq C|p - q|.$$

Coercivity part:

Let $M > 0$. Pick $|p|$ big enough so that $H(x, p) > M$.

Let u be the solution to the cell problem

$$H(x, Du + p) = \bar{H}(p).$$

Since u is continuous and periodic, u has a max at x_0 . So consider the test function $\phi(x) = u(x_0)$. We have that, by definition of viscosity solution,

$$M < H(x, D\phi(x_0) + p) = H(x, p) \leq \bar{H}(p).$$

□

Exercise 5. Let $H \in \text{Lip}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$(H1) \quad \lim_{|p| \rightarrow \infty} H(y, p) = +\infty \text{ uniformly in } y,$$

$$(H2) \quad H(y + k, p) = H(y, p) \text{ for all } k \in \mathbb{Z}^n.$$

Assume $p \mapsto H(y, p)$ is convex for any $y \in \mathbb{T}^n$, then $p \mapsto \bar{H}(p)$ is also convex.

Proof 1. We will prove the convexity of $p \mapsto \bar{H}(p)$ by the representation formula of \bar{H} , that is

$$\begin{aligned} \bar{H}(q) &= \inf \{ c \in \mathbb{R} : \exists v \in C(\mathbb{T}^n) : H(x, q + Dv(x)) \leq c \text{ in } \mathbb{T}^n \text{ in viscous sense} \} \\ &= \inf \left\{ \sup_{x \in \mathbb{T}^n} H(x, q + D\varphi(x)) : \varphi \in C^1(\mathbb{T}^n) \right\}. \end{aligned}$$

Let $u \in \text{Lip}(\mathbb{T}^n)$ is a viscosity solution of the problem

$$(E_p) \quad H(x, p + Du(x)) = \bar{H}(p).$$

Let $\varphi \in C^1(\mathbb{T}^n)$ arbitrary and let

$$C = \max_{x \in \mathbb{R}^n} H(x, q + D\varphi(x)).$$

For $\lambda \in (0, 1)$, consider the function

$$w = \lambda u + (1 - \lambda)\varphi.$$

We will show that (in the viscous sense)

$$(30) \quad H(x, \lambda p + (1 - \lambda)q + Dw) \leq \lambda \bar{H}(p) + (1 - \lambda)C.$$

To do this, we need to show that for any $\zeta \in D^+w(x)$, then

$$H(x, \lambda p + (1 - \lambda)q + \zeta) \leq \lambda \bar{H}(p) + (1 - \lambda)C.$$

Lemma. Let $u \in C(\Omega)$, then

$$D^+w(x) = \lambda D^+u(x) + (1 - \lambda)D\varphi(x)$$

for any $\varphi \in C^1(\Omega)$, $w = \lambda u + (1 - \lambda)\varphi$ and $\lambda \in [0, 1]$.

Proof of lemma. It's easy to see that $\lambda D^+u(x) + (1 - \lambda)D\varphi(x) \subseteq D^+w(x)$. For the converse, let $\zeta \in D^+u(x)$, setting $\alpha = \zeta - (1 - \lambda)D\varphi(x)$, then

$$\begin{aligned} \frac{\lambda u(y) - \lambda u(x) - \langle \alpha, y - x \rangle}{\|y - x\|} &= \frac{w(y) - w(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \\ &\quad - (1 - \lambda) \frac{\varphi(y) - \varphi(x) - \langle D\varphi(x), y - x \rangle}{\|y - x\|}. \end{aligned}$$

Taking lim sup of both sides, we have $\alpha \in D^+(\lambda u)(x)$, therefore $\frac{\alpha}{\lambda} \in D^+u(x)$, then

$$\delta = \lambda \left(\frac{\alpha}{\lambda} \right) + (1 - \lambda)D\varphi(x) \subseteq \lambda D^+u(x) + (1 - \lambda)D\varphi(x).$$

The proof of lemma is complete. \square

By above lemma, there exists $\alpha \in D^+u(x)$ such that $\zeta = \lambda\alpha + (1 - \lambda)D\varphi(x)$, using the convexity in p of H we obtain

$$\begin{aligned} H(x, \lambda p + (1 - \lambda)q + \zeta) &= H(x, \lambda p + (1 - \lambda)q + \lambda\alpha + (1 - \lambda)D\varphi(x)) \\ &\leq \lambda H(x, p + \alpha) + (1 - \lambda)H(x, q + D\varphi) \\ &\leq \lambda \bar{H}(p) + (1 - \lambda)C. \end{aligned}$$

Thus in sense of viscosity, we obtain (30) is true. By representation formula for $\bar{H}(\lambda p + (1 - \lambda)q)$ we have

$$\bar{H}(\lambda p + (1 - \lambda)q) \leq \lambda \bar{H}(p) + (1 - \lambda)C.$$

In other words, we have

$$\bar{H}(\lambda p + (1 - \lambda)q) - \lambda \bar{H}(p) \leq (1 - \lambda)C = (1 - \lambda) \max_{x \in \mathbb{T}^n} H(x, q + D\varphi(x))$$

and since $\varphi \in C^1(\mathbb{T}^n)$ is arbitrary, we get

$$\bar{H}(\lambda p + (1 - \lambda)q) - \lambda \bar{H}(p) \leq (1 - \lambda) \inf_{\varphi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} H(x, q + D\varphi(x)) = (1 - \lambda) \bar{H}(q).$$

Thus,

$$\bar{H}(\lambda p + (1 - \lambda)q) \leq \lambda \bar{H}(p) + (1 - \lambda) \bar{H}(q)$$

and \bar{H} is convex. \square

Proof 2. Suppose $\bar{H}(p)$ is not convex. There are p, q such that

$$(31) \quad \bar{H}((1-k)p + kq) > (1-k)\bar{H}(p) + k\bar{H}(q)$$

for some $k \in (0, 1)$. Let u be a solution of

$$(32) \quad H(x, Du + (1-k)p + kq) = \bar{H}((1-k)p + kq)$$

and v_p, v_q be solutions of

$$H(x, Dv + p) = \bar{H}(p)$$

and

$$H(x, Dv + q) = \bar{H}(q)$$

respectively. (32) Implies that

$$\begin{aligned} \bar{H}((1-k)p + kq) &> (1-k)H(x, Dv_p + p) + kH(x, Dv_q + q) \\ &\geq H(x, (1-k)(Dv_p + p) + k(Dv_q + q)) \\ &= H(x, (1-k)Dv_p + kDv_q + (1-k)p + kq). \end{aligned}$$

So, $(1-k)v_p + kv_q$ is a subsolution to (33). Therefore, $u \geq (1-k)v_p + kv_q$. But if u is a solution to (33), $u + C$ is also a solution to (33), which makes

$$u + C \geq (1-k)v_p + kv_q$$

for all C , which is a contradiction since v_p and v_q are real-valued functions. \square

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