Some asymptotic problems on the theory of viscosity solutions of Hamilton–Jacobi equations

by

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Abstract

Viscosity solutions arise naturally in many fields of study from engineering, physics, and operations research to economics. The study of viscosity solutions on its own has uncovered many new and interesting research problems, including the study of the asymptotic behavior of solutions with respect to the changing of parameters. In this dissertation, I present some new problems following the line of the asymptotic behavior of solutions. Each of the problems is related to the other through the old underlying theme of optimal control theory, yet presents many new problems on their own that are yet to be studied.

The first direction is on homogenization of Hamilton–Jacobi equations. Using deep analysis of the dynamics of minimizers corresponding to the solution, I established in [113] the optimal rate of convergence under the multi-scale setting in one dimension, which could not be obtained by the previous pure PDEs technique.

The second direction concerns various asymptotic problems for equations with state-constraint. In [75], my co-authors and I established some *first quantitative results* on the rate of convergence of the solution to the Hamilton–Jacobi equations with state-constraint on a nested domain setting. Utilizing the weak KAM theory, in [114], I established qualitatively various convergence results for the vanishing discount procedure with changing domains together with a new description of the regularity of the additive eigenvalues with respect to domain perturbation. Lastly, in [61], my co-author and I established the rate of convergence for the vanishing viscosity procedure, concerning the viscous state-constraint viscosity (*large*) solution that blows on the boundary of the underlying domain. This is the first-rate established for blow-up solutions in the literature as far as we know.

Dedication

To my Mom, Dad, and Sister, for being loving family members.

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Madison, Wisconsin

Son Tu

Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where states otherwise by reference or acknowledgment, the work presented is entirely my own.

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Chapter 1

Introduction

Mathematics is a place where you can do things which you can't do in the real world.
– Marcus du Sautoy

Mathematics is a game of imagination.
– Hung Vinh Tran

1.1 Overview

In applications ranging game theory, engineering, physics, operations research to economics, a dynamical system usually can be used to embed research problems within the framework of optimal control theory. One could tackle the model by either studying the dynamics of solution to the control problem, or by studying the value function associated with it. The latter approach has been a key idea in a revolution of studying such problems. It has been known for a long time that the value function satisfies a certain partial differential equation (PDE) if it is smooth – however the value function is known to be not differentiable in many cases. The viscosity solution Crandall, Lions [42] and Evans [46] developed is perfectly matched the value function, and since then this notion of solution has been the de facto one to consider when facing a nonlinear problem.

From a PDE point of view, viscosity solutions are the natural for a couple of reasons. Firstly, the vanishing viscosity procedure along which viscosity solutions are obtained is natural. Secondly, for many nonlinear equations, weak solutions via integration by parts are unavailable and usually all one has is the maximum principle. Finally, viscosity solution and weak solution are usually equivalent whenever there is overlap. The study of viscosity solution on it own has open many new interesting research problems, including the study of asymptotic behavior of solutions with respect to the changing of parameters (viscosity solution itself is a result of such an asymptotic analysis – the vanishing viscosity procedure).

In this dissertation, I present some new problems following the line of asymptotic behavior of solutions. Each of the problem is related to the other through the old underlying theme of optimal control theory, yet presents many new problems on their own that yet to be studied. We refer the readers to the recent survey [63] for an overview on Hamilton–Jacobi equations and the above-mentioned two approaches when studying the optimal control problems, as well as the introduction of viscosity solutions.

1.1.1 Homogenization

Homogenization theory is motivated from physical problems in physics and it has been studied intensively over the last century. In this dissertation, we address the optimal rate of convergence for homogenization of Hamilton–Jacobi equations in one dimension under the multi-scale setting. Let $H = H(x, y, p)$ be a Hamiltonian that is $\mathbb{Z}^n$-periodic. Roughly speaking, under some mild conditions, for each ε, the viscosity solution $u^\varepsilon(x,t)$ to the highly oscillating initial value (u_0 is given) Hamilton–Jacobi equation

$$\begin{cases} u_t^\varepsilon(x,t) + H\left(x, \frac{x}{\varepsilon}, Du^\varepsilon(x,t)\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{C$_\varepsilon$}$$

converges locally uniformly to a function $u(x,t)$ as $\varepsilon \to 0^+$ and u is a viscosity solution to an *effective* equation ([47, 48, 85])

$$\begin{cases} u_t(x,t) + \overline{H}\left(x, Du(x,t)\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{C}$$

Here $\overline{H}(x,p)$ is the *effective Hamiltonian* and is determined by H in a nonlinear way through the so-called cell problem: $\overline{H}(x,p)$ is the unique real number such that ([36, 37, 47, 48, 85, 86, 103]) the following cell problem

$$H\big(x, y, p + D_y v(y)\big) = \overline{H}(x,p) \qquad \text{in } \mathbb{T}^n \tag{CP}$$

has a continuous and periodic solution $v(y) = v(y; x, p)$. It is known that the rate of convergence is at least $\mathcal{O}(\varepsilon^{1/3})$ (see [28], by using a PDE approach). Nearly optimal and optimal rate $\mathcal{O}(\varepsilon)$ have been obtained recently for the simpler regime of single-scale $H(x,y,p) = H(y,p)$ (see [38, 94, 112]), however the multi-scale case remains open. We obtain an optimal rate $\mathcal{O}(\varepsilon)$ in one dimension in the multi-scale by using tools from optimal control theory and ergodic theory.

1.1.2 Changing domain with state-constraint

The state-constraint boundary condition is sometime referred to as the *natural boundary condition* when viewing through optimal control formula. In short, for a given domain Ω and a cost function $L(x,v)$ where x is the position and v is the velocity, one seeks to minimize the cost functional of the *infinite horizontal* problem

$$J[x,\eta] = \inf_{\eta} \int_0^\infty e^{-\lambda s} L\big(\eta(s), -\dot{\eta}(s)\big)\, ds$$

among all paths η starting from x with bounded velocity that do not exist $\overline{\Omega}$. The value function is defined in such a way $u(x) = \inf_\eta J[x,\eta]$ will satisfy the state-constraint boundary condition, which can be written in the PDE form

$$\begin{cases} \lambda u(x) + H(x, Du(x)) \le 0 & \text{in } \Omega, \\ \lambda u(x) + H(x, Du(x)) \ge 0 & \text{on } \overline{\Omega}. \end{cases} \tag{S$_\lambda$}$$

where H is the Legendre transform[1] of the (convex) cost function L. An natural question arises when viewing the problem in this form: what is the relation of an minimizer with state-constraint and an minimizer without any constraint? When viewing through the PDE point of view, it becomes a nested domain problem. Let u_k be the solution on the ball centered at 0 with radius k, the question becomes what happen as $k \to \infty$? Roughly speaking, $u_k \to u$ locally uniformly where u is the viscosity solution to the global problem

$$\lambda u(x) + H(x, Du(x)) = 0 \qquad \text{in } \Omega. \tag{S}$$

In some sense, the state-constraint boundary conditions vanishes as the domains get larger. In [75], together with co-authors we are interested in such a rate of convergence. Using classical tools in viscosity solutions we obtain a general (nonconvex) $\mathcal{O}(k^{-2})$. However, if H is convex then we can viewing the problem through the optimal control setting. We obtain a minimizer with bounded velocity to the global problem. When restricting that minimizer to smaller domains we can get good estimates that allows an optimal rate of $\mathcal{O}(e^{-Ck})$ to be obtained.

1.1.3 Vanishing discount and the additive eigenvalue with state-constraint

The factor $\lambda > 0$ in (S_λ) is called a *discounted factor*. When varying λ another question arises. What happens if $\lambda \to 0^+$? When fixing Ω, the limiting equation is of the form

$$\begin{cases} H(x, Du(x)) \leq c & \text{in } \Omega, \\ H(x, Du(x)) \geq c & \text{on } \overline{\Omega}, \end{cases} \tag{S_0}$$

where c is a unique constant called *additive eigenvalue*[2]

$$c = -\lim_{\lambda \to 0} \lambda u_\lambda(\cdot).$$

This problem is interesting since the limiting equation does not have a unique solution (even up to adding a constant). Such a behavior is called *selection principle* and indeed under the convex setting and appropriate normalization ([1, 44, 69, 92, 70, 71]) one can prove the convergence of $u_\lambda + \lambda^{-1}c \to u$ where u is a solution to (S_0) and some description of u (using weak KAM theory) is also available.

In [114], we are interested in the question: what happen if we both vanish $\lambda \to 0^+$ while also perturb Ω by scaling. Roughly speaking, let us consider the problem

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } \Omega_\lambda = (1 + r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } \overline{\Omega}_\lambda = (1 + r(\lambda))\overline{\Omega}. \end{cases} \tag{$\widehat{\mathrm{S}}_\lambda$}$$

where $r(\lambda), \phi(\lambda)$ are parameters that will be changing. Let $c(\lambda)$ be the additive eigenvalue on Ω_λ. We show the convergence $u_\lambda + \phi(\lambda)^{-1}c(\lambda) \to u^\gamma$ and provide a characterization of the limiting function u^γ given that

$$\gamma = \lim_{\lambda \to 0^+} \frac{r(\lambda)}{\phi(\lambda)}$$

[1] See Chapter 4 for a formulation of the transform.

[2] it is indeed the effective Hamiltonian $\overline{H}(0)$ in (CP) if $\Omega = \mathbb{T}^n$

exists and is finite. Moreover, we construct an example showing the nonconvergence if γ is infinite. Furthermore, the study of this convergence leads to a rather interesting description of the derivative of the function $\lambda \to c(\lambda)$, which is rather new in the literature.

1.1.4 Vanishing viscosity from blow-up solutions

Viscosity solutions historically arise from the vanishing procedure. When posing with state-constraint boundary condition, interesting questions can be asked. Let $\Omega \subset \mathbb{R}^n$ be a *nice* domain and $f : \overline{\Omega} \to \mathbb{R}$ be a *nice* source data[3]. Roughly speaking, for $\varepsilon > 0$ and $1 < p \leq 2$, the solution to u^ε to the following singular problem

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon \Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

converges locally uniformly to u, the solution to a state-constraint problem ([7, 52, 78, 89, 101, 102])

$$\begin{cases} u(x) + |Du(x)|^p - f(x) \leq 0 & \text{in } \Omega, \\ u(x) + |Du(x)|^p - f(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{PDE$_0$}$$

It turns out that (PDE$_\varepsilon$) is an equivalent way of viewing state-constraint for second-order equations, thus the problem is natural in the sense of the state-constraint boundary condition is preserved. The solution that blows up uniformly on the boundary is also called a *large solution* and has been studied independently for various kinds of equations.

The problem is interesting since in the limit we no longer have blowing up behavior, as u is bounded. For the case where (PDE$_\varepsilon$) is equipped with the Dirichlet boundary condition, a rate $\mathcal{O}(\sqrt{\varepsilon})$ is well known with multiple proofs (see [8, 43, 111]). In [61], (together with co-author) we show a rate of $\mathcal{O}(\sqrt{\varepsilon})$ for a class of source data f, and with some more assumptions the *one-sided* rate can be improved to $\mathcal{O}(\varepsilon^{1/p})$ or even $\mathcal{O}(\varepsilon)$ with compactly supported data.

1.2 Organization of the dissertation

The organization of the dissertation is as follows. In Chapter 2 we give a background with well-posedness theory for viscosity solutions, including the celebrated second-order equations as well as introducing some boundary conditions. Chapter 3 is devoted to the study of the optimal homogenization rate in one dimension with multi-scale structure, with materials taken from [113]. Chapter 4 contains material from [75] (joint work with Yeon-Eung Kim and Hung Tran) on various quantitative estimates of state-constraint viscosity solution on nested domains. Chapter 5 is devoted on the study of the vanishing discount problem on changing domains, with materials taken from [114]. Lastly, Chapter 6 contains materials from [61] (joint work with Yuxi Han) on the vanishing viscosity rate of blow-up solutions. A list of the main assumptions that being referred to through out the thesis is provided in Appendix, together with some relevant results.

[3]Complete assumptions on what is *nice* here are given in Chapter 6.

Chapter 2

Preliminaries

In this Chapter, we give a short introduction to the theory of viscosity solutions of Hamilton–Jacobi equations, which was introduced by Crandall and Lions [42] (see also Crandall, Evans, and Lions [40]). We start with a simple example of first order equation for simplicity, then the rest of the chapter will be presented for second order equations and the state-constraint problem. The materials in this chapter are taken from [8, 20, 50, 111] and many other courses collectively as the author went through them during the time at UW-Madison.

Let us consider the initial-value problem

$$\begin{cases} u_t(x,t) + H(x, Du(x,t)) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{C}$$

where the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given, as is the initial value $g : \mathbb{R}^n \to \mathbb{R}$. The general problem could be time-independent, on a bounded domain, or with different boundary conditions. In general (see the example below with a one dimensional eikonal equation) solutions of (C) are not unique. The original approach [] to study (C) is to consider the approximated equation

$$\begin{cases} u^\varepsilon_t(x,t) + H(x, Du^\varepsilon(x,t)) = \varepsilon\Delta u^\varepsilon(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{C$_\varepsilon$}$$

for $\varepsilon > 0$. This approach is usually referred to as *vanishing viscosity* process, as the term $\varepsilon\Delta$ regularizes the equations is commonly referred to as a viscosity term. We then let $\varepsilon \to 0^+$ and study the limit of the family $\{u^\varepsilon\}_{\varepsilon>0}$. Under some mild assumptions, it is often the case that $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded and locally equicontinuous on $\mathbb{R}^n \times (0,T)$. We then use Arzelà-Ascoli Theorem to deduce that

$$u^{\varepsilon_j} \to u, \qquad \text{locally uniformly in } \mathbb{R}^n \times (0,T),$$

for some subsequence $\{u^{\varepsilon_j}\}$ and some limit function $u \in \mathrm{C}(\mathbb{R}^n \times (0,T)$. We expect that u is a solution to (C) in some special senses since we have very little information about u_t and Du beside u is continuous. Since (C) is fully nonlinear, weak convergence techniques with integration by parts are usually not applicable, thus the notion of weak solution

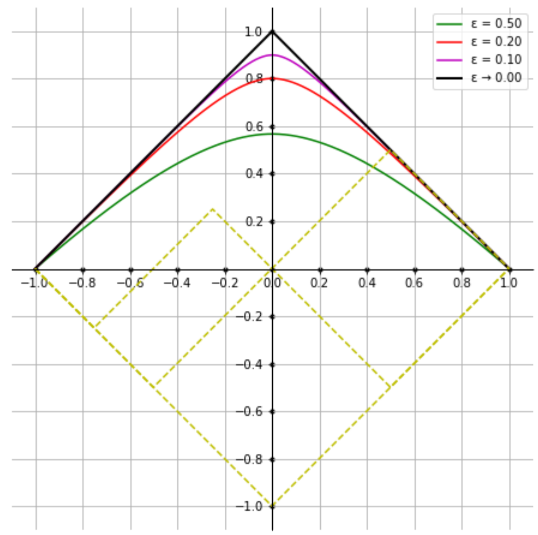

Figure 2.1: The approximated solutions u^ε and the limit solution u

here is obtained by maximum principle. The term *viscosity solutions* is used historically as those are solutions obtained from the vanishing viscosity process. We will see that the rigorous definition of viscosity solutions does not involve viscosity of any kind but the names remains because of this historical aspect.

In the modern approach, existence of viscosity solutions can be obtained by Perron's method, which is more robust and general (the Perron's method can be extended to second order equation while the vanishing viscosity faces nontrivial difficulties).

Example 1. *The one dimensional eikonal equation*

$$|u'(x)| = 1 \qquad \text{in } (-1,1) \qquad \text{with} \qquad u(-1) = u(1) = 0. \tag{2.0.1}$$

There are infinitely many Lipschitz almost everywhere solutions of (2.0.1)*, as being shown in Figure* 2.1*. However* (2.0.1) *does not have a classical solution. The vanishing viscosity procedure, however, selects a special solution. Solution of the regularized problem*

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon (u^\varepsilon)'' & \text{in } (-1,1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

satisfies

$$u^\varepsilon(x) = 1 - |x| + \varepsilon \left(e^{-\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}|x|} \right) \to u(x) = 1 - |x|$$

uniformly on $[-1,1]$.

Equation (2.0.1) is a special case of the following physical problem: *escape of a light ray* or *continuous shortest-path problems*. Suppose that Ω is an open set with suitably smooth boundary $\partial\Omega$, which can be viewed as the medium. A light ray starting from $x \in \Omega$ is a path $\gamma : [0,t] \to \Omega$ with $\gamma(0) = x$ for some $t > 0$. Let $c : \overline{\Omega} \to [0, +\infty)$ be the medium constraint of the speed of light at each point in the medium (the inhomogeneity of the medium). For a light ray one can define $T_\gamma = \inf\{s \geq 0 : \gamma(s) \notin \Omega\}$ as the first time the light ray exists the medium and $T_\gamma = +\infty$ if $\gamma([0,\infty)) \subset \Omega$. Naturally, the light ray takes the path that exists the medium in the least amount of time with the speed constraint

$$|\dot{\gamma}(s)| \leq c(\gamma(s)), \qquad s \geq 0.$$

This leads to the introduction of the minimum time function

$$u(x) = \inf\left\{T_\gamma : \gamma(0) = x, |\dot{\gamma}(s)| \leq c(\gamma(s))\right\} \tag{2.0.2}$$

for $x \in \Omega$. If one assumes that $\nabla u(x)$ exists at all points, then using Bellman's optimality principle and a Taylor expansion.

$$\begin{cases} c(x)|Du(x)| = 1 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

This is an example of Hamilton–Jacobi–Bellman equation. We have already seen that $\nabla u(x)$ is not smooth at every point from the previous example. This led to the development of viscosity solutions in the 1980s [42, 46]. We refer the readers to [26, 34, 35, 99] and the references therein for more results on the regularity of the minimum time function.

Instead of analyzing the *optimal* path for the above problem (2.0.2) directly, one could try so solve the corresponding PDE for the value function $u(\cdot)$, and obtain the optimal path as a consequence from $u(\cdot)$ (using a set of conditions due to Pontryagin, see [57, Chapter 1]). For that reason, solving such an equation numerically is of great interest. Many important methods to solve Hamilton–Jacobi equations have been developed in the literature. We refer the readers to [13, 43, 100, 106, 109] for finite difference monotone schemes of first-order equations and the references therein for recent developments. In the case where the equation if of the form $F(x, u(x), Du(x)) = 0$ and F is convex in Du, under some conditions there is also some semi-Lagrangian approximations schemes using the discretization of the Dynamical Programming Principle associated to the problem. We refer the readers to [53, 54] and the references therein. We also mention the recent developments using the Carleman estimate and the convexification method in [76, 80].

2.1 Viscosity solutions

2.1.1 Definitions

Let $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^n$. We consider the general second order nonlinear partial differential equation

$$F(x, u, Du(x), D^2u(x)) = 0 \qquad \text{in } \mathcal{O} \tag{2.1.1}$$

where $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is continuous, here $\mathbb{S}^n$ is the set of all real symmetric matrix in $\mathbb{R}^n \times \mathbb{R}^n$. Viscosity solution theory is a way to define solutions to (2.1.1) using ideas from maximum principle. A function $u : \mathcal{O} \to \mathbb{R}$ is *upper* (resp. *lower*) *semicontinuous* at $x \in \mathcal{O}$ provided

$$\limsup_{\mathcal{O} \ni y \to x} u(y) \leq u(x) \qquad \left(\text{resp.} \qquad \liminf_{\mathcal{O} \ni y \to x} u(y) \geq u(x)\right).$$

Let $\mathrm{USC}(\mathcal{O})$ (resp. $\mathrm{LSC}(\mathcal{O})$) be the collection of all functions that are upper (resp. lower) semicontinuous at all point in $\mathcal{O}$.

The basic assumption for the operator F are described in the following definitions. We say $A \preceq B$ for $A, B \in \mathbb{M}^n$, the set of real matrices in $\mathbb{R}^n \times \mathbb{R}^n$, if $A - B$ is nonpositive definite (resp. $A \succeq B$ if $A - B$ is nonnegative definite). A matrix M is nonpositive definite if all of its eigenvalues are nonnegative.

Definition 1. *We say that*

(F1) F *is* degenerate elliptic *if* $X \mapsto F(x, r, p, X)$ *is non-increasing, i.e.,* $A \preceq B$ *implies* $F(x, r, p, B) \leq F(x, r, p, A)$*,*

(F2) F *is* proper *if it is degenerate and* $r \mapsto F(x, r, p, X)$ *is non-decreasing, i.e.,* $s \leq r$ *and* $B \succeq A$ *then* $F(x, s, p, B) \leq F(x, r, p, A)$*.*

Example 2. *The linear equation*

$$-\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x)u - f(x) = 0 \qquad \text{in } \Omega$$

can be written as $F(x, r, p, X) = 0$ *where*

$$F(x, r, p, X) = -\mathrm{Trace}(A(x)X) + \langle b(x), p \rangle - c(x)r - f(x). \tag{2.1.2}$$

Definition 2 (Viscosity solution)**.** *We say that* $u \in \mathrm{USC}(\mathcal{O})$ *is a viscosity subsolution of* (2.1.1) *if for every* $x \in \mathcal{O}$ *and every* $\varphi \in C^\infty(\mathbb{R}^n)$ *such that* $u - \varphi$ *has a local maximum at* x *with respect to* $\mathcal{O}$ *then*

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0.$$

Similarly, we say that $u \in \mathrm{LSC}(\mathcal{O})$ *is a viscosity supersolution of* (2.1.1) *if for every* $x \in \mathcal{O}$ *and every* $\varphi \in C^\infty(\mathbb{R}^n)$ *such that* $u - \varphi$ *has a local minimum at* x *with respect to* $\mathcal{O}$ *then*

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq 0.$$

We say that u *is a viscosity solution to* (2.1.1) *if* u *is both a viscosity subsolution and a viscosity supersolution. We often say that* $u \in \mathrm{USC}(\mathcal{O})$ *(resp.* $\mathrm{LSC}(\mathcal{O})$*) is a viscosity solution to* $F(x, u, Du, D^2u) \leq 0$ *(resp.* $F(x, u, Du, D^2u) \leq 0$*) in* $\mathcal{O}$ *if* u *is a viscosity subsolution (resp. viscosity supersolution) of* (2.1.1)*.*

Remark 1. We will use subsolution (supersolution) when refering to viscosity subsolution (viscosity supersolution).

***Remark* 2.** If $u-\varphi$ has a maximum (resp., minimum) at x_0, we say that u is touched from above (resp. from below) by φ at x_0. It is intuitive to visualize the picture of two subsolution test and supersolution test. In some sense, we test the subsolution at a point by touching it from above by a smooth test function at that point, then perform the equation test on the test function.

A similar definition can be given for time-dependent problem, which we introduce briefly in Section 2.6. We refer to [41, 39, 50, 111] for more details on time-dependent problem. The two classes of equation that we will focus on are:

- First-order monotone equation $F(x,r,p,X)=\lambda r+H(x,p)$ where $\lambda\geq 0$ and H is called the Hamiltonian.
- Second-order equation $F(x,r,p,X)=\lambda r+H(x,p)-\gamma\,\mathrm{tr}(X)$ where $\gamma,\lambda\geq 0$. A typical case is the viscous case where $X=\mathbb{I}$.

2.1.2 Semijets

For $x,y\in\mathbb{R}^n$, we denote by $x\cdot y=\langle x,y\rangle$ the dot product of x and y. It is convenient to introduce the generalized notion of gradient for viscosity solutions as follows.

Definition 3. *For a real valued function $w(x)$ define for $x\in\mathcal{O}$ where $\mathcal{O}$ is locally compact in $\mathbb{R}^n$, we define the semi super-jet and w at x as*

$$J^{2,+}_{\mathcal{O}}w(x)=\left\{\begin{array}{l}(p,X)\in\mathbb{R}^n\times\mathbb{S}^n:\\ \limsup\limits_{y\to x}\dfrac{w(y)-w(x)-p\cdot(y-x)-\frac{1}{2}(y-x)\cdot X(y-x)}{|y-x|^2}\leq 0\end{array}\right\}.$$

Similarly, we define the semi sub-jet of w at x as $J^{2,-}_{\mathcal{O}}w(x)=-J^{2,+}_{\mathcal{O}}(-w)(x)$. The closures of semi jets are defined as follows.

$$\overline{J}^{2,\pm}_{\mathcal{O}}w(x)=\Big\{(p,X),\exists\, x_k\in\mathcal{O},\ (p_k,X_k)\in J^{2,\pm}_{\mathcal{O}}w(x_k)\quad\text{and}\quad(p_k,X_k)\to(p,X)\Big\}.\tag{2.1.3}$$

***Remark* 3.** Some authors (see [39] for example) write semijets slightly different as a triple $(w(x),p,X)$ where $(p,X)\in J^{2,\pm}_{\mathcal{O}}w(x)$ in Definition 3. We will use them interchangeably in this thesis.

***Remark* 4.** The definition can be modified to require only $\varphi\in C^2(\mathcal{O})$ for second-order equation or $\varphi\in C^1(\mathcal{O})$ for first-order equation ([39]).

***Remark* 5.** If $u:\mathcal{O}\to\mathbb{R}$ is twice differentiable at x in the interior of $\mathcal{O}$ then

$$J^{2,+}_{\mathcal{O}}u(x)\cap J^{2,-}_{\mathcal{O}}u(x)=\{Du(x),D^2u(x)\}.$$

A geometric way of visualizing semijets of u is by using smooth functions that touch u from above and below. If $\varphi\in C^2(\mathcal{O})$ and $u-\varphi$ has a local maximum relative to $\mathcal{O}$ at $x_0\in\mathcal{O}$ then

$$u(x)\leq u(x_0)+D\varphi(x_0)\cdot(x-x_0)+\frac{1}{2}\big\langle D^2\varphi(x_0)(x-x_0),(x-x_0)\big\rangle+\circ(|x-x_0|^2)\tag{2.1.4}$$

as $x \to x_0$ and $x \in \mathcal{O}$. In other words, $u - \varphi$ has a local maximum at x_0 means

$$(D\varphi(x_0), D^2\varphi(x_0)) \in J^{2,+}_{\mathcal{O}} u(x_0).$$

The converse is also true (see [39]), which enables us to use either semijets as as an equivalent definition of viscosity solutions (see Theorem 2.1.3 below).

Proposition 2.1.1. *Let $u : \mathcal{O} \to \mathbb{R}$ with $\mathcal{O}$ is open.*

- $J^{2,+}_{\mathcal{O}} u(x) = \{(D\varphi(x), D^2\varphi(x)) : \varphi \in C^2(\mathbb{R}^n)$ *and $u - \varphi$ has a local max at x_0*$\}$.
- $J^{2,-}_{\mathcal{O}} u(x) = \{(D\varphi(x), D^2\varphi(x)) : \varphi \in C^2(\mathbb{R}^n)$ *and $u - \varphi$ has a local min at x_0*$\}$.

Theorem 2.1.2. *$u \in \mathrm{USC}(\mathcal{O}; \mathbb{R})$ is a subsolution to (2.1.1) if and only if $F(x, r, p, X) \leq 0$ for all $(p, X) \in J^{2,+}_{\mathcal{O}} u(x)$ and for all $x \in \mathcal{O}$. Similarly, $u \in \mathrm{LSC}(\mathcal{O}; \mathbb{R})$ is a supersolution to (2.1.1) if and only if $F(x, r, p, X) \geq 0$ for all $(p, X) \in J^{2,-}_{\mathcal{O}} u(x)$ and for all $x \in \mathcal{O}$.*

***Remark* 6.** The definition can also be stated with the closure of semijets $\overline{J}^{2,\pm}_{\mathcal{O}}$ in place of $J^{2,\pm}_{\mathcal{O}}$ as we assume F is continuous. Using closure of semijets is more convenient as they are closed under taking limit ([39]).

2.1.3 First order equations

For the first-order equation, semijets become subdifferential and superdifferential.

Definition 4. *For a real valued function $w : \mathcal{O} \to \mathbb{R}$, we define the super-differential and sub-differential of w at x as*

$$J^{1,+} w(x) = D^+ w(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{w(y) - w(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\},$$
$$J^{1,-} w(x) = D^- w(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{w(y) - w(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}.$$

Theorem 2.1.3. *Assume that $F(x, r, p, X) = F(x, r, p)$, then*

- *$u \in \mathrm{USC}(\mathcal{O}; \mathbb{R})$ is a viscosity subsolution to (2.1.1) if and only if*

$$F(x, u(x), p) \leq 0 \quad \textit{for all } p \in D^+ u(x) \textit{ and for all } x \in \mathcal{O}.$$

- *$u \in \mathrm{LSC}(\mathcal{O}; \mathbb{R})$ is a viscosity supersolution to (2.1.1) if and only if*

$$F(x, u(x), p) \geq 0 \quad \textit{for all } p \in D^- u(x) \textit{ and for all } x \in \mathcal{O}.$$

***Remark* 7.** Observe that $(p, X) \in J^{2,+} u(x)$ implies $p \in D^+ u(x)$, but the converse may fail: $J^{2,+} u(x)$ may be empty while $D^+ u(x)$ is nonempty ([39]).

***Remark* 8.** We also write $(x, r, p, X) \in J^{2,+}_{\mathcal{O}} u(x)$ or $(u(x), p, X) \in J^{2,+}_{\mathcal{O}} u(x)$ instead of $(p, X) \in J^{2,+}_{\mathcal{O}} u(x)$ only to indicate that $r = u(x)$. This notation will come in handy later.

We refer the readers to [11, 39, 50, 111] for the equivalent definition of viscosity solution using semijets (or super-differential and sub-differential for the first-order equation).

2.1.4 Discontinuous viscosity solutions

In all the previous definitions of viscosity solutions we need to start with upper semicontinuous functions for subsolutions or lower semicontinuous functions for supersolutions. We state the definition of viscosity solutions without these assumptions using the notions of semicontinuous envelopes.

Definition 5. *Let $u : \mathcal{O} \to \mathbb{R}$.*

- *The upper semicontinuous envelope of u is the smallest upper semicontinuous function that is pointwise greater than or equal to u, and is defined by*

$$u^*(x) := \limsup_{\mathcal{O}\ni y\to x} u(y).$$

- *The lower semicontinuous envelope of u is the greatest lower semicontinuous function that is pointwise smaller than or equal to u, and is defined by*

$$u_*(x) := \liminf_{\mathcal{O}\ni y\to x} u(y),$$

It is clear that $u^* = -(-u)^*$ and $u_* \leq u \leq u^*$, and $u^* = u = u_*$ if and only if u is continuous.

Definition 6 (Discontinuous viscosity solutions)**.** *Given $u : \mathcal{O} \to \mathbb{R}$, we say that u is a viscosity subsolution of* (2.1.1) *if u^* is a subsubsolution and likewise, u is a supersolution if u_* is a supersolution. We say u is a solution if u_* is a subsolution and u^* is a supersolution.*

In what follows, we will summarize the stability and well-posedness theory for Dirichlet boundary value problem with a key ingredient, the comparison principle. Then, we switch gears to the state-constraint boundary condition and focus mostly on the first-order equation. We will turn back to the second-order equation with state-constraint in the last chapter. We refer to [41] for a more in dept introduction to the theory in the general setting and [111, 8] for the first-order case.

2.2 Stability of viscosity solutions

We will assume further that $\mathcal{O}$ is *open*, locally compact in this section. We state two basic stability results:

(i) If $\mathcal{F}$ is a collection of subsolutions of $F(x,u,Du,D^2u) = 0$ in $\mathcal{O}$ then $u^0 = \left(\sup_{u\in\mathcal{F}} u\right)^*$ is another subsolution provided that u^0 is finite on $\mathcal{O}$.

(ii) If u_n is a subsolution of $F_n(x,u_n,Du_n,D^2u_n) = 0$ in $\mathcal{O}$ for $n = 1,2,\ldots$, and $u_n \to u, F_n \to F$ in a suitable sense, then u is a subsolution of $F(x,u,Du,D^2u) = 0$ in $\mathcal{O}$.

Let us emphasize the importance of these properties. The first property is crucial in showing the existence by Perron's method which will see later. In short, the Perron's method says that the *maximal subsolution* is also so supersolution. The second property is remarkable (see [39]) in the sense that it produces a subsolution of the limit problem from an arbitrary sequence of subsolutions of approximate problems, without any assumption on the derivatives.

Proposition 2.2.1. *Suppose $\mathcal{F}$ is a nonempty collection of subsolutions of* (2.1.1)*. Let $u = \sup\{v : v \in \mathcal{F}\}$ and assume that u^* is finite on $\mathcal{O}$, then u^* a viscosity subsolution of* (2.1.1)*.*

Proof. By definition $u^* \in \mathrm{USC}(\overline{\mathcal{O}})$ and every function in $\mathcal{F}$ is also in $\mathrm{USC}(\overline{\mathcal{O}})$. Let $x_0 \in \mathcal{O}$ is an interior point and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u^* - \varphi$ has a local maximum with respect to $\mathcal{O}$ at x_0. As an interior point, there exists $r > 0$ such that

$$u^*(x) - \varphi(x) \le u^*(x_0) - \varphi(x_0) \qquad \text{for all } x \in \overline{B(x,r)} \subset \mathcal{O}.$$

Define $\psi(x) = \varphi(x) - \varphi(x_0) + u^*(x_0) + |x - x_0|^4$ for $x \in \mathcal{O}$ then $\psi(x_0) = u^*(x_0)$ and

$$u^*(x) - \psi(x) \le -|x - x_0|^4 < 0 \qquad \text{for all } x \in \overline{B(x_0,r)} \backslash \{x_0\}.$$

In other words, $u^* - \psi$ has a strict local zero maximum at $x = x_0$ with respect to $\mathcal{O}$. By definition of u^*, there exists a sequence $x_k \in \mathcal{O}$ such that $x_k \to x_0$ and $u(x_k) \to u^*(x_0)$. We can find $u_k \in \mathcal{F}$ such that

$$u(x_k) - \frac{1}{k} \le u_k(x_k) \le u(x_k)$$

Since $u_k \in \mathrm{USC}(\overline{\mathcal{O}})$, $u_k - \psi$ attains its maximum over $\overline{B(x,r)} \cap \overline{\mathcal{O}}$ at some $y_k \in \overline{B(x_0,r)} \cap \overline{\mathcal{O}}$. We obtain

$$\begin{aligned} u(x_k) - \frac{1}{k} - \psi(x_k) &\le u_k(x_k) - \psi(x_k) \\ &\le u_k(y_k) - \psi(y_k) \\ &\le \ \ u(y_k) - \psi(y_k) \\ &\le u^*(y_k) - \psi(y_k) \le -|y_k - x_0|^4. \end{aligned}$$

Let $k \to \infty$, using the fact that $u(x_k) \to u^*(x_0)$ we deduce that $y_k \to x_0$. Thus for k large enough $y_k \in B(x_0,r) \subset \mathcal{O}$ and $u_k - \psi$ has a local maximum with respect to $\mathcal{O}$ at y_k, the subsolution test gives us

$$F\Big(y_k, u_k(y_k), D\psi(y_k), D^2\psi(y_k)\Big) \le 0.$$

Let $k \to \infty$, observe that the nested inequality above also gives us $u_k(x_k) \to u^*(x_0)$, hence

$$F\Big(x_0, u^*(x_0), D\psi(x_0), D^2\psi(x_0)\Big) \le 0.$$

Since $D\psi(x_0) = D\varphi(x_0)$ and $D^2\psi(x_0) = D^2\varphi(x_0)$, we complete the subsolution test for u^* at x_0. $\square$

Let us define the proper limit before proving the stability in terms of sequences.

Definition 7. *Let $u_n : \mathcal{O} \to \mathbb{R}$ for $n = 1, 2, \ldots$. The smallest function $\overline{u}$ such that if $x_n \to x$ and $x_n \in \mathcal{O}$ then $\limsup_{n\to\infty} u_n(x_n) \le \overline{u}(x)$ is defined by*

$$\overline{u}(x) = \limsup_{\mathcal{O} \ni y \to x,\ n \to \infty} u_n(y) = \lim_{m\to\infty} \sup\left\{u_n(y) : n \ge m, y \in \mathcal{O}, |y - x| \le \frac{1}{m}\right\};$$

We denote

$$\overline{u} = \limsup_{n\to\infty} {}^*u_n.$$

In the opposite sense, we define

$$\liminf_{n\to\infty} {}_*u_n = -\limsup_{n\to\infty} {}^*(-u_n).$$

Proposition 2.2.2. *Let u_n be a subsolution of a proper equation $F_n\left(x, u(x), Du_n(x), D^2u_n(x)\right) = 0$ in $\mathcal{O}$. Let $u = \limsup_{n\to\infty} {}^*u_n$ and F be proper such that*

$$F \leq \liminf_{n\to\infty} {}_*F_n.$$

If u is finite, then it is a subsolution to $F\left(x, u, Du, D^2u\right) = 0$ in $\mathcal{O}$. In particular, if $u_n \to u$ and $F_n \to F$ locally uniformly, then u is a subsolution of $F\left(x, u, Du, D^2u\right) = 0$ in $\mathcal{O}$.

Proof. Let $x \in \mathcal{O}$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has local a maximum with respect to $\mathcal{O}$ at x_0. Let $\psi(x) = u(x_0) - \varphi(x_0) + \varphi(x) + |x - x_0|^4$ we deduce that $u - \psi$ has a strict local maximum with respect to $\mathcal{O}$ at x_0. In other words, there exists $r > 0$ such that

$$u(x) - \psi(x) = u(x) - \varphi(x) - |x - x_0|^4 \qquad \text{for all } x \in \overline{B(x_0, r)} \subset \mathcal{O}$$

and thus $u - \psi < 0$ on $\overline{B(x_0, r)}$ unless $x = x_0$. By definition of u, there exists a sequence $n_k \in \mathbb{N}$ such that

$$u(x_0) \leq \sup\left\{u_n(y) : n \geq n_k : y \in \mathcal{O}, |y - x_0| \leq \frac{1}{n_k}\right\} < u(x_0) + \frac{1}{n_k}.$$

We can find $m_k \geq n_k$ and $x_{m_k} \in \mathcal{O}$ such that $|x_{m_k} - x_0| \leq \frac{1}{m_k}$ and

$$u(x_0) - \frac{1}{m_k} \leq u_{m_k}\left(x_{m_k}\right) \leq u(x_0) + \frac{1}{n_k}.$$

Therefore $u_{m_k}\left(x_{m_k}\right) \to u(x_0)$ as $n_k \to \infty$. Let $\hat{x}_{m_k} \in \overline{B(x_0, r)}$ such that $u_{m_k} - \psi$ attains its maximum over $\overline{B(x_0, r)}$ then

$$u_{m_k}(x) - \psi(x) \leq u_{m_k}\left(\widehat{x}_{m_k}\right) - \psi\left(\widehat{x}_{m_k}\right) \qquad \text{for all } x \in \overline{B(x_0, r)}.$$

Plug in $x = x_{m_k}$ and assume $x_{m_k} \to \widehat{x}$ (passing to subsequence) for some $\widehat{x} \in \overline{B(x_0, r)}$, we deduce that

$$\begin{aligned} u_{m_k}\left(x_{m_k}\right) - \psi\left(x_{m_k}\right) &\leq u_{m_k}\left(\widehat{x}_{m_k}\right) - \psi\left(\widehat{x}_{m_k}\right) \\ u(x_0) - \psi(x_0) &\leq \liminf_{m_k\to\infty} u_{m_k}\left(\widehat{x}_{m_k}\right) - \psi(\widehat{x}) \leq u(\widehat{x}) - \psi(\widehat{x}). \end{aligned}$$

By the strict maximum of $u - \psi$ at x_0, we deduce that $\widehat{x} = x_0$ and thus $\widehat{x}_{m_k} \in B(x_0, r) \subset \mathcal{O}$ for k large. We obtain

$$F_{m_k}\left(x_{m_k}, u\left(x_{m_k}\right), D\psi\left(x_{m_k}\right), D^2\psi\left(x_{m_k}\right)\right) \leq 0.$$

let $k \to \infty$ we obtain the conclusion since $F \leq \liminf_{n\to\infty} {}_*F_n$. $\square$

***Remark* 9.** The main argument can be summarized as a strict maximum of $u - \varphi$ perturbs to maxima of $u_n - \varphi$ which converge, etc (see [39, Remark 8.5]).

Lemma 2.2.3. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. If $u : \mathcal{O} \to \mathbb{R}$ is upper semicontinuous and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a strict local maximum with respect to $\mathcal{O}$ at $x_0 \in \mathcal{O}$. Suppose u_n is a sequence of upper semicontinuous functions on $\mathcal{O}$ such that*

(i) There exists $x_n \in \mathcal{O}$ such that $x_n \to x_0$ and $u_n(x_n) \to u(x_0)$.

(ii) If $z_n \in \mathcal{O}$ and $z_n \to z$ in $\mathcal{O}$ then $\limsup_{n\to\infty} u_n(z_n) \le u(z)$.

Then there exists $\widehat{x}_n \in \mathcal{O}$ such that $u_n - \varphi$ has a local maximum with respect to $\mathcal{O}$ at $\widehat{x}_n$ and also $(\widehat{x}_n, u_n(\widehat{x}_n)) \to (x_0, u(x_0))$.

2.3 Existence via Perron's method

The Perron's method is a powerful method technique for constructing solutions. The use of Perron's method with viscosity solution is introduced by Ishii in [64]. The idea is the maximal subsolution (constructed using the envelope) is a supersolution. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, we consider the equation (2.1.1). Let $w \in \mathrm{LSC}(\overline{\mathcal{O}})$ be a supersolution, we define

$$\mathcal{F} = \left\{ v \in \ \mathrm{USC}(\overline{\mathcal{O}}) : v \text{ is a subsolution of (2.1.1) and } v \le w \text{ on } \overline{\mathcal{O}} \right\} \tag{2.3.1}$$

and

$$u(x) = \sup\{v(x) : v \in \mathcal{F}\}. \tag{2.3.2}$$

Proposition 2.3.1. *Suppose $\mathcal{F}$ is nonempty. Then the upper semicontinuous function u^* is a viscosity subsolution of* (2.1.1).

Proposition 2.3.1 follows from stability of u (Proposition 2.2.1), thus u is a candidate for a solution of (2.1.1). We need to show that u_* is a supersolution.

Proposition 2.3.2. *If $u \in \mathcal{F}$ and u_* is not a supersolution of* (2.1.1)*, then there exists $v \in \mathcal{F}$ such that $v(x) > u(x)$ for some $x \in \mathcal{O}$.*

Proof. If u_* is not a supersolution of (2.1.1) then there exists $x_0 \in \mathcal{O}$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u_* - \varphi$ has a local minimum with respect to $\mathcal{O}$ at x_0 such that

$$F\left(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) < 0. \tag{2.3.3}$$

By continuity we can find $\delta > 0$ such that

$$F\left(x_0, u_*(x_0) + \delta, D\varphi(x_0), D^2\varphi(x_0)\right) + 2\kappa < 0$$

for some $\kappa > 0$. By adding a constant to φ, we can assume $\varphi(x_0) = u_*(x_0) \le w(x_0)$ and $u_* - \varphi$ has a local minimum at x_0 means $u_* \ge \varphi$ locally near x_0.

If $u_*(x_0) = w(x_0)$ then $w(x_0) = \varphi(x_0)$ and $w(x) \ge u(x) \ge u_*(x) \ge \varphi(x)$ in a neighborhood of x_0, i.e., $w - \varphi$ has a local minimum at x_0, a contradiction to (2.3.3) since

w is a supersolution. Therefore $u_*(x_0) = \varphi(x_0) < w(x_0)$. Assume $\varphi(x_0) + 2\varepsilon = w(x_0)$ for $\varepsilon > 0$, there exists $r > 0$ such that $B(x_0, r) \subset \mathcal{O}$ and

$$\varphi(x) + \varepsilon < w(x) \qquad \text{on } B(x_0, r).$$

This is true since otherwise there exists a sequence x_{r_k} where $r_k \to 0$ and $|x_{r_k} - x_0| < r_k$ such that $\varphi(x_{r_k}) + \varepsilon \geq w(x_{r_k})$, which implies that $\varphi(x_0) + \varepsilon \geq w(x_0)$ since $w \in \mathrm{LSC}(\mathcal{O})$. We can shrink r and ε such that

$$\begin{cases} F\left(x, \varphi(x) + \delta, D\varphi(x), D^2\varphi(x)\right) + \varepsilon < 0, \\ \varphi(x) + \varepsilon < w(x) \end{cases}$$

for $x \in B(x_0, r)$. Define

$$\widehat{\varphi}(x) = \varphi(x) + \eta \left(\left(\frac{r}{2}\right)^4 - |x - x_0|^4 \right), \qquad x \in B(x_0, r)$$

where $\eta > 0$ is small such that $\widehat{\varphi}(x) \leq w(x)$ and $F(x, \widehat{u}(x), D\widehat{\varphi}(x), D^2\widehat{\varphi}(x)) \leq 0$ in $B(x_0, r)$. Define

$$\widehat{v}(x) = \begin{cases} \max\{u(x), \varphi(x)\}, & x \in B(x_0, r), \\ u(x), & x \notin B(x_0, r). \end{cases}$$

If $\frac{r}{2} \leq |x - x_0| \leq r$ then $\widehat{\varphi}(x) \leq \varphi(x) \leq u(x)$, thus $\widehat{v}(x) = u(x)$ for $x \notin B\left(x_0, \frac{r}{2}\right)$, therefore $\widehat{v}$ is a subsolution of (2.1.1) in $\mathcal{O}$ and $\widehat{v} \leq w$ on $\mathcal{O}$, thus $\widehat{v} \in \mathcal{F}$. To show that $\widehat{v}(x) > u(x)$ for some $x \in \mathcal{O}$, take a sequence $x_n \in \mathcal{O}$ such that $u_*(x_0) = \lim_{n\to\infty} u(x_n)$. Since $\widehat{v} \geq \varphi$ in $B\left(x_0, \frac{r}{2}\right)$, we have

$$\liminf_{n\to\infty} \widehat{v}(x_n) \geq \widehat{\varphi}(x_0) = u_*(x_0) + \eta \left(\frac{r}{2}\right)^4.$$

Therefore for n large enough $\widehat{v}(x_n) > u(x_n)$, which completes the proof. □

We are ready to state the Perron's method.

Proposition 2.3.3. *Assume there exists a supersolution $w \in \mathrm{LSC}(\mathcal{O})$ of (2.1.1) and $\mathcal{F}$ as defined in (2.3.1) is nonempty. Then u, defined in (2.3.2), is a viscosity solution of (2.1.1) in $\mathcal{O}$.*

Proof. From Proposition 2.3.1 we have u^* is a subsolution of (2.1.1) in $\mathcal{O}$ and $u^* \in \mathcal{F}$. By definition of u we also have $u^* \leq u$, thus together with $u \leq u^*$ we deduce that $u = u^*$ and thus $u \in \mathcal{F}$. If u_* is not a supersolution then by Proposition 2.3.2 there exists $x \in \mathcal{O}$ and $v \in \mathcal{F}$ such that $v(x) > u(x)$, a contradiction to the definition of u. Therefore u_* is a supersolution of (2.1.1) in $\mathcal{O}$ and thus u is a viscosity solution of (2.1.1) in $\mathcal{O}$ (in the sense of Definition 6). □

Remark 10. We note that at this point we have no information about the continuity of the viscosity solution u we just constructed except that $u = u^*$ is upper semicontinuous. Extra information is needed to gain more regularity on u.

2.4 Uniqueness with Dirichlet boundary condition

We will focus on the equation of the form $F(x, r, p, X) = \lambda r + H(x, p, X)$. Under certain assumptions, we have the existence of solution via the so-called Perron's method. Then, by a comparison principle we obtain the uniqueness of solution. We start with the comparison principle first. Generally it is usually stated in the following form. We use Ω instead of $\mathcal{O}$ to denote an open subset of $\mathbb{R}^n$ in this section to emphasize that a such a comparison principle is usually accompanied by a boundary condition in some senses.

Comparison principle with Dirichlet boundary condition. Assume that $u \in \mathrm{USC}(\overline{\Omega})$ is a subsolution to $F(x, u, Du, D^2u) \leq 0$ in Ω and $v \in \mathrm{LSC}(\overline{\Omega})$ is a supersolution to $F(x, v, Dv, D^2v) \geq 0$ in Ω. If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.

In most textbooks it is usually stated and proved for first-order equation, using the celebrated *doubling variable* technique. It is intuitive to prove the result using the language of *touching from below or above* (see [39, 50, 111] and the references therein). The comparison principle for second-order equation is more complicated and is usually stated using the semijets notation ([41]). In what follows, we present such a comparison principle using the notions of touching from below or above. Most of the materials are taken from [39] and [20].

2.4.1 A heuristic argument

Let us start with a heuristic argument by assuming $F(x, r, p, X) = \lambda r + H(x, p, X)$ for some $\lambda > 0$, and $u, v \in \mathrm{C}^2(\overline{\Omega})$ be subsolution and supersolution with $u \leq v$ on $\partial\Omega$. Assume $u - v$ has a maximum over $\overline{\Omega}$ at $x_0 \in \Omega$ with $u(x_0) - v(x_0) > 0$, we will derive a contradiction. We have

$$Du(x_0) = Dv(x_0) = p_0 \qquad \text{and} \qquad D^2u(x_0) \preceq D^2v(x_0). \tag{2.4.1}$$

Since they are classical solutions, we deduce that

$$\begin{aligned} \lambda u(x_0) + F(x_0, p, D^2u(x_0)) &\leq 0, \\ \lambda v(x_0) + F(x_0, p, D^2v(x_0)) &\geq 0. \end{aligned}$$

Therefore

$$0 < \lambda\big(u(x_0) - v(x_0)\big) \leq F(x_0, p, D^2v(x_0)) - F(x_0, p, D^2u(x_0)) \leq 0$$

since F is degenerate since $D^2u(x_0) \preceq D^2v(x_0)$ and thus we have a contradiction. To make the argument rigorous, the doubling variable method is employed. For each $\varepsilon > 0$ let us define the auxiliary functional

$$\Phi(x, y) = u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon}, \qquad (x, y) \in \overline{\Omega} \times \overline{\Omega}.$$

Assume it has a maximum over $\overline{\Omega} \times \overline{\Omega}$ at an interior point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$, we now can use $(2\varepsilon)^{-1}|x - y|^2$ as a common test function to perform the subsolution test and supersolution test. This is the main idea for the proof of comparison principle for the first-order equation. Another difficulty with second order-equation is the fact that, we need more information about the semijets at the maximum point. Indeed, let us simplify furthere that $F(x, p, X) = F(p, X) - f(x)$ for a continuous function f, from the subsolution test and supersolution test are have

$$u(x_\varepsilon) + F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \frac{1}{\varepsilon}I\right) \leq f(x_\varepsilon), \qquad \text{and} \qquad v(y_\varepsilon) + F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -\frac{1}{\varepsilon}I\right) \geq f(y_\varepsilon).$$

However, as $I \geq -I$ we cannot derive any useful comparison. In fact, the full Hessian of Φ as $(x_\varepsilon, y_\varepsilon)$ is nonpositive, i.e.,

$$\begin{bmatrix} D^2u(x_\varepsilon) & 0 \\ 0 & -D^2v(y_\varepsilon) \end{bmatrix} \preceq \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}. \tag{2.4.2}$$

This implies $D^2u(x_\varepsilon) \preceq D^2v(y_\varepsilon)$, however we have to do some work to make the right matrices appear in the subsolution test and supersolution test since in general u, v are not smooth. We summarize the strategy as follows.

- u, v are not smooth, thus one can use the doubling variable method to tackle this problem.
- However, it turns out that to capture more useful information to deal with the second-order case as in (2.4.2), we have to make sense of the appropriate $D^2u(x_\varepsilon)$ and $D^2v(y_\varepsilon)$. Therefore, a smoothing step is needed before performing the doubling variables.
- The sup-convolution and inf-convolution are appropriate smoothing steps. From u, v we obtain the sup-convolution u^ε and the inf-convolution v_ε. Then we perform the doubling variable on u^ε and v_ε instead of u and v, i.e.,

$$\Phi(x, y) = u^\varepsilon(x) - v_\varepsilon(y) - \frac{|x-y|^2}{2\eta}, \qquad (x, y) \in \overline{\Omega} \times \overline{\Omega}.$$

- In turns, a maximum principle for semiconvex functions (an analog of (2.4.1) but for nonsmooth functions) is needed to make sense of D^2u and D^2v in (2.4.2).

2.4.2 Smoothing viscosity solutions

We will use sup-convolution and inf-convolution as a tool to smooth up viscosity solutions, since these operations preserve the properties of subsolution and supersolution (sup-inf stability).

Definition 8 (Sup-convolution). *Let $\mathcal{O}$ be a closed subset of $\mathbb{R}^n$, $\psi \in \mathrm{USC}(\mathcal{O}; [-\infty, +\infty))$ such that $\psi \not\equiv -\infty$, we define for $\varepsilon > 0$*

$$\psi^\varepsilon(\xi) = \sup_{z \in \mathcal{O}} \left(\psi(z) - \frac{1}{2\varepsilon}|z - \xi|^2\right) \qquad \xi \in \mathbb{R}^n.$$

Definition 9 (Inf-convolution)**.** *Let $\mathcal{O}$ be a closed subset of $\mathbb{R}^n$, $\varphi \in \mathrm{LSC}(\mathcal{O}; (-\infty, +\infty])$ such that $\psi \not\equiv +\infty$, we define for $\kappa > 0$*

$$\psi_\varepsilon(\xi) = \inf_{z\in\mathcal{O}} \left(\psi(z) + \frac{1}{2\varepsilon}|z-\xi|^2\right) \qquad \xi \in \mathbb{R}^n.$$

Lemma 2.4.1. *For $\psi, \varphi \in \mathrm{USC}(\mathcal{O}; [-\infty, +\infty))$:*

(i) If $\psi \le \varphi$ then $\psi^\varepsilon \le \varphi^\varepsilon$ and $\psi \le \psi^\varepsilon$ and $\psi^\varepsilon(\xi) + \frac{1}{2\varepsilon}|\xi|^2$ is convex.

(ii) $\lim_{\varepsilon\to 0} \psi^\varepsilon(\xi) = \psi(\xi)$ for $\xi \in \mathbb{R}^n$, provided that $\limsup_{z\in\mathcal{O}, |z|\to\infty} \frac{\psi(z)}{|z|^2}$ is finite.

2.4.3 A maximum principle

Lemma 2.4.2 (Magic property)**.** *Let $u \in \mathrm{USC}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u^\varepsilon - \varphi$ has a local maximum at x_0. If $x_\varepsilon \in \mathbb{R}^n$ which realizes $u^\varepsilon(x_0) = u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - x_0|^2$ then*

$$u(x) - \varphi(x + (x_0 - x_\varepsilon)) \text{ has a local maximum at } x_\varepsilon \qquad \text{and} \qquad D\varphi(x_0) = \frac{x_\varepsilon - x_0}{\varepsilon}.$$

Remark 11*.* In the language of semijets, Lemma 2.4.2 gives a *direction to go from u^ε to u* in terms of finding semijets of u. Specifically, let $\psi(x) = \varphi(x + (x_0 - x_\varepsilon))$ then it reads

$$\boxed{\begin{aligned} \Big(x_0, D\varphi(x_0), D^2\varphi(x_0)\Big) \in J^{2,+}u^\varepsilon(x_0) \qquad &\Longrightarrow \qquad \Big(x_\varepsilon, D\psi(x_\varepsilon), D^2\psi(x_\varepsilon)\Big) \in J^{2,+}u(x_\varepsilon) \\ &\Longrightarrow \qquad \Big(x_\varepsilon, D\varphi(x_0), D^2\varphi(x_0)\Big) \in J^{2,+}u(x_\varepsilon) \end{aligned}}$$

where $x_\varepsilon = \operatorname{argmax}\left((u(x) - \frac{1}{2\varepsilon}|x - x_0|^2\right)$ such that $|x_\varepsilon - x_0| \to 0$ with some rate provided we know more about regularity of u. This is significant, as a magic property since passing $\varepsilon \to 0$ we obtain

$$\boxed{\Big(x_0, D\varphi(x_0), D^2\varphi(x_0)\Big) \in \overline{J}^{2,+}u(x_0).}$$

As subsolution test and supersolution test work with the closure of semijets instead of semijets only, this is magic!

Proof of Lemma 2.4.2. Let $r > 0$ such that $u^\varepsilon(x) - \varphi(x) \le u^\varepsilon(x_0) - \varphi(x_0)$ for $x \in B(x_0, r)$, we have

$$\begin{aligned} \left(u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - x_0|^2\right) - \varphi(x_0) &= u^\varepsilon(x_0) - \varphi(x_0) \\ &\ge u^\varepsilon(x) - \varphi(x) \ge \left(u(y) - \frac{1}{2\varepsilon}|y - x|^2\right) - \varphi(x), \qquad y \in \mathbb{R}^n. \end{aligned} \tag{2.4.3}$$

In equation (2.4.3) we do the following.

1. Choose $y = x_\varepsilon$ then $x \mapsto \varphi(x) + \frac{1}{2\varepsilon}|x - x_\varepsilon|^2$ has a local minimum at x_0, thus $D\varphi(x_0) = \frac{x_\varepsilon - x_0}{\varepsilon}$.

2. Choose $y \in \mathbb{R}^n$ such that $y - x = x_\varepsilon - x_0$, then

$$u(x_\varepsilon) - \varphi(x_0) \geq u(y) - \varphi(y + x_0 - x_\varepsilon)$$

for all $y \in \mathbb{R}^n$ such that $x = y + (x_0 - x_\varepsilon) \in B(x_0, r)$, i.e., $y \in B(x_\varepsilon, r)$. In other words, $y \mapsto u(y) - \varphi(y + (x_0 - x_\varepsilon))$ has a local minimum at x_ε.

□

2.4.4 Maximum principle for semiconvex functions

Motivated from the heuristic discussion above, one has to smooth up u, v and then find an appropriate maximum principle, i.e., an analog of the smooth version: $u - v$ has a max at x and $u, v \in C^2$ then $D^2 u \preceq D^2 v$.

Definition 10 (Semiconvex and semiconcave functions)**.** *Let $u : \mathcal{O} \to \mathbb{R}$.*

(i) u is semiconvex if there exists $\lambda > 0$ such that $u(x) + \frac{|x|^2}{2\lambda}$ is convex.

(ii) u is semiconcave if there exists $\lambda > 0$ such that $u(x) - \frac{|x|^2}{2\lambda}$ is concave.

It is clear that that the modified function $u^\varepsilon, u_\varepsilon$ given by sup-convolution and inf-convolution are semiconvex and semiconcave functions. We state the Jensen's Lemma, a key ingredient in establishing the maximum principle for semiconvex functions. Intuitively, Jensen's Lemma says that near a strict maximum of a semiconvex function, there are a lot of points with very small gradients, so the function in some sense *round* near its maximum. Two important ingredients are the Area formula (see [51]) and Alexandroff's theorem.

Theorem 2.4.3 (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map. Then for every Lebesgue measurable set $K \subset \mathbb{R}^n$ we have*

$$\int_{f(K)} \# \left(K \cap f^{-1}(\{x\}) \right) \, dx = \int_K |\det Df(x)| \, dx.$$

Theorem 2.4.4 (Alexandroff's theorem, [2])**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then for almost every $x \in \mathbb{R}^n$ there is $Df(x) \in \mathbb{R}^n$ and a symmetric $(n \times n)$ matrix $D^2 f(x)$ such that*

$$\lim_{y \to x} \frac{|f(y) - f(x) - Df(x)(y - x) - \frac{1}{2}(y - x)^T D^2 f(x)(y - x)|}{|y - x|^2} = 0.$$

Now we are ready to state Jensen's Lemma.

Theorem 2.4.5 (Jensen's Lemma)**.** *Assume $u : \mathbb{R}^n \to \mathbb{R}$ is semiconvex that has a strict local maximum at $x_0 \in \mathbb{R}^n$ then for $r > 0$ small and $\delta > 0$ the set*

$$K = \left\{ y \in B(x_0, r) : \ \exists\, p \in B(0, \delta) \text{ such that the linear line } p \cdot (x - x_0) \text{ touches } u \text{ from above at } y \right\}$$

has positive measure.

Proof. Without loss of generality, we can assume $u(x_0) = 0$. Let $\lambda > 0$ be a constant such that $u(x) + \frac{\lambda}{2}|x|^2$ is convex. Since u has a strict local maximum at x_0, there exists $r > 0$ such that $u(x_0) - u(x) > 0$ for $x \in \overline{B(x_0, r)}$. Let

$$\eta = \min\left\{u(x_0) - u(x) : x \in \partial B(x_0, r)\right\} > 0.$$

Let $\delta > 0$ and $p \in B(0, \delta)$. Consider $\varphi(x) = u(x) - p \cdot (x - x_0)$ for $x \in \overline{B(x_0, r)}$, we see that $\varphi(x_0) = 0$ while

$$\varphi(x) - \varphi(x_0) = u(x) - u(x_0) - p \cdot (x - x_0) \leq -\eta + \delta r < 0$$

if $\delta \leq \delta_0 = \frac{r}{2\eta}$. There fore $\varphi(x)$ has a local maximum over $\overline{B(x_0, r)}$ at an interior point $y \in B(x_0, r)$, hence by definition $y \in K$.

- Heuristically, if $u \in C^2(\mathbb{R}^n)$ then $Du(y) = p$ and $-\lambda \mathbb{I}_n \preceq D^2u(y) \preceq 0$ (by convexity). Consider the map $\mathcal{F} = Du : K \to \mathbb{R}^n$, using the Area formula we have

  $$|\mathcal{F}(K)| \leq \int_{\mathcal{F}(K)} \#\left(K \cap \mathcal{F}^{-1}(\{x\})\right)\, dx$$

  since if $x \in K$ then $\#\left(K \cap \mathcal{F}^{-1}(\{x\})\right) \geq 1$. On the other hand

  $$\int_{\mathcal{F}(K)} \#\left(K \cap \mathcal{F}^{-1}(\{x\})\right)\, dx = \int_K |\det(D^2u(x))|\, dx \leq \lambda^n |K|.$$

  Therefore

  $$|Du(K)| \leq \lambda^n |K|.$$

  However, for all $p \in B(0, \delta)$ the equation $Du(y) = y$ has a solution $y \in B(x_0, r)$, thus

  $$B(0, \delta) \subset Du(K) \qquad \Longrightarrow \qquad |K| \geq \frac{|B(0, \delta)|}{\lambda^n} = \left(\frac{\delta}{\lambda}\right)^n \alpha(n) > 0$$

  where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

- For general u we use convolution to make it smooth. For $\varepsilon > 0$ let $u^\varepsilon = u * \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $u^\varepsilon \to u$ uniformly on $\overline{B(x_0, r)}$. By stability there exists $x_\varepsilon \to x_0$ such that u^ε has a local maximum at x_ε. We also have that u^ε is semiconvex with the same constant $\lambda > 0$. Indeed, for $\tau \in (0, 1)$ and $x, y \in \mathbb{R}^n$ we have

  $$u(\tau x + (1 - \tau)y) \leq \tau u(x) + (1 - \tau)u(y) + \frac{\lambda}{2}\tau(1 - \tau)|x - y|^2.$$

  Therefore

  $$\begin{aligned} &u^\varepsilon(\tau x + (1 - \tau)y) - \tau u^\varepsilon(x) - (1 - \tau)u^\varepsilon(y) \\ &\quad = \int_{\mathbb{R}^n} \Big(u(\tau x + (1 - \tau)y - \xi) - \tau u(x - \xi) - (1 - \tau)u(y - \xi)\Big)\eta_\varepsilon(\xi)\, d\xi \\ &\quad \leq \int_{\mathbb{R}^n} \frac{\lambda\tau(1 - \tau)}{2}|x - y|^2 \eta_\varepsilon(\xi)\, d\xi = \frac{\lambda\tau(1 - \tau)}{2}|x - y|^2. \end{aligned}$$

Therefore $u^\varepsilon(x) + \frac{\lambda}{2}|x|^2$ is convex as well. Define

$$K^\varepsilon = \Big\{ y \in B\,(x_0, r) : \ \exists\, p \in B(0,\delta) \text{ s.t. } \ p \cdot (x - x_\varepsilon) \text{ touches } u^\varepsilon \text{ from above at } y \Big\}.$$

We observe also that

$$u^\varepsilon(x) - u^\varepsilon(x_0) \le 2\|u^\varepsilon - u\|_{L^\infty(\overline{B(x_0,r)})} + u(x) - u(x_0) \le -\eta + 2\|u^\varepsilon - u\|_{L^\infty(\overline{B(x_0,r)})} \le -\frac{\eta}{2}$$

if $\varepsilon \le \varepsilon_0$ small enough. We can apply the previous result to obtain

$$|K^\varepsilon| \ge \left(\frac{\delta}{\lambda}\right)^n \alpha(n)$$

for a fixed small $\delta = \delta_1 > 0$ and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. We claim that

$$\Big\{ x \in \mathbb{R}^n : x \in K^{1/m} \text{ infinitely often} \Big\} \subset K.$$

Indeed, if $y \in K^{1/m}$ infinitely oftern then there exists $\varepsilon_m \to 0$ such that $y \in K^{1/m}$ for all $m \in \mathbb{N}$. There exists $p_m \in B(0,\delta)$ such that

$$u^{\varepsilon_m}(x) - p_m \cdot (x - x_{\varepsilon_m}) \le u^{\varepsilon_m}(y) - p_m \cdot (y - x_{\varepsilon_m}) \qquad \text{for all } x \in B(x_0, r).$$

Let $m \to \infty$ and $\varepsilon \to 0$ we deduce that, assuming $p_{m_j} \to p_0 \in \overline{B(0,\delta)}$ by compactness

$$u(x) - u(x_0) - p_0 \cdot (x - x_0) \le u(y) - u(x_0) - p_0 \cdot (y - x_0) \qquad \text{for all } x \in B(x_0, r).$$

Thus $y \in K$, hence

$$|K| \ge \left(\frac{\delta}{\lambda}\right)^n \alpha(n)$$

and thus the proof is complete.

□

Now we state an analog of the classical maximum principle for semiconvex functions.

Theorem 2.4.6 (Maximum principle for semiconvex functions)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be semiconvex with a strict maximum at x_0. Then there exists $x_k \to x_0$ such that u is twice differentiable at x_k, $Du(x_k) \to 0$ and $D^2u(x_k) \prec \varepsilon_k \mathbb{I}_n$ as $k \to \infty$, $\varepsilon_k \to 0$.*

Proof. Let $\tilde{u}(x) = u(x) - |x - x_0|^4$ so that $\tilde{u}$ has a strict local maximum at x_0. By Jensen's lemma, for a sequence $r_\kappa \to 0$ there exists $\delta_\kappa \to 0$ such that

$$K_\kappa = \big\{ \hat{x} \in B(x_0, r_\kappa) : \ \exists\, p \in B(0, \delta_\kappa) \text{ s.t. } \tilde{u}(x) - p \cdot (x - x_0) \text{ has a local max at } \hat{x} \big\}$$

has positive measure. By Alexandrov's Theorem, $\tilde{u}$ is twice differentiable almost everywhere, therefore we can find $\hat{x}_\kappa \in K_\kappa$ and $p_\kappa \in B(0, \delta_\kappa)$ such that $\tilde{u}$ is differentiable at $\hat{x}_\kappa$ and $D\tilde{u}(\hat{x}_\kappa) = p_\kappa$ and

$$D^2\tilde{u}(\hat{x}_\kappa) = D^2u(\hat{x}_\kappa) - 12|\hat{x}_\kappa - x_0|^2\, \mathbb{I}_n \preceq 0,$$

which implies that $D^2u(\hat{x}_\kappa) \preceq 12 r_\kappa\, \mathbb{I}_n$, and thus the proof is complete. □

2.4.5 Comparison principle

Theorem 2.4.7 (Comparison principle with discontinuous solutions)**.** *Let $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ be open bounded and $F(x,r,p,X) = \lambda r + H(x,p,X)$ from $\mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is degenerate elliptic satisfying*

$$|H(x,p,X) - H(y,p,X)| \le \omega(|x-y|(1+|p|)) \tag{2.4.4}$$

for some modulus $\omega(\cdot)$. Assume that $u \in \mathrm{USC}(\overline{\Omega})$ is a subsolution to $F(x,u,Du,D^2u) \le 0$ in Ω and $v \in \mathrm{LSC}(\overline{\Omega})$ is a supersolution to $F(x,v,Dv,D^2v) \ge 0$ in Ω. If $u \le v$ on $\partial\Omega$ then $u \le v$ in Ω.

Remark 12. The difficulty of the proof is to handle the sup-convolution and inf-convolution in the doubling variable, as well as treating the coupling term $|x-y|^2$ *at the same time*.

Proof. Note that $u - v \in \mathrm{USC}(\overline{\Omega})$, therefore it is bounded above and achieves maximum over $\overline{\Omega}$ (since Ω is bounded). Assume the contradiction that $\max_{\overline{\Omega}}(u-v) = u(\hat{x}) - v(\hat{x}) > 0$, we define the auxiliary functional

$$\Phi(x,y) = u(x) - v(y) - \frac{\alpha}{2}|x-y|^2, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{2.4.5}$$

It is clear that for α large enough, $\Phi(x,y)$ has a maximum at $(x_\alpha, y_\alpha) \in \Omega \times \Omega$ and $|x_\alpha - y_\alpha| \to 0$ and $\alpha|x_\alpha - y_\alpha|^2 \to 0$ as $\alpha \to \infty$. We do a reduction to make $(x_\alpha, y_\alpha) \sim (0,0)$ and the maximum $\Phi(x_\alpha, y_\alpha) \sim 0$ as follows. Let $p_\alpha = \alpha(x_\alpha - y_\alpha)$, we define

$$\tilde{u}(x) = u(x + x_\alpha) - u(x_\alpha) - p_\alpha \cdot x \qquad \text{and} \qquad \tilde{v}(y) = v(y + y_\alpha) - v(y_\alpha) - p_\alpha \cdot y$$

for (x,y) near $(0,0)$. It is clear that $\tilde{u}(0) = \tilde{v}(0) = 0$ and $\Phi(x + x_\alpha, y + y_\alpha) \le \Phi(x_\alpha, y_\alpha)$ for $(x,y) \sim (0,0)$ which gives us

$$u(x + x_\alpha) - v(y + y_\alpha) - \frac{\alpha}{2}|(x + x_\alpha) - (y + y_\alpha)|^2 \le u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2.$$

This implies that

$$\tilde{u}(x) - \tilde{v}(y) \le \frac{\alpha}{2}|x-y|^2$$

for (x,y) in a neighborhood of $(0,0)$. We apply the sup-convolution jointly in $\mathbb{R}^n \times \mathbb{R}^n$ to obtain

$$\tilde{u}^\varepsilon(x) - \tilde{v}^\varepsilon(y) \le (1 - 2\alpha\varepsilon)^{-1}\left(\frac{\alpha}{2}|x-y|^2\right)$$

for all $(x,y) \sim (0,0)$. Therefore the map

$$\boxed{(x,y) \mapsto \tilde{u}^\varepsilon(x) - \tilde{v}^\varepsilon(y) - (1 - 2\alpha\varepsilon)^{-1}\left(\frac{\alpha}{2}|x-y|^2\right) \text{ is semiconvex, has a maximum at } (0,0).}$$

We can use the magic property (Lemma 2.4.2), however using the Maximum principle of semi-convex functions (Theorem 2.4.6) is better as we gain extra information about second derivatives. We need to modify so that the local maximum becomes local strict maximum. We define

$$\Phi(x,y) = \tilde{u}^\varepsilon(x) - \tilde{v}^\varepsilon(y) - (1 - 2\alpha\varepsilon)^{-1}\left(\frac{\alpha}{2}|x-y|^2\right) - |x|^4 - |y|^4.$$

We have $(x,y) \mapsto \Phi(x,y)$ (semi-convex) has a strict local maximum at $(0,0)$. Thus there exists $(x_k, y_k) \to (0,0)$ so that $\tilde{u}^\varepsilon$ is differentiable at x_k, $\tilde{v}^\varepsilon$ is differentiable at y_k, $D\Phi(x_k, y_k) \to 0$ and $D^2\Phi(x_k, y_k) \prec \gamma_k \mathbb{I}_{2n}$ where $\gamma_k \to 0$. In other words, let $p^k = D\tilde{u}^\varepsilon(x_k)$ and $q^k = D\tilde{v}^\varepsilon(y_k)$ we have

$$\begin{cases} D\tilde{u}^\varepsilon(x_k) - (1-2\alpha\varepsilon)^{-1}\alpha(x_\alpha - y_\alpha) - 4|x_k|^2 x_k \to 0, \\ -D\tilde{v}^\varepsilon(y_k) + (1-2\alpha\varepsilon)^{-1}\alpha(x_\alpha - y_\alpha) - 4|y_k|^2 y_k \to 0, \end{cases} \quad \Longrightarrow \quad \begin{cases} p^k \to (1-2\alpha\varepsilon)^{-1} p_\alpha, \\ q^k \to (1-2\alpha\varepsilon)^{-1} p_\alpha, \end{cases}$$

and with $X^k = D^2\tilde{u}^\varepsilon(x_k)$ and $Y^k = D^2\tilde{v}^\varepsilon(y_k)$ we have

$$\begin{bmatrix} X^k - \alpha(1-2\alpha\varepsilon)^{-1}\mathbb{I}_n - 12|x_k|^2\mathbb{I}_n & \alpha(1-2\alpha\varepsilon)^{-1}\mathbb{I}_n \\ \alpha(1-2\alpha\varepsilon)^{-1}\mathbb{I}_n & -Y^k + \alpha(1-2\alpha\varepsilon)^{-1}\mathbb{I}_n - 12|y_k|^2\mathbb{I}_n \end{bmatrix} \prec \gamma_k \begin{bmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_n \end{bmatrix}.$$

Therefore, let $r_k = \max\{|x_k|, |y_k|\} \to 0$ then

$$\begin{bmatrix} X^k & 0 \\ 0 & Y^k \end{bmatrix} \prec \alpha(1-2\alpha\varepsilon)^{-1} \begin{bmatrix} \mathbb{I}_n & -\mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{I}_n \end{bmatrix} + (\gamma_k + 12 r_k)\mathbb{I}_{2n}.$$

Since $\tilde{u}^\varepsilon$ is semi-convex and $\tilde{v}^\varepsilon$ is semi-concave, we also have

$$-\frac{1}{\varepsilon}\mathbb{I}_{2n} \prec \begin{bmatrix} X^k & 0 \\ 0 & Y^k \end{bmatrix} \prec \alpha(1-2\alpha\varepsilon)^{-1} \begin{bmatrix} \mathbb{I}_n & -\mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{I}_n \end{bmatrix} + (\gamma_k + 12 r_k)\mathbb{I}_{2n}. \tag{2.4.6}$$

Estimate (2.4.6) is *crucial*, since

- (2.4.6) implies the compactness of X^k and Y^k, thus we can assume $(X^k, Y^k) \to (X_\alpha, Y_\alpha)$.
- Multiply (2.4.6) by vector $(\xi, \xi)^T \in \mathbb{R}^{2n}$ we see that $\boxed{X^k \prec Y^k}$, which implies that $X_\alpha \prec Y_\alpha$.

The proof is pretty much finished. From Lemma 2.4.2, there exists $x_k^\varepsilon \to 0$ such that

$$\tilde{u}^\varepsilon(x_k) = \tilde{u}(x_k^\varepsilon) - \frac{1}{2\varepsilon}|x_k^\varepsilon - x_k|^2$$

and

$$\begin{aligned} \left(x_k, p^k, X^k\right) \in J^{2,+}\tilde{u}^\varepsilon(x_k) \qquad &\Longrightarrow \qquad \left(x_k^\varepsilon, p^k, X^k\right) \in J^{2,+}\tilde{u}(x_k^\varepsilon) \\ &\Longrightarrow \qquad \left(x_k^\varepsilon + x_\alpha, p^k + p_\alpha, X^k\right) \in J^{2,+}u(x_k^\varepsilon + x_\alpha). \end{aligned}$$

Taking the limit in k we obtain

$$(x_\alpha, p_\alpha, X_\alpha) \in \overline{J}^{2,+}u(x_\alpha) \qquad \Longrightarrow \qquad \lambda u(x_\alpha) + H(x_\alpha, p_\alpha, X_\alpha) \le 0.$$

Similarly we obtain

$$(y_\alpha, p_\alpha, Y_\alpha) \in \overline{J}^{2,-}v(y_\alpha) \qquad \Longrightarrow \qquad \lambda v(y_\alpha) + H(y_\alpha, p_\alpha, Y_\alpha) \ge 0.$$

Subtract these equations we deduce that

$$\begin{aligned}\lambda\big(u(x_\alpha)-v(y_\alpha)\big) &\le H(y_\alpha,p_\alpha,Y_\alpha)-H(x_\alpha,p_\alpha,X_\alpha)\\ &\le H(y_\alpha,p_\alpha,Y_\alpha)-H(x_\alpha,p_\alpha,Y_\alpha)+H(x_\alpha,p_\alpha,Y_\alpha)-H(x_\alpha,p_\alpha,X_\alpha)\\ &\le \omega\big(|x_\alpha-y_\alpha|(1+|p_\alpha|)\big)\to 0\end{aligned}$$

as $\alpha\to\infty$ since $\alpha|x_\alpha-y_\alpha|^2\to 0$ and $X_\alpha\prec Y_\alpha$. This contradicts to $u(x_\alpha)-v(y_\alpha)\ge \max_{\overline{\Omega}}(u-v)>0$ and thus the proof is complete. □

Remark 13. There is another way to package this argument in the proof, which is called *Theorem on Sums* in the language of semi-jets (see [39]). It is usually stated in the general form where the test function $\frac{\alpha}{2}|x-y|^2$ is replaced by $\varphi(x,y)$, a general C^2 function. Theorem 2.4.8 also usually appears under the name *Ishii's Lemma* ([41]).

Theorem 2.4.8 (Theorem on Sums)**.** *Let $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^n$. Let $u,v:\mathcal{O}\to\mathbb{R}$ and φ be a twice continuously differentiable function in a neighborhood of $\mathcal{O}\times\mathcal{O}$ in $\mathbb{R}^n\times\mathbb{R}^n$. If $w(x,y)=u(x)+v(y)$ in $\mathcal{O}\times\mathcal{O}$ such that $w-\varphi$ has a local maximum over $\mathcal{O}\times\mathcal{O}$ at $(\hat{x},\hat{y})$ then, if $\kappa D^2\varphi(\hat{x},\hat{y})\prec \mathrm{I}_{2n}$ there exist $X,Y\in\mathcal{S}(n)$ such that*

$$(\hat{x},D_x\varphi(\hat{x},\hat{y}),X)\in\overline{J}^{2,+}u(\hat{x}),\qquad (\hat{y},D_y\varphi(\hat{x},\hat{y}),Y)\in\overline{J}^{2,+}v(\hat{y}),$$

and

$$-\left(\frac{1}{\kappa}\right)\mathrm{I}_{2n}\preceq\begin{pmatrix}X&0\\0&Y\end{pmatrix}\preceq\Big(\mathrm{I}_{2n}-\kappa D^2\varphi(\hat{x},\hat{y})\Big)^{-1}D^2\varphi(\hat{x},\hat{y}).$$

Remark 14. In Theorem 2.4.8 choose $\varphi(x,y)=\frac{1}{2\varepsilon}|x-y|^2$ and $\kappa=\frac{\varepsilon}{3}$ we recover

$$\left(\hat{x},\frac{\hat{x}-\hat{y}}{\varepsilon}\right)\in\overline{J}^{2,+}u(\hat{x}),\qquad \left(\hat{y},-\frac{\hat{x}-\hat{y}}{\varepsilon}\right)\in\overline{J}^{2,+}v(\hat{y})$$

and

$$-\left(\frac{3}{\varepsilon}\right)\begin{pmatrix}\mathrm{I}_n&0\\0&\mathrm{I}_n\end{pmatrix}\preceq\begin{pmatrix}X&0\\0&Y\end{pmatrix}\preceq\left(\frac{3}{\varepsilon}\right)\begin{pmatrix}\mathrm{I}_n&-\mathrm{I}_n\\-\mathrm{I}_n&\mathrm{I}_n\end{pmatrix}$$

as we used in the proof of Theorem 2.4.7.

Remark 15. Problems on unbounded domain can also be handed using some modification, for example by adding some penalization terms into the auxiliary functional (2.4.5) like the *Japanese bracket*

$$\langle x\rangle=\sqrt{|x|^2+1}.$$

Depend on situation the auxiliary functional can be modified to satisfy a special need (see [6, 50, 111] or the later Chapters of this thesis).

2.5 Perron's method revisited

Let us come back to the existence of solutions of (2.1.1) equipped with a Dirichlet boundary condition. We have already obtained the existence of a viscosity solution u where $u=u^*$ is a subsolution and u_* is a supersolution. If comparison principle holds (provided some necessary conditions are available) then $u_* \geq u^*$, which implies that $u = u^* = u_*$ and thus u is continuous. In what follows we will mention a special second-order situations where this is indeed the case. More on first-order cases will be provided in the next section.

Let $g : \partial\Omega \to \mathbb{R}$ be a continuous function, we consider the following problem.

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.5.1}$$

Comparison principle holds for (2.5.1) provided that $F(x, r, p, X) = \lambda r + H(x, p, X)$ and H satisfies (2.4.4). We can relax these conditions further but for simplicity, we will state the theorem in the following form.

Theorem 2.5.1 ([64])**.** *Let comparison hold for* (2.5.1)*. Suppose there is a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of* (2.5.1) *which satisfy the boundary condition $(\underline{u})_*(x) = (\overline{u})^*(x) = g(x)$ for $x \in \partial\Omega$. Then*

$$W(x) = \sup \left\{ w(x) : \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ is a subsolution of } (2.5.1) \right\} \tag{2.5.2}$$

is a solution to (2.5.1)*.*

Proof. We recall that $\underline{u}$ is a subsolution means $(\underline{u})^* \in \mathrm{USC}(\overline{\Omega})$ is a subsolution and similarly, $(\overline{u})_* \in \mathrm{LSC}(\overline{\Omega})$ is a supersolution. It is clear that $(\underline{u})_*(x) \leq W_*(x) \leq W(x) \leq W^*(x) \leq (\overline{u})^*(x)$ for $x \in \overline{\Omega}$ and in particular $W_* = W = W^* = g$ on $\partial\Omega$.

- As $W^* \in \mathrm{USC}(\overline{\Omega})$ is a subsolution and $(\overline{u})_* \in \mathrm{LSC}(\overline{\Omega})$ is a supersolution, comparison principle gives $W^* \leq (\overline{u})_* \leq \overline{u}$. On the other hand, it is clear that $W \geq \underline{u}$, thus W^* belongs to the admissible set in the definition of W, hence $W = W^* \in \mathrm{USC}(\overline{\Omega})$.

- If $W_* \in \mathrm{LSC}(\overline{\Omega})$ fails to be a supersolution at $x_0 \in \Omega$, then there exists $\widehat{x} \in \Omega$ near x_0 and a subsolution $\widehat{W} \in \mathrm{USC}(\overline{\Omega})$ such that $\widehat{W}(\widehat{x}) > W(\widehat{x})$, $\widehat{W} \geq W$ on Ω and $\widehat{W} = g$ on $\partial\Omega$ (recall in the Perron's method we can construct a subsolution that equals to W outside a ball near x_0). Again, by comparison principle $\widehat{W} \leq (\overline{u})_* \leq \overline{u}$ and thus $\widehat{W}$ belongs to the admissible set of the definition of W, which implies $W \geq \widehat{W}$, a contradiction. Thus $W_* \in \mathrm{LSC}(\overline{\Omega})$ is a supersolution.

By comparison principle $W^* \leq W_*$, which implies that $W_* = W = W^*$ is a continuous solution to (2.5.1). □

For the existence of subsolution and supersolution that agree on the boundary to some function g, we refer to [39].

2.6 Time-dependent problem

The results in the previous sections can be extended to time-dependent problem

$$u_t + F(x, u, Du, D^2u) = 0 \qquad \text{in } \mathcal{O} \times (0, T), \tag{2.6.1}$$

where $u = u(x,t)$ and Du, D^2u mean $D_x u(x,t), D_x^2 u(x,t)$. For $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^n$ and $T > 0$, we define $\mathcal{O}_T = \mathcal{O} \times (0,T)$. The notions of viscosity solutions on an open set can be defined in the same manner as in Definition 2.

Definition 11. *We say that $u \in \mathrm{USC}(\mathcal{O}_T)$ is a viscosity subsolution of (2.6.1) if for every $(x,t) \in \mathcal{O}_T$ and every $\varphi \in \mathrm{C}^\infty(\mathbb{R}^n) \times (0,T)$ such that $u - \varphi$ has a local maximum at (x,t) with respect to $\mathcal{O}_T$ then*

$$\varphi_t(x,t) + F(x, u(x,t), D\varphi(x,t), D^2\varphi(x,t)) \leq 0.$$

Similarly, we say that $u \in \mathrm{LSC}(\mathcal{O}_T)$ is a viscosity supersolution of (2.6.1) if for every $(x,t) \in \mathcal{O}_T$ and every $\varphi \in \mathrm{C}^\infty(\mathbb{R}^n) \times (0,T)$ such that $u - \varphi$ has a local minimum at (x,t) with respect to $\mathcal{O}_T$ then

$$\varphi_t(x,t) + F(x, u(x,t), D\varphi(x,t), D^2\varphi(x,t)) \geq 0.$$

We say u is a viscosity solution to (2.6.1) if u is both a viscosity subsolution and a viscosity supersolution. We often say that $u \in \mathrm{USC}(\mathcal{O})$ (resp. $\mathrm{LSC}(\mathcal{O})$) is a viscosity solution to $u_t + F(x,u,Du,D^2u) \leq 0$ (resp. $u_t + F(x,u,Du,D^2u) \leq 0$) in $\mathcal{O}_T$ if u is a viscosity subsolution (resp. viscosity supersolution) of (2.6.1).

We also define the semijets for time-dependent problem as follows.

Definition 12. *Let $u : \mathcal{O}_T \to \mathbb{R}$ and $(x,t) \in \mathcal{O}_T$. The parabolic semi super-jets $\mathcal{P}^{2,+}_{\mathcal{O}} u(x_0,t_0)$ is the set of all $(\alpha, p, X) \in \ \times \mathbb{R}^n \times \mathbb{S}^n$ such that*

$$\begin{aligned} u(x,t) \leq u(x_0,t_0) &+ \alpha(t - t_0) \\ &+ \langle p, x - x_0 \rangle + \frac{1}{2}\langle x - x_0, X(x - x_0)\rangle + o\left(|t - t_0| + |x - x_0|^2\right) \end{aligned}$$

as $\mathcal{O} \ni (x,t) \to (x_0,t_0)$. Similarly, $\mathcal{P}^{2,+}_{\mathcal{O}} u(x,t) = -\mathcal{P}^{2,+}_{\mathcal{O}}(-u(x,t))$. Similarly we define $\mathcal{P}^{2,\pm}_{\mathcal{O}} u(x,t)$ in the same manner as in (2.1.3).

With the notions of semijets, we can reformulate the definitions of viscosity solutions as follows.

Definition 13. *The upper semicontinuous function $u \in \mathrm{USC}(\mathcal{O}_T)$ is a subsolution of (2.6.1) if*

$$\alpha + F(x, u(t,x), p, X) \leq 0 \qquad \text{for all } (\alpha, p, X) \in \mathcal{P}^{2,+}_{\mathcal{O}} u(x,t), \text{ for all } (x,t) \in \mathcal{O}_T.$$

Likewise, the lower semicontinuous function $u \in \mathrm{LSC}(\mathcal{O}_T)$ is a supersolution of (2.6.1) if

$$\alpha + F(x, u(t,x), p, X) \geq 0 \qquad \text{for all } (\alpha, p, X) \in \mathcal{P}^{2,-}_{\mathcal{O}} u(x,t), \text{ for all } (x,t) \in \mathcal{O}_T.$$

We will focus on the following problem.

$$\begin{cases} u_t + F(x,u,Du,D^2u) = 0 & \text{in } \mathcal{O} \times (0,T), \\ u(x,t) = 0 & \text{in } \partial\mathcal{O} \times [0,T) \\ u(x,0) = u_0(x) & \text{on } \overline{\mathcal{O}} \end{cases} \tag{2.6.2}$$

where $\mathcal{O}$ is open, $T > 0$ and $u_0 \in \mathrm{C}(\overline{\Omega})$ are given.

Definition 14. *A subsolution to* (2.6.2) *is an upper semicontinuous function* $u : \overline{\mathcal{O}} \times [0, T)$ *such that:*

- u *is a subsolution in* $\mathcal{O}_T$*,*
- $u(x, t) \leq 0$ *on* $\partial\mathcal{O} \times [0, T)$*,*
- $u(x, 0) \leq u_0(x)$ *on* $\overline{\mathcal{O}}$*.*

A supersolution is defined similarly for lower semicontinuous function on $\overline{\mathcal{O}} \times [0, T)$*.*

The Perron's method can be carried over similarly for this problem. The comparison principle can be stated as follows.

Theorem 2.6.1 (Comparison principle)**.** *Let* $\mathcal{O} \subset \mathbb{R}^n$ *be open, bounded. Let* $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ *be open bounded and* $F(x, r, p, X) = \lambda r + H(x, p, X)$ *from* $\mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is degenerate elliptic satisfying*

$$|H(x, p, X) - H(y, p, X)| \leq \omega(|x - y|(1 + |p|))$$

for some modulus $\omega(\cdot)$*. Assume that* $u \in \mathrm{USC}(\overline{\Omega})$ *is a subsolution to* (2.6.2) *and* $v \in \mathrm{LSC}(\overline{\Omega})$ *is a supersolution to* (2.6.2)*, then* $u \leq v$ *in* $\Omega \times [0, T)$*.*

A proof is similar to the the one in Theorem 2.4.7 with a key modification in the auxiliary functional. For each $\varepsilon > 0$ we define

$$\Phi(x, y, t, s) = u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon} - \frac{\varepsilon}{(T - t)^2}$$

for $(x, y, t, s) \in \overline{\Omega} \times \overline{\Omega} \times [0, T) \times [0, T)$. We refer the readers to [39, 41] for a complete proof and [50, 111] for a simplified proof on the first-order equations.

2.7 First-order equations and relations to optimal control theory

Let Ω be an open subset of $\mathbb{R}^n$ (could be the whole space) and

$$F(x, r, p, X) = \lambda r + H(x, p)$$

where $H : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ satisfies some standard assumptions to be described. The comparison principle in this case is usually easier to obtained. We listed some scenarios where the asymptotic problems will be considered later. We refer to [111, Appendix 4] or [8] for other kinds of boundary conditions. For simplicity, we consider the following three problems:

1. Time-dependent first-order problem on $\mathbb{R}^n$.
2. Static problem on $\mathbb{R}^n$.
3. Static problem on a bounded domain with state-constraint.

The existences of uniformly bounded subsolution and supersolution in these first-order cases are usually easier to obtain due to the *finite speed of propagation* (see [50]). We also note that the Perron's method can be modified to produce a Lipschitz solution (see [111]). One can modify the results from previous sections to reproduce the well-posedness and comparison principles for the three problems above under appropriate assumptions on H. For example the following strong Lipschitz condition: There exists a constant $C > 0$ such that, for all $x, y, p, q \in \mathbb{R}^n$,

$$\begin{cases} |H(x,p) - H(y,p)| \leq C(1+|p|)|x-y|, \\ |H(x,p) - H(x,q)| \leq C|p-q|. \end{cases} \tag{2.7.1}$$

This assumption (H) is rather strong, however in this case the proof of a comparison principle is easier to obtained. For more general assumptions we refer the readers to [8, 50, 111] and the references therein. The following coercivity is also considered standard.

$$\begin{cases} H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R)), & \text{for all } R > 0, \\ \lim\limits_{|p|\to\infty} \left(\inf\limits_{x\in\mathbb{R}^n} H(x,p) \right) = +\infty. \end{cases} \tag{2.7.2}$$

For first-order equation, solutions can be written as an optimal control formula. We refer the readers to [8, 50, 111] and the references therein for the theory behind these formula. In what follows, we will summarize equation, some assumptions on which a comparison principle hold and an optimal control formula.

1. First-order equation with time-dependent in the whole space

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2.7.3}$$

   where $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$, the space of bounded, uniformly continuous functions on $\mathbb{R}^n$.

   - Assume (2.7.1) then a comparison principle holds. The statement is as follows: If $u, v \in \mathrm{BUC}(\mathbb{R}^n \times [0,T])$ are viscosity subsolution and supersolution of (2.7.3), respectively, then $u(x,t) \leq v(x,t)$ on $\mathbb{R}^n \times [0,T]$. Thus the uniqueness of a solution u follows, while the existence follows from Perron's method.
   - Assume also (2.7.2) together with (2.7.1), the unique solution satisfies a gradient estimate $|u_t| + |Du| \leq C$ for some $C > 0$.
   - Optimal control formula

$$u(x,t) = \inf\left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s))ds + u_0(\gamma(t)) : \gamma \in \mathrm{AC}([0,t];\mathbb{R}^n), \gamma(t) = x \right\}$$

     where $\mathrm{AC}([0,t];\mathbb{R}^n)$ is the space of absolute continuous functions from $[0,T]$ to $\mathbb{R}^n$.

2. The static problem on the whole space $\Omega = \mathbb{R}^n$

$$\lambda u(x) + H(x, Du(x)) = 0 \qquad \text{in } \mathbb{R}^n. \tag{2.7.4}$$

A boundary condition is not needed for the well-posedness of viscosity solution in this case.

- Assume (2.7.1) then a comparison principle holds. The statement is as follows: If $u, v \in \mathrm{BUC}(\mathbb{R}^n)$ are viscosity subsolution and supersolution of (2.7.4), respectively, then $u(x) \leq v(x)$ in $\mathbb{R}^n$. Thus the uniqueness of a solution u follows, while the existence follows from Perron's method.
- Assume also (2.7.2) together with (2.7.1), the unique solution satisfies a gradient estimate $|Du| \leq C$ for some $C > 0$, hence the solution is uniformly Lipschitz.
- Optimal control formula

$$u(x) = \inf\left\{\int_0^\infty e^{-\lambda s} L(\gamma(s), -\dot{\gamma}(s))ds : \gamma \in \mathrm{AC}([0,\infty); \mathbb{R}^n), \gamma(0) = x\right\}.$$

3. The static problem on an open, bounded domain with state-constraint, which is written as follows.

$$\begin{cases} \lambda u(x) + H(x, Du(x)) \leq 0 & \text{in } \Omega, \\ \lambda u(x) + H(x, Du(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{2.7.5}$$

The state-constraint boundary condition is a hidden boundary condition, which is motivated from optimal control theory (see [8, 50, 111]). At a matter of fact, the boundary condition means, beside u is a viscosity solution in Ω, we only require the supersolution property of u on the boundary $\partial\Omega$.

- Well-posedness and gradient bound are stated in the next section.
- Optimal control formula

$$u(x) = \inf\left\{\int_0^\infty e^{-\lambda s} L(\gamma(s), -\dot{\gamma}(s))ds : \gamma \in \mathrm{AC}([0,\infty); \overline{\Omega}), \gamma(0) = x\right\}.$$

  We note that the only difference comparing to the optimal control formula of the static problem in $\mathbb{R}^n$ is that, the path γ is admissible if it does not exist $\overline{\Omega}$. This is the source of the name state-constraint.

A more detailed introduction on state-constraint problem (2.7.5) is given in the next section.

We refer the readers to [8, 50, 111] for proofs of those properties for (2.7.3), (2.7.4) (the first and second problems above) as they are standard, while the properties of solutions with state-constraint (the third problem) will be presented in the next section as it is one of the main concern of this thesis.

2.8 First-order static equation with state-constraint

2.8.1 Formal definition

For an open subset $\Omega \subset \mathbb{R}^n$, we denote the space of bounded uniformly continuous functions defined in Ω by $\mathrm{BUC}(\Omega;\mathbb{R})$. We will consider $H : \overline{\mathcal{O}} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous Hamiltonian where $\mathcal{O}$ is an open neighborhood of $\overline{\Omega}$. We list the main assumptions that (some of them will be used depending on the situation) as follows.

($\mathcal{H}_0$) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$.

($\mathcal{H}_1$) There exists $C_1 > 0$ such that $H(x,p) \geq -C_1$ for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$.

($\mathcal{H}_2$) There exists $C_2 > 0$ such that $|H(x,0)| \leq C_2$ for all $x \in \overline{\Omega}$.

($\mathcal{H}_3$) For each $R > 0$ there exists a constant C_R such that

$$\begin{cases} |H(x,p) - H(y,p)| \leq C_R|x-y|, \\ |H(x,p) - H(x,q)| \leq C_R|p-q|, \end{cases} \tag{2.8.1}$$

for $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$ with $|p|, |q| \leq R$.

($\mathcal{H}_4$) H satisfies the coercivity assumption

$$\lim_{|p|\to\infty} \left(\inf_{x\in\overline{\Omega}} H(x,p) \right) = +\infty. \tag{2.8.2}$$

($\mathcal{H}_5$) $p \mapsto H(x,p)$ is convex for each $x \in \overline{\Omega}$.

($\mathcal{H}_6$) $p \mapsto H(x,p)$ is superlinear uniformly for $x \in \overline{\Omega}$, that is,

$$\lim_{|p|\to\infty} \left(\inf_{x\in\Omega} \frac{H(x,p)}{|p|} \right) = +\infty. \tag{2.8.3}$$

Definition 15. *Assume ($\mathcal{H}_0$), ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_3$), ($\mathcal{H}_4$). We consider the following equation with $\delta \geq 0$:*

$$\delta u(x) + H(x, Du(x)) = 0 \qquad \text{in } \Omega. \tag{2.8.4}$$

We say that

(i) $v \in \mathrm{BUC}(\Omega;\mathbb{R})$ is a viscosity subsolution of (2.8.4) in Ω if, for every $x \in \Omega$ and $\varphi \in \mathrm{C}^1(\Omega)$ such that $v - \varphi$ has a local maximum over Ω at x, $\delta v(x) + H\big(x, D\varphi(x)\big) \leq 0$ holds.

(ii) $v \in \mathrm{BUC}(\overline{\Omega};\mathbb{R})$ is a viscosity supersolution of (2.8.4) on $\overline{\Omega}$ if, for every $x \in \overline{\Omega}$ and $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $v - \varphi$ has a local minimum over $\overline{\Omega}$ at x, $\delta v(x) + H\big(x, D\varphi(x)\big) \geq 0$ holds.

If v is a viscosity subsolution to (2.8.4) in Ω, and is a viscosity supersolution to (2.8.4) on $\overline{\Omega}$, that is, v is a viscosity solution to

$$\begin{cases}\delta v(x) + H(x, Dv(x)) \leq 0 & \text{in } \Omega,\\ \delta v(x) + H(x, Dv(x)) \geq 0 & \text{on } \overline{\Omega},\end{cases} \tag{HJ$_\delta$}$$

then we say that v is a state-constraint viscosity solution of (2.8.4).

We will summarize basic properties on existence, well-posedness and the Lipschitz bound on solutions. We refer the readers to [30, 75, 107] for more details.

Remark 16. As pointed out in [107], the state-constraint implicitly imposes a boundary condition to solutions. Indeed, when $\partial\Omega$ is smooth, we can define an outward normal vector $\vec{v}(x)$ at $x \in \partial\Omega$. Moreover, if the state-constraint solution $v \in C^1(\overline{\Omega})$, then v solves $v(x) + H(x, Dv(x)) = 0$ in Ω and satisfies

$$H(x, Dv(x)) \leq H\big(x, Dv(x) + \beta\vec{v}(x)\big) \qquad \text{for any } \beta \geq 0, x \in \partial\Omega.$$

If H is differentiable in p, the above condition can also be phrased as a constraint on the normal derivative on the boundary as

$$D_p H\big(x, Dv(x)\big) \cdot \vec{v}(x) \geq 0 \qquad \text{for any } x \in \partial\Omega. \tag{2.8.5}$$

We will give a more detailed derivations in the next section.

2.8.2 State-constraint boundary condition from optimal control theory

Let U be a compact metric space and by control, we mean a Borel measurable map $u : [0, \infty) \longmapsto U$. Let $y(x, t, u) = y^{x,u}(t)$ be the controlled process, i.e., the solution of

$$\begin{cases}\frac{d}{dt} y^{x,u}(t) &= b\left(y^{x,u}(t), u(t)\right), \qquad t > 0\\ \quad y^{x,u}(t) &= x.\end{cases} \tag{2.8.6}$$

Let Ω be an open subset of $\mathbb{R}^n$ and $\mathcal{A}_X$ be the set of strategies under which $y^{x,u}(t) \in \overline{\Omega}$ for all $t \geq 0$, refered as the set of admissible controls. The structure of $\mathcal{A}_x$ constitutes a state-space constraint.
We now associate a discounted cost to every admissible control u and $x \in \overline{\Omega}$, given by

$$J_\infty[x, u] = \int_0^\infty e^{-t} f\big(y^{x,u}(t), u(t)\big)\, dt. \tag{2.8.7}$$

Given these the optimal value function is

$$v(x) = \inf_{u \in \mathcal{A}_x} \int_0^\infty e^{-t} f\big(y^{x,u}(t), u(t)\big)\, dt. \tag{2.8.8}$$

Under some assumptions, one can define the Hamiltonian

$$H(x, p) := \sup_{\alpha \in U} \{-b(x, \alpha) \cdot p - f(x, \alpha)\} \in C\left(\overline{\Omega} \times \mathbb{R}^n; \mathbb{R}\right). \tag{2.8.9}$$

We will show that

$$\begin{cases} v(x) + H(x, Dv(x)) \leq 0 & \text{in } \Omega,\\ v(x) + H(x, Dv(x)) \geq 0 & \text{on } \overline{\Omega}.\end{cases}$$

- The function $v(x)$ is not necessarily continuous, due to the complicated structure of the set valued function $x \mapsto \mathcal{A}_x$.
- As a solution to the Hamilton-Jacobi equation $v(x) + H(x, Dv(x)) = 0$ in Ω, the function $v(x)$ is not priori defined on $\partial\Omega$, the only information on $\partial\Omega$ is given by the state-space constraint.

The motivation for that definition on the boundary can be deduced from the fact that, in order to have the path belongs to $\overline{\Omega}$, let say for $x \in \partial\Omega$ we have an optimal control $\alpha^*(x) \in U$, then we must have

$$b\left(x, \alpha^*(x)\right) \cdot \nu(x) \le 0$$

where $\nu(x)$ is the outward normal vector to Ω. Therefore one can see that for all $x \in \partial\theta$ then

$$H(x, Dv(x)) \le H(x, Dv(x) + \beta\nu(x)) \qquad \text{for all} \qquad \beta \ge 0. \tag{2.8.10}$$

Hence, if $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $v - \varphi$ has a minimum relative to $\overline{\Omega}$ at $x \in \partial\Omega$, then by Lagrange multiplier, assuming the boundary $\partial\Omega$ locally around x can be described as $g(z) = 0$, we must have $\nabla(v-\varphi)(x) = \lambda\nabla g(x)$ and $\nabla g(x)$ is the normal. By a simple argumen, let $\nabla(v-\varphi)(x) = -\beta\nu(x)$, then by consider $y = x - \varepsilon\nu(x) \in \Omega$ and $(v-\varphi)(y) - (v-\varphi)(x) \ge 0$ we deduce that

$$\frac{(v-\varphi)(y) - (v-\varphi)(x) - \nabla(v-\varphi)(x)\cdot(y-x)}{|y-x|} \ge -\beta.$$

Taking the limit (assuming $v \in \mathrm{C}^1$)) then $\beta \ge 0$. Also note that in the Lagrange multiplier we must have $\beta > 0$, therefore for any $\varphi \in \mathrm{C}^1(\overline{\Omega})$ with $v - \varphi$ has a min over $\overline{\Omega}$ at $x \in \partial\Omega$ then $\nabla\varphi(x) = \nabla v(x) + \beta\nu(x)$, thus

$$0 = v(x) + H(x, Dv(x)) \le v(x) + H(x, D\varphi(x)).$$

Another motivation is from (2.8.10) if H is differentiable in p then differentiable both side in β we have

$$H_p(x, Dv(x)) \cdot \nu(x) \ge 0 \qquad \text{for all} \qquad x \in \partial\Omega.$$

2.8.3 Dynamic Programming Principle

Let Ω be an open subset of $\mathbb{R}^n$ with a connected boundary $\partial\Omega$. We have the following assumptions:

($\mathcal{A}_0$) Let $\mathcal{A}$ be the set of all controls from $[0,\infty) \longrightarrow U$, the author only allows the controls which leave $y(x,\cdot,u) \in \overline{\Omega}$.

$$\mathcal{A}_x = \left\{ u \in \mathcal{A} : y(x,t,u) \in \overline{\Omega} \text{ for all } t \ge 0 \right\} \neq \emptyset \qquad \text{for all } x \in \overline{\Omega}. \tag{2.8.11}$$

($\mathcal{A}_2$) There exists a universal pair of positive numbers (r,h) and $\eta \in \mathrm{BUC}(\overline{\Omega};\mathbb{R}^n)$ such that

$$B(x + t\eta(x), rt) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0,h].$$

$(\mathcal{B}_1)$ Let $U \subset \mathbb{R}^n$ be a compact set, $b : \mathbb{R}^n \times U \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times U \to \mathbb{R}$ such that

$$\sup_{\alpha \in U} |b(x,\alpha) - b(y,\alpha)| \leq L(b)|x-y| \qquad \text{for all } x, y. \tag{1.1}$$

$$\sup_{\alpha \in U} |b(x,\alpha)| \leq K(b) \qquad \text{for all } x. \tag{1.2}$$

$$\sup_{\alpha \in U} |f(x,\alpha) - f(y,\alpha)| \leq \omega_f(|x-y|) \qquad \text{for all } x, y. \tag{1.3}$$

$$\sup_{\alpha \in U} |f(x,\alpha)| \leq K(f) \qquad \text{for all } x. \tag{1.4}$$

where ω_f is a nondecreasing continuous function with $\omega_f(0) = 0$.

Under these assumptions, let us define the Hamiltonian to be

$$H(x,p) = \sup_{\alpha \in U} \Big\{ -b(x,\alpha) \cdot p - f(x,\alpha) \Big\}.$$

It is easy to see that

$$|H(x,p) - H(y,p)| \leq \big(L(b)|p|\big) \cdot |x-y| + \omega_f(|x-y|) \qquad \text{for all} \qquad x, y.$$

Lemma 2.8.1 (Dynamic Programming Principle)**.** *Suppose that $(\mathcal{A}_0)$, $(\mathcal{A}_2)$,* (1.1)–(1.4) *holds and $v(x)$ be the optimal value function defined as in* (2.8.8)*. Then for any $x \in \overline{\Omega}$ and $T > 0$ there holds*

$$v(x) = \inf_{u \in \mathcal{A}_x} \left\{ \int_0^T e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt + e^{-t} v\left(y^{x,u}(t)\right) \right\}. \tag{2.8.12}$$

Proof. For simplicity let us define $\mathbf{x}(\cdot) \equiv y^{x,u}(\cdot)$, i.e., $\mathbf{x}(\cdot)$ solves uniquely

$$\begin{cases} \dot{\mathbf{x}}(t) & = b(\mathbf{x}(t), u(t)) \\ \mathbf{x}(0) & = x \end{cases} \implies \begin{cases} \dot{\mathbf{x}}(t+T) & = b(\mathbf{x}(t+T), \beta(t)) \\ \mathbf{x}(T) & = y^{x,u}(T) \end{cases} \tag{2.8.13}$$

where $\beta(t) = u(t+T)$. Therefore by uniqueness of (2.8.13) we have

$$\mathbf{x}(t+T) \equiv y^{x,u}(t+T) \equiv y^{\mathbf{x}(T),\beta}(t)$$

and thus $\beta(\cdot)$ is an admissible control in the sense that $\beta \in \mathcal{A}_{\mathbf{x}(T)}$. Now we have

$$\begin{aligned} \int_0^\infty e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt &= \int_0^T e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt + \int_T^\infty e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt \\ &= \int_0^T e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt + e^{-T} \int_0^\infty e^{-s} f\left(y^{\mathbf{x}(t),\beta}(s), \beta(s)\right)\, ds \\ &\geq \int_0^T e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt + e^{-T} v\left(y^{x,u}(T)\right). \end{aligned}$$

Taking infimum over all control $u \in \mathcal{A}_x$ we have LHS $\geq$ RHS in (2.8.12). For the other inequality, take $u \in \mathcal{A}_x$ and $\beta \in \mathcal{A}_{\mathbf{x}(T)}$, we can define a new control

$$\overline{u}(t) := \begin{cases} u(t) & 0 \leq t \leq T \\ \beta(t-T) & T < t < \infty. \end{cases}$$

Since $\beta \in \mathcal{A}_{\mathbf{x}(T)}$ and $y^{x,u}(T) = \mathbf{x}(T)$, we have $\overline{u} \in \mathcal{A}_x$. Furthermore it is clear that

$$\int_0^\infty e^{-t} f\left(y^{x,\overline{u}}(t), \overline{u}(t)\right)\, dt = \int_0^T e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt + e^{-T}\int_0^\infty e^{-s} f\left(y^{\mathbf{x}(t),\beta}(s), \beta(s)\right)\, ds.$$

Therefore for all $\beta \in \mathcal{A}_{\mathbf{x}(T)}$ then

$$v(x) \le \int_0^T e^{-t} f\left(y^{x,u}(t), u(t)\right)\, dt + e^{-T}\int_0^\infty e^{-s} f\left(y^{\mathbf{x}(t),\beta}(s), \beta(s)\right)\, ds.$$

Taking infimum over all $\beta \in \mathcal{A}_{\mathbf{x}(T)}$ we obtain LHS $\le$ RHS in (2.8.13). □

Theorem 2.8.2. *Suppose that* $(\mathcal{A}_0)$, $(\mathcal{A}_2)$, (1.1)–(1.4) *holds and that that the optimal value function* $v \in \mathrm{BUC}(\overline{\Omega})$. *Then* v *is a constrained viscosity solution of* $v(x) + H(x, Dv(x)) = 0$ *on* $\overline{\Omega}$.

Proof.

- Subsolution on Ω. Let $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $v(x_0) = \varphi(x_0)$ and $v - \varphi$ has a maximum at $x_0 \in \Omega$, we show $v(x_0) + H(x_0, D\varphi(x_0)) \le 0$. Let $u \in \mathcal{A}_{x_0}$ then for all $t > 0$ by the DPP with $y^{x_0,u}(t) = \mathbf{x}(t)$ we have

$$e^{-0} v(\mathbf{x}(0)) \le \int_0^t e^{-s} f\left(\mathbf{x}(s), u(s)\right) ds + e^{-t} v\left(\mathbf{x}(t)\right).$$

Therefore

$$-\int_0^t \frac{d}{ds}\left(e^{-s}\varphi(\mathbf{x}(s))\right)\, ds = e^{-0}\varphi(\mathbf{x}(0)) - e^{-t}\varphi(\mathbf{x}(t)) \le \int_0^t e^{-s} f\left(\mathbf{x}(s), u(s)\right) ds.$$

Thus

$$e^{-s}\frac{1}{t}\int_0^t \varphi(\mathbf{x}(s))\, ds + \frac{1}{t}\int_0^t e^{-s}\Big[-b(\mathbf{x}(s), u(s)) \cdot \varphi(\mathbf{x}(s)) - f(\mathbf{x}(s), u(s))\Big] ds \le 0. \tag{2.8.14}$$

Let $t_0 = (K(b))^{-1}\operatorname{dist}(x_0, \partial\Omega)$, then for any control w not necessarily in $\mathcal{A}_{x_0}$ we still see that $y^{x_0,w}(t) \in \Omega$ for all $0 < t < t_0$. Thus for $\alpha \in U$, let us define

$$u(t) := \begin{cases} \alpha & 0 \le t < t_0, \\ \tilde{u}(t - t_0) & t_0 \le t < \infty \end{cases}$$

for some $\tilde{u} \in \mathcal{A}_{y^{x_0,\alpha}(t_0)}$ then $u \in \mathcal{A}_{x_0}$ and for $t < t_0$ (2.8.14) becomes

$$\frac{1}{t}\int_0^t e^{-s}\varphi(\mathbf{x}(s))\, ds + \frac{1}{t}\int_0^t e^{-s}\Big[-b(\mathbf{x}(s), \alpha) \cdot \varphi(\mathbf{x}(s)) - f(\mathbf{x}(s), \alpha)\Big] ds \le 0.$$

Let $t \longrightarrow 0$ then take supremum over all $\alpha \in U$ we obtain the result.

- Supersolution. Let $\psi \in \mathrm{C}^1(\overline{\Omega})$ such that $v(x_0) = \psi(x_0)$ and $v - \psi$ has a minimum at $x_0 \in \Omega$, we show $v(x_0) + H(x_0, D\psi(x_0)) \ge 0$. By the DPP with $t_m = \frac{1}{m}$ we can find the corresponding control $u_m \in \mathcal{A}_{x_0}$ such that

$$e^{-0}\psi\left(y^{x_0,u_m}(0)\right) - e^{-1/m}\psi\left(y^{x_0,u_m}(1/m)\right) - \int_0^{1/m} e^{-s} f\left(y^{x_0,u_m}(s), u_m(s)\right) ds \ge -\frac{1}{m^2}.$$

I.e.,

$$-\int_0^{1/m}\frac{d}{ds}\left(e^{-s}\psi\left(y^{x_0,u_m}(s)\right)\right)ds-\int_0^{1/m}e^{-s}f\left(y^{x_0,u_m}(s),u_m(s)\right)ds\geq-\frac{1}{m^2}.$$

Hence

$$\begin{aligned}m&\int_0^{1/m}e^{-s}\varphi\left(y^{x_0,u_m}(s)\right)ds\\&+m\int_0^{1/m}e^{-s}\Big[-b\left(y^{x_0,u_m}(s),u_m(s)\right)\cdot D\psi\left(y^{x_0,u_m}(s)\right)-f\left(y^{x_0,u_m}(s),u_m(s)\right)\Big]ds\geq-\frac{1}{m}.\end{aligned}$$

Look at the first term, we note that $|y^{x_0,u_m}(s)-y^{x_0,u_m}(0)|\leq K(b)s$, thus

$$\lim_{m\longrightarrow\infty}m\int_0^{1/m}e^{-s}\varphi\left(y^{x_0,u_m}(s)\right)ds=\varphi(x_0).$$

For the second term, we need some kind of compactness. The term $D\psi\left(y^{x_0,u_m}(s)\right)$ can be handled easily with the modulus of continuity of $D\psi$ and $|y^{x_0,u_m}(s)-y^{x_0,u_m}(0)|\leq K(b)s$, therefore

$$\begin{aligned}m\int_0^{1/m}e^{-s}\varphi\left(y^{x_0,u_m}(s)\right)ds+&\left(m\int_0^{\frac{1}{m}}-b\left(x_0,u_m(s)\right)e^{-s}\,ds\right)\cdot D\psi(x_0)\\&+\left(m\int_0^{\frac{1}{m}}-f\left(x_0,u_m(s)\right)e^{-s}\,ds\right)\geq-\frac{1}{m}+\mathcal{O}(1/m).\end{aligned}$$

Note that

$$\begin{aligned}(b_m,f_m):=&\left(m\int_0^{\frac{1}{m}}b\left(x_0,u_m(s)\right)e^{-s}\,ds,m\int_0^{\frac{1}{m}}f\left(x_0,u_m(s)\right)e^{-s}\,ds\right)\\&\in\overline{\mathrm{conv}}\left(\Big\{\,(b(x_0,\alpha),f(x_0,\alpha)):\alpha\in U\Big\}\right):=\mathrm{BF}(x_0)\end{aligned}$$

which is the closed convex hull, and is compact. Therefore upto subsequence $(b_m,f_m)\longrightarrow(b_0,f_0)\in\mathrm{BF}(x_0)$. Hence

$$\varphi(x_0)+\Big[-b_0\cdot D\psi(x_0)-f_0\Big]\geq 0.$$

By definition of H we get the result, since

$$H(x_0,p)=\sup\Big\{-b\cdot p-f:(b,f)\in\mathrm{BF}(x_0)\Big\}.$$

$\square$

2.8.4 Existence of solutions via Perron's method and a priori estimate

Now we construct a state-constraint viscosity solution based on Perron's method. It is a variant of the classical result in [64] but we include the proof here for the sake of the readers' convenience. Note that the Lipschitz regularity of subsolutions is encoded directly into the admissible class $\mathcal{F}$.

Theorem 2.8.3. *Assume $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ and $\delta > 0$. Then, there exists a state-constrained viscosity solution $u \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{W}^{1,\infty}(\Omega)$ to (2.8.4) with $\delta|u(x)| + |Du(x)| \le C_H$ for $x \in \Omega$ where C_H only depends on H.*

Proof of Theorem 2.8.3. From the assumption C_1 and $-C_2$ are a supersolution on $\overline{\Omega}$ and a subsolution in Ω of (2.8.4), respectively. By the coercivity assumption $(\mathcal{H}_4)$, we can find a constant $C_3 > 0$ such that

$$H(x,p) \le \max\{C_1, C_2\} \quad \text{for some } x \in \overline{\Omega} \qquad \Longrightarrow \qquad |p| \le C_3.$$

Let us define

$$\begin{aligned}\mathcal{F} = \big\{ w \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{W}^{1,\infty}(\Omega) :\ -C_2 \le w(x) \le C_1,\ \|Dw\|_{L^\infty(\overline{\Omega})} \le C_3, \\ \text{and } w \text{ is a viscosity subsolution to } w(x) + H(x, Dw(x)) \le 0 \text{ in } \Omega \big\}\end{aligned}$$

and for each $x \in \overline{\Omega}$, we define

$$u(x) := \sup\{w(x) : w \in \mathcal{F}\}.$$

By the stability of viscosity subsolutions, we have that u is a viscosity subsolution to (2.8.4) in Ω. Thus, $u \in \mathcal{F}$ as well. We now check that u is a viscosity supersolution to (2.8.4) on $\overline{\Omega}$. Assume that u is not a supersolution on $\overline{\Omega}$. Then, there exists $x_0 \in \overline{\Omega}$, $\varphi \in \mathrm{C}^1(\overline{\Omega})$ with $\|D\varphi\|_{L^\infty(B(x_0,r))} \le C_3$ and $r > 0$ such that $u(x_0) = \varphi(x_0)$ and $(u-\varphi)(x) \ge |x - x_0|^2$ for all $x \in B(x_0, r) \cap \overline{\Omega}$, and

$$\varphi(x_0) + H(x_0, D\varphi(x_0)) < 0. \tag{2.8.15}$$

From the boundedness below by $-C_1$ of H and (2.8.15), we obtain $\varphi(x_0) = u(x_0) < C_1$. By continuity of φ and H, one can choose $\delta, \varepsilon \in \left(0, \frac{r}{2}\right)$ small enough so that $\varepsilon < \delta^2$ and

$$\begin{cases} \varphi(x) + \delta^2 < C_1, \\ \varphi(x) + \delta^2 + H(x, D\varphi(x)) < 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \varphi(x) + \varepsilon^2 < C_1, \\ \varphi(x) + \varepsilon^2 + H(x, D\varphi(x)) < 0 \end{cases}$$

for all $x \in B(x_0, 2\varepsilon) \cap \overline{\Omega}$. Clearly, $x \mapsto \varphi(x) + \varepsilon^2$ is a viscosity subsolution to (2.8.4) in $B(x_0, 2\varepsilon) \cap \overline{\Omega}$ and $u(x) \ge \varphi(x) + \varepsilon^2$ for $x \in B(x, 2\varepsilon) \backslash B(x_0, \varepsilon)$. Let us define $w : \overline{\Omega} \to \mathbb{R}$ by

$$w(x) = \begin{cases} \max\{u(x), \varphi(x) + \varepsilon^2\} & x \in B(x_0, \varepsilon) \cap \overline{\Omega}, \\ u(x) & x \in \overline{\Omega} \backslash B(x_0, \varepsilon). \end{cases}$$

Then, $w(x) = \max\{u(x), \varphi(x) + \varepsilon^2\}$ in $B(x_0, 2\varepsilon) \cap \overline{\Omega}$ belongs to $\mathcal{F}$. Therefore, $w(x)$ is a viscosity subsolution to (2.8.4) . However, $w(x_0) = \varphi(x_0) + \varepsilon^2 = u(x_0) + \varepsilon^2 > u(x_0)$, which is a contradiction to the definition of u. $\square$

The argument used in the proof of Perron's method implies the following corollary as well, see also [30] for a similar corollary.

Corollary 2.8.4. *Let $u \in \mathrm{C}(\overline{\Omega})$ be a viscosity subsolution to (2.8.4) in Ω. Assume further that $v \le u$ on $\overline{\Omega}$ for all viscosity subsolutions $v \in \mathrm{C}(\overline{\Omega})$ of (2.8.4) in Ω. Then, u is a viscosity supersolution to (2.8.4) on $\overline{\Omega}$.*

Remark 17. The converse of Corollary 2.8.4 also true provided a comparison principle holds (thus uniqueness follows), that is, if u is the state-constraint solution of (2.8.4) then it is the maximal subsolution among all subsolution of (2.8.4) in Ω.

2.8.5 Comparison principle with state-constraint

By a domain, we mean an open, bounded, connected subset of $\mathbb{R}^n$. Without loss of generality, we will always assume $0 \in \Omega$. We will need some additional structural assumptions on Ω to ensure a comparison principle holds.

($\mathcal{A}_1$) Ω a bounded star-shaped (with respect to the origin) open subset of $\mathbb{R}^n$ and there exists some $\kappa > 0$ such that $\mathrm{dist}(x, \overline{\Omega}) \ge \kappa r$ for all $x \in (1+r)\partial\Omega$ and $r > 0$.

($\mathcal{A}_2$) There exists a universal pair of positive numbers (r, h) and $\eta \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R}^n)$ such that

$$B(x + t\eta(x), rt) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, h].$$

Remark 18. Condition ($\mathcal{A}_2$) was introduced in [107] by M. Soner. This can be understood as an *interior sphere condition* or *interior ball condition*, while ($\mathcal{A}_1$) was introduced in [30] which allows an easier proof of comparison principle due to the scaling structure. We also note that in ($\mathcal{A}_2$) the domain can be unbounded.

Remark 19. The assumption Ω is star-shaped can be removed in ($\mathcal{A}_1$), that is any bounded, open subset of $\mathbb{R}^n$ containing the origin that satisfies

$$\mathrm{dist}(x, \overline{\Omega}) \ge \kappa r \qquad \text{for all } x \in (1+r)\partial\Omega, \text{ for all } r > 0.$$

for some $\kappa > 0$ is star-shaped, and ($\mathcal{A}_2$) is a consequence of ($\mathcal{A}_1$) (see Lemma 2.8.8).

Remark 20. The condition ($\mathcal{A}_2$) can be generalized to a weaker *interior cone* condition instead, that is there exists $\sigma \in (0,1)$ such that $B(x + t\eta(x), rt^\sigma) \subset \Omega$ for all $x \in \overline{\Omega}$ and $t \in (0, h]$. The author was awared of this fact thanks to H. Mitake. We give a sketch of the proof for this fact in Theorem 2.8.7. For convenience we will define the interior condition in a similar manner as in ($\mathcal{A}_2$).

($\mathcal{A}_3$) There exists a universal pair of positive numbers (r, h), $\eta \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R}^n)$ and $\sigma \in (0, 1]$ such that

$$B(x + t\eta(x), rt^\sigma) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, h].$$

In what follows, we will state some more relaxed conditions on the continuity of H under which a comparison principle can be established. We then give a simple proof for comparison principle under assumption $(\mathcal{A}_1)$ and a more involved proof using the weaker assumption $(\mathcal{A}_2)$.

$(\mathcal{H}3a)$ There exists a modulus $\omega_H : [0,\infty) \to [0,\infty)$, which is a nondecreasing function satisfying $\omega_H(0^+) = 0$ and

$$\begin{cases} |H(x,p) - H(y,p)| \le \omega_H\big(|x-y|(1+|p|)\big), \\ |H(x,p) - H(x,q)| \le \omega_H(|p-q|), \end{cases} \tag{$\mathcal{H}3a$}$$

for $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$.

$(\mathcal{H}3b)$ For every $R > 0$, there exists a modulus $\omega_R : [0,+\infty) \to [0,+\infty)$, which is nondecreasing with $\omega_R(0^+) = 0$ and

$$\begin{cases} |H(x,p) - H(y,p)| \le \omega_R\big(|x-y|\big), \\ |H(x,p) - H(x,q)| \le \omega_R(|p-q|), \end{cases} \tag{$\mathcal{H}3b$}$$

for $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$ with $|p|, |q| \le R$.

For clarity we state two separate versions of the comparison principle, since the proofs are somewhat different.

Theorem 2.8.5. *Assume $(\mathcal{A}_1)$ and $v_1, v_2 \in \mathrm{BUC}(\overline{\Omega};\mathbb{R})$ are a viscosity subsolution and supersolution of (2.8.4) in Ω, respectively. If either*

- *$(\mathcal{H}3a)$ holds, or*
- *$(\mathcal{H}3b)$ holds and v_1 is Lipschitz,*

then $v_1(x) \le v_2(x)$ for all $x \in \overline{\Omega}$.

Proof of Theorem 2.8.5. Let us assume that $\max_{x\in\overline{\Omega}}(v_1(x) - v_2(x)) = v_1(x_0) - v_2(x_0)$ for some $x_0 \in \overline{\Omega}$. For $\varepsilon > 0$ we define

$$\tilde{v}_1^\varepsilon(x) = (1+\varepsilon)v_1\left(\frac{x}{1+\varepsilon}\right) \qquad \text{for } x \in (1+\varepsilon)\overline{\Omega}$$

then (in the viscosity sense)

$$\frac{\delta}{1+\varepsilon}\tilde{v}_1^\varepsilon(x) + H(x, D\tilde{v}_1^\varepsilon(x)) \le 0 \qquad \text{in } (1+\varepsilon)\Omega.$$

Let us define the auxiliary functional

$$\Phi^\varepsilon(x,y) = \tilde{v}_1^\varepsilon(x) - v_2(y) - \frac{|x-y|^2}{\varepsilon}, \qquad (x,y) \in (1+\varepsilon)\overline{\Omega} \times \overline{\Omega}.$$

Assume that Φ^ε achieves its maximum over $(1+\varepsilon)\overline{\Omega}\times\overline{\Omega}$ at $(x_\varepsilon, y_\varepsilon)\in(1+\varepsilon)\overline{\Omega}\times\overline{\Omega}$. Since $(y_\varepsilon, y_\varepsilon)\in(1+\varepsilon)\overline{\Omega}\times\overline{\Omega}$, we have $\Phi^\varepsilon(x_\varepsilon,y_\varepsilon)\geq\Phi^\varepsilon(y_\varepsilon,y_\varepsilon)$ to obtain

$$\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon}\leq\tilde{v}_1^\varepsilon(x_\varepsilon)-\tilde{v}_1^\varepsilon(y_\varepsilon)\leq(1+\varepsilon)\omega_1\left(\frac{|x_\varepsilon-y_\varepsilon|}{1+\varepsilon}\right)\leq 2\omega_1(|x_\varepsilon-y_\varepsilon|)\tag{2.8.16}$$

where $\omega_1(\cdot)$ is the modulus of continuity of v_1. Therefore $|x_\varepsilon-y_\varepsilon|\leq\sqrt{2\varepsilon\,\omega_1(|x_\varepsilon-y_\varepsilon|)}$. Thus $|x_\varepsilon-y_\varepsilon|=\mathcal{o}(\varepsilon)$. This ensures that $x_\varepsilon\in(1+\varepsilon)\Omega$ for all ε small enough. Indeed, if there exist $\varepsilon_k\to 0$ such that $y_{\varepsilon_k}\in(1+\varepsilon_k)\partial\Omega$ for all ε_k, then by assumption $(\mathcal{A}_1)$ reads

$$|x_\varepsilon-y_\varepsilon|\geq\operatorname{dist}(x_\varepsilon,\overline{\Omega})\geq\kappa\varepsilon$$

which is a contradiction to $|x_\varepsilon-y_\varepsilon|=\mathcal{o}(\varepsilon)$. Hence $x_\varepsilon\in(1+\varepsilon)\Omega$ for all ε small. Using the subsolution test and supersolution test we obtain

$$\frac{\delta}{1+\varepsilon}\tilde{v}_1^\varepsilon(x_\varepsilon)+H\left(x_\varepsilon,\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon}\right)\leq 0$$

and

$$\delta v_2(y_\varepsilon)+H\left(y_\varepsilon,\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon}\right)\geq 0.$$

Therefore

$$\delta v_1\left(\frac{x_\varepsilon}{1+\varepsilon}\right)-\delta v_2(y_\varepsilon)\leq H\left(y_\varepsilon,\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon}\right)-H\left(x_\varepsilon,\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon}\right).$$

Since Ω is bounded, we can assume $(x_\varepsilon,y_\varepsilon)\to(\hat{x},\hat{x})$ as $\varepsilon\to 0$ for $\hat{x}\in\overline{\Omega}$. Let $\varepsilon\to 0$ we deduce that

$$v_1(\hat{x})-v_2(\hat{x})\leq 0.\tag{2.8.17}$$

On the other hand, we have $\Phi^\varepsilon(x_\varepsilon,y_\varepsilon)\geq\Phi^\varepsilon(x_0,x_0)$, which implies

$$(1+\varepsilon)v_1\left(\frac{x_\varepsilon}{1+\varepsilon}\right)-v_2(y_\varepsilon)\geq(1+\varepsilon)v_1\left(\frac{x_0}{1+\varepsilon}\right)-v_2(y_0).$$

Let $\varepsilon\to 0$, we obtain that

$$v_1(\hat{x})-v_2(\hat{x})\geq v_1(x_0)-v_2(x_0)$$

which implies that

$$v_1(\hat{x})-v_2(\hat{x})=v_1(x_0)-v_2(x_0)$$

since x_0 is a maximum point of v_1-v_2 over $\overline{\Omega}$. If $(\mathcal{H}3a)$ holds then

$$\delta v_1\left(\frac{x_\varepsilon}{1+\varepsilon}\right)-\delta v_2(y_\varepsilon)\leq\omega_H\left(|x_\varepsilon-y_\varepsilon|\left(1+\left|\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon}\right|\right)\right).$$

Let $\varepsilon\to 0$ we obtain

$$\max_{\overline{\Omega}}(v_1-v_2)=v_1(x_0)-v_2(x_0)\leq 0.$$

If $(\mathcal{H}3b)$ holds then (2.8.16) can be refined to

$$\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon}\leq C|x_\varepsilon-y_\varepsilon|\qquad\Longrightarrow\qquad\frac{|x_\varepsilon-y_\varepsilon|}{\varepsilon}\leq C.$$

Therefore we can now use $(\mathcal{H}3b)$ in place of $(\mathcal{H}3a)$ and the proof carries out similarly. $\square$

Remark 21. The modulus of continuity of v_2 does not really play a role here. If we assume ($\mathcal{H}3b$), a comparison principle only with and without the assumption that v_1 is Lipschitz is still not known at this moment.

Theorem 2.8.6. *Assume* $(\mathcal{A}_2)$ *and* $v_1, v_2 \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R})$ *are a viscosity subsolution and supersolution of* (2.8.4) *in* Ω*, respectively. If either*

- $(\mathcal{H}3a)$ *holds, or*
- $(\mathcal{H}3b)$ *holds and* v_1 *is Lipschitz,*

then $v_1(x) \le v_2(x)$ *for all* $x \in \overline{\Omega}$.

Remark 22. When the uniqueness of (2.8.4) is guaranteed, the unique viscosity solution to (2.8.4) is the maximal viscosity subsolution of (2.8.4). If we assume the coercivity $(\mathcal{H}_4)$ then any subsolution of (2.8.4) is automatically Lipschitz.

Remark 23. We note that ($\mathcal{H}3b$) is weaker than ($\mathcal{H}3a$) and in most standard situations the solution constructed from Perron's method is Lipschitz, which is enough to show uniqueness.

Proof of Theorem 2.8.6. Since v_1, v_2 are bounded on $\overline{\Omega}$, fix $\zeta > 0$ small, there exists $z_\zeta \in \overline{\Omega}$ such that

$$\sup_{\overline{\Omega}}(v_1 - v_2) \ge v_1(z_\zeta) - v_2(z_\zeta) > \sup_{\overline{\Omega}}(v_1 - v_2) - \zeta.$$

Choose ε small such that $\frac{2\varepsilon}{r} < h$, then

$$B\left(z + \frac{2\varepsilon}{r}\eta(z), 2\varepsilon\right) \subset \Omega \qquad \text{for all } z \in \overline{\Omega}. \tag{2.8.18}$$

If we define some auxiliary functional $\Phi(x, y)$ to use the doubling variable technique and assume heuristically that its maximum achieves at (x^*, y^*). The supersolution test at y^* holds easily, while in order to use the subsolution test at x^* we need $x^* \in \Omega$. We will manage to obtain that by using (2.8.18), i.e.,

$$x^* \in B\left(y^* + \frac{2\varepsilon}{r}\eta(y^*), 2\varepsilon\right) \subset \Omega.$$

That means we need to control the distance

$$\left| x^* - \left(y^* + \frac{2\varepsilon}{r}\eta(y^*) \right) \right| < 2\varepsilon.$$

Motivated from that, we define the new auxiliary functional as

$$\Phi^\varepsilon(x, y) = v_1(x) - v_2(y) - \left| \frac{x - y}{\varepsilon} - \frac{2}{r}\eta(z_\zeta) \right|^2 - \left| \frac{y - z_\zeta}{\rho} \right|^2, \qquad (x, y) \in \overline{\Omega} \times \overline{\Omega}$$

where $\rho > 0$ is chosen such that $|x - y| < \rho$ implies $|\eta(x) - \eta(y)| < \frac{r}{2}$.

We claim that Φ^ε achieves its maximum over $\overline{\Omega} \times \overline{\Omega}$ at some $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$. This is not trivial since Ω can be unbounded. We observe that

$$\Phi^\varepsilon\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta), z_\zeta\right) = v_1\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta)\right) - v_2(z_\zeta)$$

We claim that

$$\Phi^\varepsilon(x,y) \le \Phi^\varepsilon\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta), z_\zeta\right) \tag{2.8.19}$$

if $|x|, |y| \ge R$ for some $R > 0$. Indeed, let us look at the reverse inequality of (2.8.19), i.e.,

$$\Phi^\varepsilon(x,y) \ge \Phi^\varepsilon\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta), z_\zeta\right) \ge v_1(z_\zeta) - v_2(z_\zeta) - \omega_1\left(\frac{2\varepsilon}{r}|\eta(z_\zeta)|\right) \tag{2.8.20}$$

where ω_1, ω_2 are modulus of continuity of v_1, v_2 respectively. Using (2.8.20) we have

$$\begin{aligned}
\sup_{\overline{\Omega}}(v_1 - v_2) + \omega_1(|x-y|) - \left|\frac{y - z_\zeta}{\rho}\right|^2 - \left|\frac{x-y}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right|^2 \\
&\ge \Phi^\varepsilon(x,y) \ge \Phi^\varepsilon\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta), z_\zeta\right) \\
&\ge v_1(z_\zeta) - v_2(z_\zeta) - \omega_1\left(\frac{2\varepsilon}{r}|\eta(z_\zeta)|\right) \\
&\ge \sup_{\overline{\Omega}}(v_1 - v_2) - \zeta - \omega_1\left(\frac{2\varepsilon}{r}|\eta(z_\zeta)|\right).
\end{aligned} \tag{2.8.21}$$

Therefore

$$\omega_1(|x-y|) + \omega_1\left(\frac{2\varepsilon}{r}|\eta(z_\zeta)|\right) + \zeta \ge \left|\frac{y - z_\zeta}{\rho}\right|^2 + \left|\frac{x-y}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right|^2.$$

Since ω_1, ω_2 are bounded, there exists $R > 0$ such that if $x, y \notin B(z_\zeta, R)$ then (2.8.19) fails, thus Φ^ε achieves its maximum over $\overline{\Omega} \times \overline{\Omega}$ at $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$.

Using $\Phi^\varepsilon(y_\varepsilon, y_\varepsilon) \le \Phi^\varepsilon(x_\varepsilon, y_\varepsilon)$ we obtain that

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right|^2 \le v_1(x_\varepsilon) - v_1(y_\varepsilon) + \left|\frac{2}{r}\eta(z_\zeta)\right|^2.$$

We deduce that

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right|^2 - 2\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right)\cdot\left(\frac{2}{r}\eta(z_\zeta)\right) + \left|\frac{2}{r}\eta(z_\zeta)\right|^2 \le 2\|v_1\|_{L^1(\overline{\Omega})} + \left|\frac{2}{r}\eta(z_\zeta)\right|^2$$

which implies that

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right|^2 \le 2\|v_1\|_{L^1(\overline{\Omega})} + \frac{1}{2}\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right|^2 + 2\left|\frac{2}{r}\eta(z_\zeta)\right|^2.$$

Therefore

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right|^2 \le 4\|v_1\|_{L^\infty(\overline{\Omega})} + 4\left|\frac{2}{r}\eta(z_\zeta)\right|^2 \qquad \Longrightarrow \qquad |x_\varepsilon - y_\varepsilon| \le C\varepsilon. \tag{2.8.22}$$

Using $\Phi^\varepsilon\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta), z_\zeta\right) \le \Phi^\varepsilon(x_\varepsilon, y_\varepsilon)$ and (2.8.21) we have

$$\begin{aligned}
\sup_{\overline{\Omega}}(v_1 - v_2) - \zeta - \omega_1\left(\frac{2\varepsilon}{r}|\eta(z_\zeta)|\right) &\le v_1\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta)\right) - v_2(z_\zeta) \\
&= \Phi^\varepsilon\left(z_\zeta + \frac{2\varepsilon}{r}\eta(z_\zeta), z_\zeta\right) \le \Phi^\varepsilon(x_\varepsilon, y_\varepsilon) \\
&\le v_1(x_\varepsilon) - v_2(y_\varepsilon) - \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right|^2 - \left|\frac{y_\varepsilon - z_\zeta}{\rho}\right|^2 \\
&\le \sup_{\overline{\Omega}}(v_1 - v_2) + \omega_1(|x_\varepsilon - y_\varepsilon|) - \left|\frac{y_\varepsilon - z_\zeta}{\rho}\right|^2 - \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right|^2.
\end{aligned}$$

Using (2.8.22) we deduce that (we concerns small δ only)

$$\left|\frac{y_\varepsilon - z_\zeta}{\rho}\right|^2 + \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right|^2 \le \zeta + \omega_1(C\varepsilon) + \omega_1\left(\frac{2\varepsilon}{r}\eta(z_\zeta)\right) \le 2\zeta < 1 \tag{2.8.23}$$

if ε small enough. Since $|y_\varepsilon - z_\zeta| < \rho$ we have $|\eta(y_\varepsilon) - \eta(z_\zeta)| < \frac{r}{2}$. Therefore

$$\begin{aligned}
\left|x_\varepsilon - \left(y_\varepsilon + \frac{2\varepsilon}{r}\eta(y_\varepsilon)\right)\right| &\le \left|x_\varepsilon - \left(y_\varepsilon + \frac{2\varepsilon}{r}\eta(z_\zeta)\right)\right| + \frac{2\varepsilon}{r}|\eta(y_\varepsilon) - \eta(z_\zeta)| \\
&\le \varepsilon\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{2}{r}\eta(z_\zeta)\right| + \frac{2\varepsilon}{r}|\eta(y_\varepsilon) - \eta(z_\zeta)|\varepsilon + \frac{2\varepsilon}{r}\cdot\frac{r}{2} = 2\varepsilon.
\end{aligned}$$

As mentioned in (2.8.18), we deduce that $x_\varepsilon \in \Omega$. Now the subsolution test for v_1 at x_ε gives us

$$\delta v_1(x_\varepsilon) + H(x_\varepsilon, p_\varepsilon) \le 0 \qquad \text{where} \qquad p_\varepsilon = \frac{2}{\varepsilon}\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{2}{r}\eta(z_\delta)\right).$$

Similarly, the supersolution test for v_2 at y_ε gives us

$$\delta v_2(y_\varepsilon) + H(y_\varepsilon, p_\varepsilon + q_\varepsilon) \ge 0 \qquad \text{where} \qquad q_\varepsilon = -\frac{2}{\rho}\left(\frac{y_\varepsilon - z_\delta}{\rho}\right).$$

Therefore

$$\delta v_1(x_\varepsilon) - \delta v_2(y_\varepsilon) \le H(y_\varepsilon, p_\varepsilon + q_\varepsilon) - H(x_\varepsilon, p_\varepsilon).$$

From (2.8.23) we have

$$|q_\varepsilon| \le C\sqrt{\zeta}, \qquad |p_\varepsilon| \le \frac{2}{\varepsilon}\sqrt{\zeta + \omega_1(C\varepsilon) + \omega_1\left(\frac{2\varepsilon}{r}\eta(z_\zeta)\right)}.$$

If we assume $(\mathcal{H}3a)$ then

$$\begin{aligned}\delta v_1(x_\varepsilon) - \delta v_2(y_\varepsilon) &\le \omega_H(|q_\varepsilon|) + \omega_H(|x_\varepsilon - y_\varepsilon|(1+|p_\varepsilon|)) \\ &\le \omega_H\left(C\sqrt{\zeta}\right) + \omega_H\left(C\varepsilon + 2\sqrt{\zeta + \omega_1 C\varepsilon) + \omega_1\left(C\varepsilon\eta(z_\zeta)\right)}\right).\end{aligned}$$

Let $\varepsilon \to 0$, it is clear that $(x\varepsilon, y_\varepsilon) \to (z_\zeta, z_\zeta)$, thus

$$\delta \sup_{\overline{\Omega}}(v_1 - v_2) - \delta\zeta \le \delta v_1(z_\zeta) - \delta v_2(z_\zeta) \le 2\omega_H\left(C\sqrt{\zeta}\right).$$

Let $\zeta \to 0$ we obtain $v_1 \le v_2$ on $\overline{\Omega}$.

If we assume $(\mathcal{H}3b)$ then as v_1 is Lipschitz and $p_\varepsilon \in D^+ v_1(x_\sigma)$, we have $|p_\varepsilon| \le C$. This together with the boundedness of q_ε give us a modulus ω_R as in $(\mathcal{H}3b)$ and the proof carries out similarly. □

We sketch a proof in case the interior cone condition is assumed instead of the interior ball condition, as being mentioned in Remark 20.

Theorem 2.8.7. *Assume $(\mathcal{A}_3)$ and $v_1, v_2 \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R})$ are a viscosity subsolution and supersolution of (2.8.4) in Ω, respectively. If either*

- *$(\mathcal{H}3a)$ holds, or*
- *$(\mathcal{H}3b)$ holds and v_1 is Lipschitz,*

then $v_1(x) \le v_2(x)$ for all $x \in \overline{\Omega}$.

Sketch of the proof of Theorem 2.8.7. Since v_1, v_2 are bounded on $\overline{\Omega}$, fix $\zeta > 0$ small, there exists $z_\zeta \in \overline{\Omega}$ such that $v_1(z_\zeta) - v_2(z_\zeta) > \sup_{\overline{\Omega}}(v_1 - v_2) - \zeta$. Let $\theta = \sigma^{-1} \ge 1$, then for $\varepsilon > 0$ we have

$$B\left(z + \left(\frac{2\varepsilon}{r}\right)^\theta \eta(z), 2\varepsilon\right) \subset \Omega \qquad \text{for all } z \in \overline{\Omega} \text{ if } \left(\frac{2\varepsilon}{r}\right)^\theta < h. \tag{2.8.24}$$

We define the auxiliary functional as

$$\Phi^\varepsilon(x,y) = v_1(x) - v_2(y) - \left|\frac{x-y}{\varepsilon^\theta} - \left(\frac{2}{r}\right)^\theta \eta(z_\zeta)\right|^2 - \left|\frac{y - z_\zeta}{\rho}\right|^2, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega}$$

where $\rho > 0$ is chosen such that $|x-y| < \rho$ implies $|\eta(x) - \eta(y)| < \left(\frac{r}{2}\right)^\theta$. Similar to, Φ^ε achieves its maximum over $\overline{\Omega} \times \overline{\Omega}$ at some $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$. Using $\Phi^\varepsilon(y_\varepsilon, y_\varepsilon) \le \Phi^\varepsilon(x_\varepsilon, y_\varepsilon)$ and proceed similarly as in the proof of Theorem 2.8.6 we obtain that

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^\theta}\right|^2 \le 4\|v_1\|_{L^\infty(\overline{\Omega})} + 4\left|\left(\frac{2}{r}\right)^\theta \eta(z_\zeta)\right|^2 \qquad \Longrightarrow \qquad |x_\varepsilon - y_\varepsilon| \le C\varepsilon^\theta. \tag{2.8.25}$$

Using $\Phi^{\varepsilon}\left(z_\zeta + \left(\frac{2\varepsilon}{r}\right)^\theta \eta(z_\zeta), z_\zeta\right) \leq \Phi^{\varepsilon}(x_\varepsilon, y_\varepsilon)$ we have

$$\begin{aligned}
&\sup_{\overline{\Omega}}(v_1 - v_2) - \zeta - \omega_1\left(\left(\frac{2\varepsilon}{r}\right)^\theta |\eta(z_\zeta)|\right) \\
&\qquad \leq \sup_{\overline{\Omega}}(v_1 - v_2) + \omega_1(|x_\varepsilon - y_\varepsilon|) - \left|\frac{y_\varepsilon - z_\zeta}{\rho}\right|^2 - \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^\theta} - \left(\frac{2}{r}\right)^\theta \eta(z_\zeta)\right|^2.
\end{aligned}$$

Using (2.8.25) we deduce that

$$\left|\frac{y_\varepsilon - z_\zeta}{\rho}\right|^2 + \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^\theta} - \left(\frac{2}{r}\right)^\theta \eta(z_\zeta)\right|^2 \leq \zeta + \mathcal{O}(\varepsilon^\theta) < 1 \tag{2.8.26}$$

if ε small enough. As a consequence

$$|y_\varepsilon - z_\zeta| < \rho \qquad \Longrightarrow \qquad |\eta(y_\varepsilon) - \eta(z_\zeta)| < \left(\frac{r}{2}\right)^\theta.$$

Using that fact we have

$$\begin{aligned}
\left|x_\varepsilon - \left(y_\varepsilon + \left(\frac{2\varepsilon}{r}\right)^\theta \eta(y_\varepsilon)\right)\right| &\leq \left|x_\varepsilon - \left(y_\varepsilon + \left(\frac{2\varepsilon}{r}\right)^\theta \eta(z_\zeta)\right)\right| + \left(\frac{2\varepsilon}{r}\right)^\theta |\eta(y_\varepsilon) - \eta(z_\zeta)| \\
&\leq \varepsilon^\theta \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^\theta} - \left(\frac{2}{r}\right)^\theta \eta(z_\zeta)\right| + \left(\frac{2\varepsilon}{r}\right)^\theta |\eta(y_\varepsilon) - \eta(z_\zeta)| \\
&\leq \varepsilon^\theta + \left(\frac{2\varepsilon}{r}\right)^\theta \left(\frac{r}{2}\right)^\theta = 2\varepsilon^\theta \leq 2\varepsilon
\end{aligned}$$

since $\theta \geq 1$. As being mentioned in (2.8.24), we deduce that $x_\varepsilon \in \Omega$ and the rest of the proof is similar to the proof of Theorem 2.8.6. □

Remark *24.* The following Lemma show that the scaling structure $(\mathcal{A}_1)$ of a domain is rather strong, comparing to a sphere condition like $(\mathcal{A}_2)$. The author learned this result from H. Ishii during a seminar talk given in October 2020.

Lemma 2.8.8. *Assume that Ω is bounded, open and $0 \in \Omega$. Assume further that*

$$\mathrm{dist}(x, \overline{\Omega}) \geq \kappa r \qquad \text{for all } x \in (1+r)\partial\Omega, \text{ for all } r > 0. \tag{2.8.27}$$

holds for some $\kappa > 0$, then Ω is star-shaped and $(\mathcal{A}_2)$ holds.

Proof of Lemma 2.8.8. Suppose that Ω is not star-shaped, there exists $x \in \overline{\Omega}$ and $0 < \theta < 1$ such that $\theta x \notin \Omega$. Since 0 is an interior point of Ω, there exists $0 < \delta < \theta$ such that $\tau x \in \Omega$ for all $0 < \sigma \leq \delta$. Let us define $\eta = \sup\{\tau > 0 : \tau x \in \Omega\}$ then $0 < \delta \leq \eta \leq \theta$ and $\eta x \in \partial\Omega$. Set $y = \eta x \in \partial\Omega$, we see that

$$x = \eta^{-1} y = (1+r)y \in (1+r)\partial\Omega$$

where $\eta^{-1} = 1 + r$. Now (2.8.27) gives us that $0 = \mathrm{dist}(x, \Omega) \geq \kappa r$ which is a contradiction and thus Ω is star-shaped. For $0 < r < 1$, as Ω is star-shaped, $(1-r)\overline{\Omega} \subset (1+r)^{-1}\Omega$ and

$$B\left(0, \frac{\kappa r}{1+r}\right) \subset B\left(0, \frac{\kappa r}{2}\right).$$

From (2.8.27) we have $\Omega + B(0, \kappa r) \cap (1+r)\partial\Omega = \emptyset$ for all $r \in (0,1)$, therefore

$$(1+r)^{-1}\Omega + B\left(0, \frac{\kappa r}{1+r}\right) \cap \partial\Omega = \emptyset \quad \Longrightarrow \quad (1-r)\overline{\Omega} + B\left(0, \frac{\kappa r}{2}\right) \cap \partial\Omega = \emptyset.$$

From (2.8.27) we deduce that $(1-r)\overline{\Omega} + B\left(0, \frac{\kappa r}{2}\right) \subset \Omega$. We observe that

$$B\left(x - rx, \frac{\kappa r}{2}\right) = (1-r)x + B\left(0, \frac{\kappa r}{2}\right) \subset (1-r)\overline{\Omega} + B\left(0, \frac{\kappa r}{2}\right) \subset \Omega.$$

This implies $(\mathcal{A}_2)$ with

$$\eta = x, \qquad r = \frac{\kappa}{2}, \qquad \text{and} \qquad h = \frac{1}{2}$$

which completes the proof. $\square$

Chapter 3

Optimal rate of convergence for multi-scale periodic homogenization

In this chapter we study the behavior, as $\varepsilon \to 0^+$, of solutions (u^ε) of the following oscillatory initial value Hamilton–Jacobi equation

$$\begin{cases} u_t^\varepsilon + H\left(x, \dfrac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in} \quad \mathbb{R} \times [0, \infty) \\ \qquad\qquad u^\varepsilon(x, 0) = u_0(x) & \text{on} \quad \mathbb{R}, \end{cases} \tag{C$_\varepsilon$}$$

where the initial data u_0 is contained in $\mathrm{BUC}(\mathbb{R}^n)$. There is an *effective* equation that is associated with (C$_\varepsilon$), which takes the following form

$$\begin{cases} u_t + \overline{H}(x, Du) & = 0 \qquad \text{in} \quad \mathbb{R} \times [0, \infty) \\ \qquad u(x, 0) & = u_0(x) \quad \text{on} \quad \mathbb{R}. \end{cases} \tag{C}$$

The *effective Hamiltonian* $\overline{H}$ and the association between (C$_\varepsilon$) and (C) are described in the next section. In this chapter we show that there exists a limit function u, which is a solution to (C), such that $u^\varepsilon(x, t) \to u(x, t)$ locally uniformly in $\mathbb{R}^n \times (0, \infty)$ and

$$|u^\varepsilon - u| = \mathcal{O}(\varepsilon)$$

and this rate is *optimal*. The material of this chapter is mainly taken from [113] with some new literature added on the developments and recent breakthroughs in this topic for the case $H(\frac{x}{\varepsilon}, Du^\varepsilon)$ homogenizes to $\overline{H}(Du)$.

- In [38], the author combining the representation formula from optimal control theory and a theorem of Alexander from first passage percolation theory to obtain $|u^\varepsilon - u| = \mathcal{O}(\varepsilon \log(\varepsilon))$.
- In [112], the authors use optimal control formula and a curve decomposition technique to obtain the optimal rate $|u^\varepsilon - u| = \mathcal{O}(\varepsilon)$ for all dimensions, finishing the big open problem of the field.

Our result is in a different setting, which remains open whether or not the same optimal rate $\mathcal{O}(\varepsilon)$ can be obtained for all dimensions. We note that all of the results above are under the assumption of convexity on H, which is crucial to apply the optimal control formula.

3.1 Introduction to Homogenization of Hamilton–Jacobi equations

We first give a brief description of the periodic homogenization theory for Hamilton–Jacobi equations in the framework of viscosity solutions (see [8, 41, 79, 83]). Consider the problem (C_ε) with a given a Hamiltonian $H(x,y,p) \in C(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfying some conditions (H1)–(H4) below, define the effective Hamiltonian as follows: For each $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, let $\overline{H}(x,p) \in \mathbb{R}$ be the unique constant for which the cell (ergodic) problem

$$H\big(x,y,p+D_y v(y)\big) = \overline{H}(x,p) \qquad \text{in } \mathbb{T}^n \tag{CP}$$

has a continuous viscosity solution $v(y) = v(y;x,p)$. That such a constant exists and is unique is proven in [85] and [47, 48]. It is worth mentioning that in general the solution $v(y;x,p)$ to the cell problem (CP) is not unique even up to the addition of a constant. The effective Hamilton-Jacobi equation corresponding to (C_ε) is given by the following Cauchy problem:

$$\begin{cases} u_t + \overline{H}\,(x, Du) &= 0 \quad \text{ in } \quad \mathbb{R} \times [0,\infty) \\ \qquad\quad u(x,0) &= u_0(x) \;\; \text{on} \quad \mathbb{R}. \end{cases} \tag{C}$$

Some papers treating the properties of the effective Hamiltonian $\overline{H}$ are [22, 36, 37, 86, 103], and the references given therein.

The theory of periodic homogenization studies the behavior of viscosity solutions $u^\varepsilon \in C(\mathbb{R}^n \times [0,\infty)$ to (C_ε) as the period of oscillation ε approaches 0^+. The first results in the theory of periodic homogenization were proved under the following assumptions on the Hamiltonian $H = H(x,y,p) \in C(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^n)$:

(H1) For each $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, $y \mapsto H(x,y,p)$ is $\mathbb{Z}^n$-periodic.

(H2) $p \mapsto H(x,y,p)$ is uniformly coercive in $(x,y) \in \mathbb{R}^n \times \mathbb{T}^n$. That is,

$$\lim_{|p|\to+\infty} \left(\inf_{(x,y)\in\mathbb{R}^n\times\mathbb{T}^n} H(x,y,p) \right) = +\infty.$$

Here $\mathbb{T}^n = \mathbb{R}^n \backslash \mathbb{Z}^n$.

(H3) $\sup\left\{ |H(x,y,p)| : (x,y) \in \mathbb{R}^{2n}, |p| \le R \right\} < \infty$ for all $R > 0$.

(H4) For each $R > 0$, there exists $\omega_R(\cdot) \in C([0,\infty))$, with $\omega_R(0) = 0$, such that for all $x, y \in \mathbb{R}^n, p, q \in B(0,R)$ then

$$|H(x,y,p) - H(x,y,q)| \le \omega_R(|p-q|)$$

where $B(0,R)$ denotes the open ball centered at 0 with radius R in $\mathbb{R}^n$.

Under the assumptions (H1)–(H4), the viscosity solutions u^ε converge to a limit u locally uniformly on $\mathbb{R}^n \times [0,\infty)$, where u is a viscosity solution to the effective equation (C). This was first proved by P.-L. Lions, G. Papanicolau and S.R.S. Varadhan [85] in the case that H is independent of x, namely $H(x,y,p) = H(y,p)$. The more general case in which $H = H(x,y,p)$ can depend on x was established later by L. C. Evans [47, 48], who developed the perturbed test functions method for studying the homogenization problem in the framework of viscosity solutions.

The rate of convergence of $u^\varepsilon \to u$ was first studied by I. Capuzzo-Dolcetta and H. Ishii in [28] using a PDE approach. They consider the stationary problem

$$w^\varepsilon(x) + H\left(x, \frac{x}{\varepsilon}, Dw^\varepsilon(x)\right) = 0 \qquad \text{in } \mathbb{R}^n. \tag{S_ε}$$

As $\varepsilon \to 0$, $w^\varepsilon \to w$ locally uniformly on $\mathbb{R}^n$ and w solves the effective equaition

$$w(x) + \overline{H}\left(x, Dw(x)\right) = 0 \qquad \text{in } \mathbb{R}^n. \tag{S}$$

Under this stationary setting, the authors of [28] establish the rate of convergence is *at least* $\mathcal{O}(\varepsilon^{1/3})$ for general (including nonconvex) Lipschitz Hamiltonians under quite general assumptions. In the case that $H(x,y,p) = H(y,p)$, Capuzzo-Dolcetta and Ishii obtain the rate of convergence $\mathcal{O}(\varepsilon)$ of w^ε to w by a simple comparison argument. Their approach can be easily adjusted to handle the Cauchy problem (C_ε) giving the same rate $\mathcal{O}(\varepsilon^{1/3})$. This approach is quite robust, and it works for various different situations. Another example occurs in [88], where C. Marchi considers the case where H depends on more scales, and establishes the rate $\mathcal{O}(\varepsilon^{1/3}) + \omega(\varepsilon)$ for some modulus of continuity of H using the method of [28].

Heuristically, the rate of convergence $\mathcal{O}(\varepsilon)$ seems to be optimal. By using an ansatz $u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$ and plugging it into (C_ε), we can derive the following two–scale asymptotic expansion (see [79, 85, 95]),

$$u^\varepsilon(x,t) \approx u(x,t) + \varepsilon v\left(\frac{x}{\varepsilon}; x, Du(x,t)\right) + \mathcal{O}(\varepsilon^2), \tag{3.1.1}$$

in which the rate of convergence looks like $\mathcal{O}(\varepsilon)$. However, it is hard to justify (3.1.1) rigorously as the solution $u(x,t)$ to (C) is only Lipschitz in (x,t), and is usually not C^1. Also, the solution v to the ergodic problem (CP) is not unique even up to the addition of a constant (Example 6.1 in [79] or Proposition 5.4 in [83]).

3.1.1 Spatial independent effective Hamiltonians

In the case where $H(x,y,p) = H(y,p)$, i.e., $H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right)$ homogenizes to $\overline{H}(Du)$, there has been major developments recently with the biggest open problem completely solved.

H. Mitake, H. V. Tran and Y. Yu established in [95] that the rate $\mathcal{O}(\varepsilon)$ is optimal in the case that the dimension $n = 1$ and the Hamiltonian H is convex (and independent of x). They provide the following example of a family of u^ε's that converge to u at the strict rate of $O(\varepsilon)$:

Proposition 3.1.1. *Let $n = 1$ and $H(y,p) = \frac{1}{2}|p|^2 + V(y)$ where $V \in C(\mathbb{T})$ with $\max_{\mathbb{T}} V = 0$ and $V \le -1$ in $\left[-\frac{1}{3}, \frac{1}{3}\right]$. Then in this case $u \equiv 0$, $\|u^\varepsilon\|_{L^\infty(\mathbb{R}\times[0,\infty))} \le C\varepsilon$ and $u^\varepsilon(0,1) \ge \frac{1}{6}\varepsilon$ for all $\varepsilon \in (0,1)$.*

Proposition 3.1.1 and other important results in higher dimensional spaces are proved in [95] using tools from dynamical systems and weak KAM theory.

Very recently, regarding the time of writing this thesis, in [38], the author combining the representation formula from optimal control theory and a theorem of Alexander from first passage percolation theory to obtain $|u^\varepsilon - u| = \mathcal{O}(\varepsilon\ \log(\varepsilon))$. The big break through came in [112] where the authors use optimal control formula and a curve decomposition technique to obtain the optimal rate $|u^\varepsilon - u| = \mathcal{O}(\varepsilon)$ for all dimensions, finishing the big open problem of the field.

Mitake, Tran, and Yu also present in [95] an essential obstacle to improving the convergence rate $\mathcal{O}(\varepsilon^{1/3})$ by the method used by Capuzzo-Dolcetta and Ishii in [28]. More precisely, for each $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, instead of using $v(y;x,p)$ directly in (3.1.1), the authors of [28] use $v^\lambda(y) = v^\lambda(y;x,p)$ as the unique solution to the following discount problem

$$\lambda v^\lambda(y) + H\left(x,y,p+D_y v^\lambda(y)\right) = 0 \qquad \text{in } \mathbb{T}^n, \tag{S_λ}$$

and approximate $Du(x,t)$ by $\frac{x-y}{\varepsilon^\beta}$ in (3.1.1) using the doubling variable method. By optimizing λ and β, $\mathcal{O}(\varepsilon^{1/3})$ is the best convergence rate that can be obtained. In order to improve the convergence rate, it is necessary to have a nice selection of viscosity solutions $v(\cdot;x,p)$ to the ergodic problem (CP) with respect to (x,p), so that one can use directly $v(y;x,p)$ instead of $v^\lambda(y;x,p)$ in (3.1.1). In the case that $H(x,y,p) = H(y,p)$, assume that

$$\begin{cases} \text{For each } p \in \mathbb{R}^n \text{ there exists a solution } v(\cdot;p) \text{ of (CP)} \\ \text{such that } p \mapsto v(\cdot;p) \text{ is Lipschitz.} \end{cases} \tag{3.1.2}$$

Then, the convergence rate can be improved from $\mathcal{O}(\varepsilon^{1/3})$ to $\mathcal{O}(\varepsilon^{1/2})$, as one needs only introduce one parameter into the doubling variable formulation (see Section 7.2 in [111]) instead of two parameters as before. However, condition (3.1.2) is quite restrictive in general and does not always hold (see Section 5 in [95]).

3.1.2 Other related settings

Closely related to the results outlined above for the problem (C_ε) are the recent developments in the case of the viscous Hamilton–Jacobi equations. Let $H = H(x,y,p,X) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ be the Hamiltonian that is $\mathbb{Z}^n$-periodic in the y variable, and $\mathbb{S}^n$ denotes the set of $n \times n$ symmetric matrices. The associated viscous Cauchy problem is

$$\begin{cases} u_t^\varepsilon + H\left(x, \frac{x}{\varepsilon}, Du^\varepsilon, D^2u^\varepsilon\right) &= 0 \qquad \text{in} \quad \mathbb{R} \times [0,\infty) \\ \qquad\qquad u^\varepsilon(x,0) &= u_0(x) \quad \text{on} \quad \mathbb{R}, \end{cases} \tag{C_ε^*}$$

One can find the effective Hamiltonian $\overline{H}$ with a method similar to that used in the non-viscous case and obtain a solution u to the Cauchy problem associated to $\overline{H}$, such that the solutions u^ε to (C_ε^*) converge locally uniformly to u (see [24, 48]). The following analogous results on the rate of convergence of $u^\varepsilon \to u$ for the viscous Hamilton–Jacobi equation below are important to note:

- In the stationary setting, F. Camilli and C. Marchi ([24]) show that the rate is $\mathcal{O}(\varepsilon)$ if $H = H(y,p,X)$. It can be upgraded to $\mathcal{O}(\varepsilon^2)$ if $H = H(y,X)$.
- F. Camilli, C. Annalisa and C. Marchi ([21]) show that the rate is $\mathcal{O}(\varepsilon)$ for the vanishing viscosity problem $u^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon, \varepsilon D^2 u^\varepsilon\right) = 0$ in $\mathbb{R}^n$.

- For the Cauchy problem $u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon, \varepsilon D^2 u^\varepsilon\right) = 0$ in $\mathbb{R}^n \times (0,\infty)$ with initial data $u(x,0) = g(x)$ on $\mathbb{R}^n$, S. Kim and K.-A. Lee ([74]) obtain high order rates of convergence for special chosen initial data.

In both situations, viscous and nonviscous, the case when H depends on x is significantly harder. In particular, the methods used in [21, 24] provide the rate $\mathcal{O}(\varepsilon^\alpha)$ for some $\alpha < 1$.

We refer to [24, 21, 74] and the references therein for more related results on the viscous case. See also [5, 19, 87, 91] and the references therein for related results to the rate of convergence of Hamilton–Jacobi equations in stochastic homogenization and other settings.

3.1.3 Multi-scale setting

Even though the question in the case of $H\left(\frac{x}{\varepsilon}, Du^\varepsilon(x)\right)$ homogenizes to $\overline{H}(Du(x))$ has been completely settled, the problem regarding the multi-scale setting where

$$H\left(x, \frac{x}{\varepsilon}, Du^\varepsilon(x)\right) \qquad \text{homogenizes to} \qquad \overline{H}(x, Du(x))$$

remains unfinished. The best-known convergence rate in this setting is $\mathcal{O}(\varepsilon^{1/3})$, obtained in [28]. The main goal of this chapter is to obtain the optimal rate of convergence of $u^\varepsilon \to u$ in one dimension: more precisely, to obtain an optimal bound for $\|u^\varepsilon - u\|_{L^\infty([-R,R]\times[0,T])}$ for any given $R, T > 0$ as $\varepsilon \to 0^+$.

3.2 Optimal rate of convergence in one dimension

We consider the one dimensional case $n = 1$ and the convex Hamiltonian is of the form:

$$H(x,y,p) = H(p) + V(x,y) \qquad \text{for all} \qquad (x,y,p) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}.$$

For a clear presentation, we present the proof for the classical mechanics Hamiltonian first, then generalized to a more complicated Hamiltonian later. The idea of the proof is to analyze quantitatively the minimizer paths of the corresponding optimal control problem.

3.2.1 Classical Mechanics Hamiltonian

Theorem 3.2.1. *Assume $n = 1$ and $H(x,y,p) = \frac{1}{2}|p|^2 + V(x,y)$ where V is of the separable form $V(x,y) = a(x)b(y) + C_0$ where C_0 is a constant and*

(i) $a(x) \in \mathrm{C}^1(\mathbb{R})$ is bounded with $a(x) > 0$ for all $x \in \mathbb{R}$,

(ii) $b(y) \in \mathrm{C}(\mathbb{T})$ and $\max_{y\in\mathbb{T}} b(y) = 0$.

Assume $u_0 \in \mathrm{Lip}(\mathbb{R}) \cap \mathrm{BUC}(\mathbb{R})$, then for each $R, T > 0$ we have

$$\|u^\varepsilon - u\|_{L^\infty([-R,R]\times[0,T])} \leq C\varepsilon \tag{3.2.1}$$

where C is a constant depends on $R, T, \mathrm{Lip}(u_0)$, $a(x)$ and $\max|b(y)|$.

***Remark* 25.** In Theorem 3.2.1 if $V(x,y) = V(y)$ does not depend on x, then we can choose C explicitly as $C = 2\|u_0'\|_{L^\infty(\mathbb{R})} + 8\|V\|_{L^\infty}^{1/2}$. As a consequence, the convergence is uniform in the sense that $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}\times[0,\infty))} \leq C\varepsilon$ (Section 2 and Remark 26).

Proof of Theorem 3.2.1. We observe that the estimate (3.2.1) does not depend on the smoothness of $b(\cdot)$, by approximation, without loss of generality we can assume that $V \in \mathrm{C}^2(\mathbb{R}\times\mathbb{T})$. Also, by replacing u by $u + C$ we can normalize that $C_0 = 0$. Let us fix $R, T > 0$, $\varepsilon \in (0,1)$ and $(x_0, t_0) \in [-R, R] \times [0, T]$, thanks to the optimal control formula (see [8, 79]) we have

$$u^\varepsilon(x_0,t_0) = \inf_{\eta\in\mathcal{T}} \left\{ \varepsilon \int_0^{\varepsilon^{-1}t_0} \left(\frac{|\dot{\eta}(s)|^2}{2} - V\left(\varepsilon\eta(s), \eta(s)\right) \right) ds + u_0\left(\varepsilon\eta(\varepsilon^{-1}t_0)\right) \right\}, \tag{3.2.2}$$

where $\mathcal{T} = \left\{\eta(\cdot) \in \mathrm{AC}\left([0,\varepsilon^{-1}t_0]\right), \varepsilon\eta(0) = x_0\right\}$. Here $\mathrm{AC}([a,b])$ denotes the set of absolutely continuous functions from $[a,b]$ to $\mathbb{R}$. Let $\eta_\varepsilon(\cdot) \in \mathcal{T}$ be a minimizer to the optimization problem (3.2.2), it is clear that $\eta_\varepsilon(\cdot)$ must satisfy the following Euler-Lagrange equation

$$\begin{cases} \ddot{\eta}_\varepsilon(s) &= -\nabla V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right)\cdot(\varepsilon, 1) \qquad \text{on} \qquad \left(0, \varepsilon^{-1}t_0\right), \\ \eta_\varepsilon(0) &= \varepsilon^{-1}x_0. \end{cases} \tag{3.2.3}$$

Here ∇V means the full gradient of V. In particular, this implies the following conservation of energy:

$$\frac{d}{ds}\left(\frac{|\dot{\eta}_\varepsilon(s)|^2}{2} + V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right)\right) = \dot{\eta}_\varepsilon(s)\Big(\ddot{\eta}_\varepsilon(s) + \nabla V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right)\cdot(\varepsilon,1)\Big) = 0$$

for all $s \in \left(0, \varepsilon^{-1}t_0\right)$. There exists a constant $r = r(\eta_\varepsilon) \in [V(0,0), +\infty)$ such that

$$\frac{|\dot{\eta}_\varepsilon(s)|^2}{2} + V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right) = r \qquad \text{for all} \qquad s \in (0, \varepsilon^{-1}t_0). \tag{3.2.4}$$

For each $r \in \left[V(x_0, \varepsilon^{-1}x_0), \infty\right)$ the Euler-Lagrange equation (3.2.3) is

$$\begin{cases} \ddot{\eta}_\varepsilon(s) &= -\nabla V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right)\cdot(\varepsilon,1) \qquad \text{on} \qquad \left(0, \varepsilon^{-1}t_0\right), \\ |\dot{\eta}_\varepsilon(0)| &= \sqrt{2(r - V(x_0, \varepsilon^{-1}x_0))}, \\ \eta_\varepsilon(0) &= \varepsilon^{-1}x_0. \end{cases} \tag{3.2.5}$$

For simplicity, let us define the action functional

$$A^\varepsilon[\eta] = \varepsilon \int_0^{\varepsilon^{-1}t_0} \left(\frac{|\dot{\eta}(s)|^2}{2} - V\left(\varepsilon\eta(s), \eta(s)\right)\right) ds + u_0\left(\varepsilon\eta\left(\varepsilon^{-1}t_0\right)\right)$$

for $\eta(\cdot) \in \mathcal{T}$. Thanks to the conservation of energy (3.2.4), the optimization problem (3.2.2) is equivalent to

$$u^\varepsilon(x_0,t_0) = \inf_r \left\{ A^\varepsilon[\eta_\varepsilon] : \text{among all } \eta_\varepsilon(\cdot) \text{ solve (3.2.3) with energy } r \right\}. \tag{3.2.6}$$

We proceed to get different estimates for $r \leq 0$ and $r > 0$. For simplicity, let us introduce the following notation. For I be an interval of $\mathbb{R}$, we define $\inf_{r\in I} A^\varepsilon[\eta_\varepsilon]$ which means the infimum over all solutions $\eta_\varepsilon(\cdot)$ that solve (3.2.3) and with all energies $r \in I$.

Proposition 3.2.2. *When $r \leq 0$, we have the following estimate:*

$$\left| \inf_{r\leq 0} A^{\varepsilon}[\eta_{\varepsilon}] - u_0(x_0) \right| \leq \left(\sqrt{2\|V\|_{L^\infty}} + \|u_0'\|_{L^\infty} \right) \varepsilon. \tag{3.2.7}$$

Lemma 3.2.9 is crucial in establishing the proof of Proposition 3.2.2.

Proof. Let $\eta_{\varepsilon}(\cdot)$ be a solution to (3.2.5) with $r \in \left[V(x_0, \varepsilon^{-1}x_0), 0\right]$ we claim that

$$\underline{y}_0 \leq \eta_{\varepsilon}(s) \leq \overline{y}_0 \qquad \text{for all } s \in [0, \varepsilon^{-1}t_0], \tag{3.2.8}$$

where

$$\begin{aligned} \overline{y}_0 &= \min\left\{ y \in \left[\varepsilon^{-1}x_0, \varepsilon^{-1}x_0 + 1\right) : b(y) = 0 \right\}, \\ \underline{y}_0 &= \max\left\{ y \in \left(\varepsilon^{-1}x_0 - 1, \varepsilon^{-1}x_0\right] : b(y) = 0 \right\}. \end{aligned}$$

The existence of $\underline{y}_0$ and $\overline{y}_0$ is due to the periodicity of $b(\cdot)$ and $b(y_0) = 0$. Recall that $\eta_{\varepsilon}(\cdot)$ satisfies the following equation thanks to the conservation of energy (3.2.4):

$$\begin{cases} |\dot{\eta}_{\varepsilon}(s)| &= \sqrt{2\left(r - V\left(\varepsilon\eta_{\varepsilon}(s), \eta_{\varepsilon}(s)\right)\right)}, \qquad s \in (0, \varepsilon^{-1}t_0), \\ \eta_{\varepsilon}(0) &= \varepsilon^{-1}x_0. \end{cases}$$

Let us define $\gamma_+ : [0, \infty) \to \mathbb{R}$ and $\gamma_- : [0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} \dot{\gamma}_+(s) &= \sqrt{-2V\left(\varepsilon\gamma_+(s), \gamma_+(s)\right)} \quad \text{on} \quad (0, +\infty), \\ \gamma_+(0) &= \varepsilon^{-1}x_0, \end{cases} \tag{3.2.9}$$

and

$$\begin{cases} \dot{\gamma}_-(s) &= -\sqrt{-2V\left(\varepsilon\gamma_-(s), \gamma_-(s)\right)} \quad \text{on} \quad (0, +\infty), \\ \gamma_-(0) &= \varepsilon^{-1}x_0, \end{cases} \tag{3.2.10}$$

respectively. To be precise, there are two cases:

- $V\left(x_0, \varepsilon x_0\right) = 0$, by Lemma 3.2.9 we have $x \mapsto \sqrt{-V(\varepsilon x, x)}$ is Lipschitz on $\left[\varepsilon^{-1}x_0, \varepsilon^{-1}x_0 + 1\right]$. By uniqueness of solutions to (3.2.9) and (3.2.10) we have $\gamma_-(s) \equiv \gamma_+(s) \equiv \varepsilon^{-1}x_0$ for all $s \in [0, +\infty)$.

- $V(x_0, \varepsilon x_0) \neq 0$, the solution $\gamma_+(\cdot)$ exists at least until $\gamma_+(\cdot)$ goes passing $\varepsilon^{-1}x_0 + 1$. Indeed, $\gamma_+(\cdot)$ remains staying inside $\left[\varepsilon^{-1}x_0, \varepsilon^{-1}x_0 + 1\right]$ and hence solution exists on $(0, +\infty)$. To see this, we first observe that $\gamma_+(\cdot)$ is increasing and for each time $t > 0$, from (3.2.9) we have

$$t = \int_{\gamma_+(0)}^{\gamma_+(t)} \frac{dx}{\sqrt{-V(\varepsilon x, x)}}.$$

  Thus, the amount of time $\gamma_+(\cdot)$ needs to reach $\overline{y}_0$ is $\int_{\gamma_+(0)}^{\overline{y}_0} \frac{dx}{\sqrt{-V(\varepsilon x, x)}} = +\infty$ since $x \mapsto \sqrt{-V(\varepsilon x, x)}$ is Lipschitz on $\left[\varepsilon^{-1}x_0, \varepsilon^{-1}x_0 + 1\right]$ by Lemma 3.2.9. We conclude that $\gamma_+(s) \to \overline{y}_0$ and similarly $\gamma_-(s) \to \underline{y}_0$ as $s \to \infty$.

As a consequence, we have

$$\underline{y}_0 \le \gamma_-(s) \le \eta_\varepsilon(s) \le \gamma_+(s) \le \overline{y}_0 \qquad \text{for all} \qquad s \in [0, \varepsilon^{-1}t_0] \tag{3.2.11}$$

and thus (3.2.8) follows. Now we utilize (3.2.8) to estimate $A^\varepsilon[\eta_\varepsilon]$. For any η_ε which solves (3.2.5) we have

$$A^\varepsilon[\eta_\varepsilon] \ge u_0\left(\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)\right) \ge u_0\left(\varepsilon\eta_\varepsilon(0)\right) - \|u_0'\|_{L^\infty}\varepsilon. \tag{3.2.12}$$

On the other hand,

$$\begin{aligned}\inf_{r\le 0} A^\varepsilon[\eta_\varepsilon] \le A^\varepsilon[\gamma_+] &= \varepsilon\int_{\gamma_+(0)}^{\gamma_+(\varepsilon^{-1}t_0)} \sqrt{-2V(\varepsilon x, x)}dx + u_0\left(\varepsilon\gamma_+(\varepsilon^{-1}t_0)\right)\\ &\le u_0(\varepsilon\eta_\varepsilon(0)) + \left(\sqrt{2\|V\|_{L^\infty}} + \|u_0'\|_{L^\infty}\right)\varepsilon.\end{aligned} \tag{3.2.13}$$

thanks to (3.2.11). From (3.2.12) and (3.2.13) we obtain our claim (3.2.7). □

For each $r \in (0,\infty)$, equation (3.2.5) has exactly two distinct solutions $\eta_{1,r,\varepsilon}(\cdot)$ and $\eta_{2,r,\varepsilon}(\cdot)$ thanks to the conservation of energy (3.2.4). They are

$$\begin{cases}\dot\eta_\varepsilon(s) &= \sqrt{2\left(r - V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right)\right)} \qquad \text{on} \qquad (0, \varepsilon^{-1}t_0),\\ \eta_\varepsilon(0) &= \varepsilon^{-1}x_0,\end{cases} \tag{3.2.14}$$

and

$$\begin{cases}\dot\eta_\varepsilon(s) &= -\sqrt{2\left(r - V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right)\right)} \qquad \text{on} \qquad (0, \varepsilon^{-1}t_0),\\ \eta_\varepsilon(0) &= \varepsilon^{-1}x_0,\end{cases} \tag{3.2.15}$$

respectively. Let us consider the first case $\eta_\varepsilon(\cdot)$ solves (3.2.14) since the other case is similar. Since $\dot\eta_\varepsilon(s) > 0$ we have

$$t_0 = \varepsilon\int_0^{\varepsilon^{-1}t_0} \frac{\dot\eta_\varepsilon(s)}{\dot\eta_\varepsilon(s)}\, ds = \varepsilon\int_{\eta_\varepsilon(0)}^{\eta_\varepsilon(\varepsilon^{-1}t_0)} \frac{dx}{\sqrt{2(r - V(\varepsilon x, x))}}. \tag{3.2.16}$$

This holds true for every $\varepsilon > 0$, thus we deduce that $\eta_\varepsilon\left(\varepsilon^{-1}t_0\right) \to +\infty$ as $\varepsilon \to 0^+$. It is also clear that for all $\varepsilon > 0$ then

$$t_0\sqrt{2r} \le \varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) - x_0 \le t_0\sqrt{2\left(r + \|V\|_{L^\infty}\right)}. \tag{3.2.17}$$

By the conservation of energy (3.2.4) we can write the action functional as

$$A^\varepsilon[\eta_\varepsilon] = rt_0 + 2\varepsilon\int_{\eta_\varepsilon(0)}^{\eta_\varepsilon(\varepsilon^{-1}t_0)} \frac{-V(\varepsilon x, x)}{\sqrt{2(r - V(\varepsilon x, x))}}\, dx + u_0\left(\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)\right). \tag{3.2.18}$$

We observe that the infimum of the optimization problem (3.2.6) should be taken over r not too big.

Proposition 3.2.3. *There exists $r_0 > 0$ depends only on* $\mathrm{Lip}(u_0)$ *and* $\|V\|_{L^\infty}$ *such that*

$$\inf_{r \geq r_0} A^\varepsilon[\eta_\varepsilon] \geq u^\varepsilon(x_0, t_0) + t_0. \tag{3.2.19}$$

Proof. If η_ε is a solution to (3.2.14) with $r > 0$, then from (3.2.18) we have

$$\begin{aligned} A^\varepsilon[\eta_\varepsilon] &\geq r t_0 + u_0\left(\varepsilon \eta_\varepsilon(\varepsilon^{-1} t_0)\right) \\ &\geq r t_0 + u_0(x_0) - \|u_0'\|_{L^\infty} \left|\varepsilon \eta_\varepsilon(\varepsilon^{-1} t_0) - x_0\right| \\ &\geq r t_0 + u_0(x_0) - \|u_0'\|_{L^\infty} t_0 \sqrt{2(r + \|V\|_{L^\infty})} \end{aligned} \tag{3.2.20}$$

thanks to (3.2.17). On the other hand, by assumption (H3) we can define

$$\overline{C} = \sup_{(x,y)} \left\{ |H(x,y,p)| : |p| \leq \|u_0'\|_{L^\infty} \right\} < \infty$$

then $\overline{u}(x,t) = u_0(x) + \overline{C} t$ is a viscosity supersolution to (C_ε), therefore

$$u^\varepsilon(x_0, t_0) \leq \overline{u}(x_0, t_0) = u_0(x_0) + \overline{C} t_0. \tag{3.2.21}$$

There exists $r_0 > 0$ such that for $r \geq r_0$ we have

$$r \geq \overline{C} + 1 + \|u_0'\|_{L^\infty} \sqrt{2\left(r + \|V\|_{L^\infty}\right)},$$

which is equivalent to

$$r t_0 + u_0(x_0) - \|u_0'\|_{L^\infty} t_0 \sqrt{2\left(r + \|V\|_{L^\infty}\right)} \geq u_0(x_0) + (\overline{C} + 1) t_0.$$

This estimate together with (3.2.20) and (3.2.21) gives us

$$A^\varepsilon[\eta_\varepsilon] \geq u^\varepsilon(x_0, t_0) + t_0 \quad \text{for all} \quad r \geq r_0$$

which proves our claim (3.2.19), as the case η_ε solves (3.2.15) can be done similarly. $\square$

With (3.2.19), the optimization problem (3.2.6) can be reduced to

$$u^\varepsilon(x_0, t_0) = \min\left\{ \inf_{r \leq 0} A^\varepsilon[\eta_\varepsilon], \inf_{0 < r < r_0} A^\varepsilon[\eta_\varepsilon] \right\}. \tag{3.2.22}$$

Thanks to (3.2.7), we only need to focus on the case $0 < r < r_0$. For simplicity, let us define the following interval $I_0 \subset \mathbb{R}$ to be

$$I_0 = I_0(T, R) = [-R, c_0 + R] \qquad \text{where} \qquad c_0 = T\sqrt{2\left(r_0 + \|V\|_{L^\infty}\right)}.$$

Since (3.2.17) is true for all $0 < r < r_0$, for all $(x_0, t_0) \in [-R, R] \times [0, T]$ we have

$$\varepsilon \eta_\varepsilon(\varepsilon^{-1} t_0) \in I_0.$$

Let us define $c_{1,r} > 0$ and $c_{2,r} < 0$ be unique numbers such that

$$\int_{x_0}^{c_{1,r}} \int_0^1 \frac{dydx}{\sqrt{2\left(r - V(x,y)\right)}} = \int_{c_{2,r}}^{x_0} \int_0^1 \frac{dydx}{\sqrt{2(r - V(x,y))}} = t_0, \tag{3.2.23}$$

repsectively.

Proposition 3.2.4. *Let $\alpha_T = \min_{x\in I_0} a(x)$ and $\beta_T = \max_{x\in I_0} a(x)$, then*

$$\left|\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) - c_{1,r}\right| \le C_K\varepsilon \tag{3.2.24}$$

for $0 < r < r_0$ where C_K is a constant only depends on R, T and V.

Proof. Let us define $\mathcal{K}_r(x,y) = \frac{1}{\sqrt{2(r-V(x,y))}}$ for $(x,y) \in \mathbb{R}\times\mathbb{T}$. From (3.2.16) and (3.2.23) we have

$$t_0 = \int_{x_0}^{\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)} \mathcal{K}_r\left(x, \frac{x}{\varepsilon}\right) dx = \int_{x_0}^{c_{1,r}} \int_0^1 \mathcal{K}_r(x,y)\, dy\, dx. \tag{3.2.25}$$

Using Lemma 3.2.8 we obtain

$$\left|\int_{x_0}^{\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)} \mathcal{K}_r\left(x, \frac{x}{\varepsilon}\right)\, dx - \int_{x_0}^{\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)} \int_0^1 \mathcal{K}_r(x,y)\, dydx\right| \le K\varepsilon \tag{3.2.26}$$

where

$$K = 2\max_{x\in I_0} \int_0^1 \mathcal{K}_r(x,y)\, dy + c_0 \max_{x\in I_0} \int_0^1 \frac{\partial K_r}{\partial x}(x,y)\, dy. \tag{3.2.27}$$

Using (3.2.25) in (3.2.26) we have

$$\left|\int_{c_{1,r}}^{\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)} \int_0^1 \mathcal{K}_r(x,y)\, dydx\right| \le K\varepsilon$$

which implies that

$$\left(\min_{x\in I_0} \int_0^1 \mathcal{K}_r(x,y)\, dy\right) \left|\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) - c_{1,r}\right| \le K\varepsilon. \tag{3.2.28}$$

On I_0 we have $0 < \alpha_T \le a(x) \le \beta_T$, which implies that

$$\begin{aligned} \int_0^1 \frac{dy}{\sqrt{2(r-\beta_T b(y))}} &\le \min_{x\in I_0} \int_0^1 \mathcal{K}_r(x,y)\, dy \\ &\le \max_{x\in I_0} \int_0^1 \mathcal{K}_r(x,y)\, dy \le \int_0^1 \frac{dy}{\sqrt{2(r-\alpha_T b(y))}}. \end{aligned} \tag{3.2.29}$$

Since $\alpha_T \le \beta_T$, it is clear that

$$\int_0^1 \frac{dy}{\sqrt{2(r-\alpha_T b(y))}} \le \sqrt{\frac{\beta_T}{\alpha_T}} \int_0^1 \frac{dy}{\sqrt{2(r-\beta_T b(y))}}. \tag{3.2.30}$$

From direct calculation we have

$$\max_{x\in I_0} \int_0^1 \left|\frac{\partial \mathcal{K}_r}{\partial x}(x,y)\right|\, dy \le \frac{1}{2}\max_{x\in I_0}\left|\frac{a'(x)}{a(x)}\right| \int_0^1 \frac{dy}{\sqrt{2(r-\alpha_T b(y))}}. \tag{3.2.31}$$

Use (3.2.29) and (3.2.31) in (3.2.27) we deduce that

$$K \leq \left(2 + \frac{c_0}{2} \max_{x \in I_0} \left|\frac{a'(x)}{a(x)}\right|\right) \left(\int_0^1 \frac{dy}{\sqrt{2(r - \alpha_T b(y))}}\right). \tag{3.2.32}$$

Next, we use (3.2.29), (3.2.32) in (3.2.28) to deduce that

$$\int_0^1 \frac{dy}{\sqrt{2(r - \beta_T b(y))}} \left|\varepsilon \eta_\varepsilon(\varepsilon^{-1} t) - c_{1,r}\right| \leq \left(2 + \frac{c_0}{2} \max_{x \in I_0} \frac{|a'(x)|}{a(x)}\right) \left(\int_0^1 \frac{dy}{\sqrt{2(r - \alpha_T b(y))}}\right) \varepsilon.$$

From that and (3.2.30) we deduce (3.2.24) with

$$C_K = \sqrt{\frac{\beta_T}{\alpha_T}} \left(2 + \frac{c_0}{2} \max_{x \in I_0} \left|\frac{a'(x)}{a(x)}\right|\right). \tag{3.2.33}$$

It is clear that C_K depends only on R, T and $a(x)$. □

In view of (3.2.18), for $0 < r < r_0$ we aim to show that the integral term is close to its average with an error of order $\mathcal{O}(\varepsilon)$.

Proposition 3.2.5. *For $0 < r < r_0$, in view of (3.2.18) we have that*

$$\left|\varepsilon \int_{x_0}^{\eta_\varepsilon(\varepsilon^{-1} t_0)} \frac{-V(\varepsilon x, x)}{\sqrt{2(r - V(\varepsilon x, x))}} \, dx - \int_{x_0}^{c_{1,r}} \int_0^1 \frac{-V(x,y)}{\sqrt{2(r - V(x,y))}} \, dy \, dx\right| \leq C_F \varepsilon \tag{3.2.34}$$

where C_F is some constant only depends on R, T and V.

Proof. To see it, let

$$\mathcal{F}_r(x,y) = \frac{-V(x,y)}{\sqrt{2(r - V(x,y))}}, \qquad (x,y) \in \mathbb{R} \times \mathbb{T}.$$

Using Lemma 3.2.8 we obtain

$$\left|\int_{x_0}^{c_{1,r}} \frac{-V(x, \varepsilon^{-1} x)}{\sqrt{2(r - V(x, \varepsilon^{-1} x))}} \, dx - \int_{x_0}^{c_{1,r}} \int_0^1 \frac{-V(x,y)}{\sqrt{2(r - V(x,y))}} \, dy \, dx\right| \leq (2F_1 + c_0 F_2)\varepsilon \tag{3.2.35}$$

where

$$F_1 := (\|V\|_{L^\infty})^{1/2} \geq \max_{\mathbb{R} \times \mathbb{T}} |\mathcal{F}_r(x,y)| \tag{3.2.36}$$

$$F_2 := \frac{3}{2\sqrt{2}} (\|V\|_{L^\infty})^{1/2} \max_{x \in I_0} \left|\frac{a'(x)}{a(x)}\right| \geq \max_{I_0 \times \mathbb{T}} \left|\frac{\partial \mathcal{F}_r}{\partial x}(x,y)\right|. \tag{3.2.37}$$

On the other hand, we have

$$\begin{aligned} &\left|\int_{x_0}^{\varepsilon \eta_\varepsilon(\varepsilon^{-1} t_0)} \frac{-V(x, \varepsilon^{-1} x)}{\sqrt{2(r - V(x, \varepsilon^{-1} x))}} \, dx - \int_{x_0}^{c_{1,r}} \frac{-V(x, \varepsilon^{-1} x)}{\sqrt{2(r - V(x, \varepsilon^{-1} x))}} \, dx\right| \\ &\qquad \leq F_1 \left|\varepsilon \eta_\varepsilon(\varepsilon^{-1} t_0) - c_{1,r}\right| \leq F_1 C_K \varepsilon \end{aligned} \tag{3.2.38}$$

thanks to (3.2.24). From (3.2.35) and (3.2.38) we deduce that

$$\left|\int_{x_0}^{\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)} \frac{-V(x,\varepsilon^{-1}x)}{\sqrt{2(r-V(x,\varepsilon^{-1}x))}}\,dx - \int_{x_0}^{c_{1,r}}\int_0^1 \frac{-V(x,y)}{\sqrt{2(r-V(x,y))}}\,dy\,dx\right|$$
$$\leq (2F_1 + F_2c_0 + F_1C_K)\varepsilon. \tag{3.2.39}$$

From (3.2.39) we obtain our claim (3.2.34) with

$$C_F = (\|V\|_{L^\infty})^{1/2}\left(2 + 2\sqrt{\frac{\beta_T}{\alpha_T}} + c_0\left(\frac{3}{2\sqrt{2}} + \frac{1}{2}\sqrt{\frac{\beta_T}{\alpha_T}}\right)\max_{x\in I_0}\left|\frac{a'(x)}{a(x)}\right|\right). \tag{3.2.40}$$

□

Proposition 3.2.6. *We have the following estimate:*

$$\left|\inf_{\substack{0<r<r_0\\ i=1,2}} A^\varepsilon[\eta_{i,r,\varepsilon}] - \inf_{0<r<r_0} I(r)\right| \leq C\varepsilon \tag{3.2.41}$$

where C is a constant depends only on $R, T, a(x)$ and $\|V\|_{L^\infty}$, $I(r) = \min\{I_1(r), I_2(r)\}$ where

$$I_1(r) = rt_0 + 2\int_{x_0}^{c_{1,r}}\int_0^1 \frac{-V(x,y)}{\sqrt{2(r-V(x,y))}}\,dydx + u_0\left(c_{1,r}\right), \tag{3.2.42}$$
$$I_2(r) = rt_0 + 2\int_{c_{2,r}}^{x_0}\int_0^1 \frac{-V(x,y)}{\sqrt{2(r-V(x,y))}}\,dydx + u_0(c_{2,r}). \tag{3.2.43}$$

Proof. Within our notation $\eta_\varepsilon \equiv \eta_{1,r,\varepsilon}$, we have

$$\left|u_0\left(\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)\right) - u_0(c_{1,r})\right| \leq \|u_0'\|_{L^\infty}\left|\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) - c_{1,r}\right|. \tag{3.2.44}$$

since $u_0 \in \mathrm{Lip}(\mathbb{R})$. In view of (3.2.18) and (3.2.24), (3.2.34), (3.2.44) we conclude that

$$|A^\varepsilon[\eta_\varepsilon] - I_1(r)| \leq 2C_F\varepsilon + \|u_0'\|_{L^\infty}\left|\varepsilon\eta_\varepsilon(\varepsilon^{-1}t) - c_{1,r}\right|$$
$$\leq \left(2C_F + C_K\|u_0'\|_{L^\infty}\right)\varepsilon. \tag{3.2.45}$$

Taking the infimum over $0 < r < r_0$ we obtain

$$\left|\inf_{0<r<r_0} A^\varepsilon[\eta_{1,r,\varepsilon}] - \inf_{0<r<r_0} I_1(r)\right| \leq C_1\varepsilon \tag{3.2.46}$$

where $C_1 = 2C_F + C_K\|u_0'\|_{L^\infty}$. Similarly for the case $\eta_{2,r,\varepsilon}$ solves (3.2.15), we obtain

$$\left|\inf_{0<r<r_0} A^\varepsilon[\eta_{2,r,\varepsilon}] - \inf_{0<r<r_0} I_2(r)\right| \leq C_2\varepsilon \tag{3.2.47}$$

where C_2 is some constant depends on $R, T, a(x)$ and $\|V\|_{L^\infty}$ in the same manner as C_1. Thus our claim (3.2.41) is correct with $C = \max\{C_1, C_2\}$. □

From (3.2.7), (3.2.22) and (3.2.41) we conclude that

$$|u^{\varepsilon}(x_0,t_0)-u(x_0,t_0)| \le \left(\max\left\{\sqrt{2\|V\|_{L^\infty}}+\|u_0'\|_{L^\infty}, C\right\}\right)\varepsilon$$

and the proof is complete. □

Corollary 3.2.7. *We have the following representation formula*

$$u(x_0,t_0)=\min\left\{u_0(x_0), \min\left\{\inf_{0<r<r_0} I_1(r), \inf_{0<r<r_0} I_2(r)\right\}\right\}$$

where $I_1(r)$ and $I_2(r)$ are defined in (3.2.42) and (3.2.43) respectively.

Remark 26. If $V(x,y)=V(y)$ is independent of x, then the constants C_K in (3.2.40) and C_F in (3.2.40) are independent of R and T. Therefore the convergence is uniform in $\mathbb{R}\times[0,\infty)$ and by carefully keeping track of all constants, we get

$$C=2\left(\|u_0'\|_{L^\infty(\mathbb{R})}+4\sqrt{\max_{y\in\mathbb{T}}|V(y)|}\right).$$

Also Proposition 3.2.2 is no longer needed in this case.

Lemma 3.2.8. *If $F(x,y)\in C^1(\mathbb{R}\times\mathbb{T})$ then for any real numbers $a<b$ we have*

$$\left|\int_a^b F\left(x,\frac{x}{\varepsilon}\right)\,dx-\int_a^b\left(\int_0^1 F(x,y)\,dy\right)\,dx\right|\le C\varepsilon$$

where

$$C=2\max_{x\in[a,b]}\int_0^1|F(x,y)|\,dy+(b-a)\max_{x\in[a,b]}\int_0^1\left|\frac{\partial F}{\partial x}(x,y)\right|\,dy.$$

This lemma is a quantitative version of the ergodic Theorem for periodic functions in one dimension. The author was first aware of this Lemma from [98]). For the purpose of a quantitative bound, we provide a proof for this Lemma with an explicit bound. We also note that this Lemma is a generalized version of Lemma 4.2 in [95].

Proof of Lemma 3.2.8. Since $y\mapsto F(x,y)$ is periodic, we have $y\mapsto \frac{\partial F}{\partial x}(x,y)$ is also periodic. Let us define

$$G(x,y)=\int_0^y\left(F(x,z)-\int_0^1 F(x,\zeta)\,d\zeta\right)\,dz$$

then $\frac{\partial G}{\partial y}(x,y)=F(x,y)-\int_0^1 F(x,\zeta)\,d\zeta$. Since G is periodic in y, $\frac{\partial G}{\partial x}$ is also periodic in y. Thus G and $\frac{\partial G}{\partial x}$ are bounded in y. The fact that $\frac{\partial F}{\partial x}$ is bounded in x implies $\frac{\partial G}{\partial x}$ is bounded in x as well. Let $g_\varepsilon(x)=\varepsilon G\left(x,\frac{x}{\varepsilon}\right)$ we obtain

$$\frac{d}{dx}\Big(g_\varepsilon(x)\Big)=\varepsilon\frac{\partial G}{\partial x}\left(x,\frac{x}{\varepsilon}\right)+\frac{\partial G}{\partial y}\left(x,\frac{x}{\varepsilon}\right)=\varepsilon\frac{\partial G}{\partial x}\left(x,\frac{x}{\varepsilon}\right)+F\left(x,\frac{x}{\varepsilon}\right)-\int_0^1 F(x,\zeta)\,d\zeta.$$

Thus

$$\int_a^b F\left(x, \frac{x}{\varepsilon}\right) dx - \int_a^b \int_0^1 F(x,\zeta)\, d\zeta dx = \varepsilon \left[G\left(b, \frac{b}{\varepsilon}\right) - G\left(a, \frac{a}{\varepsilon}\right) - \int_a^b \frac{\partial G}{\partial x}\left(x, \frac{x}{\varepsilon}\right) dx \right].$$

Note that by the way we defined G, we also have

$$\max_{(x,y)} |G(x,y)| \le \max_{x\in[a,b]} \int_0^1 |F(x,y)|\; dy$$
$$\left| \int_a^b \frac{\partial G}{\partial x}\left(x, \frac{x}{\varepsilon}\right)\; dx \right| \le \int_a^b \max_{(x,y)} \left| \frac{\partial G}{\partial x}(x,y) \right|\; dx \le (b-a) \max_{x\in[a,b]} \int_0^1 \left| \frac{\partial F}{\partial x}(x,y) \right|\; dy$$

and hence the proof is complete. □

The following lemma is crucial in handling the minimizer paths that correspond to nonpositive energies.

Lemma 3.2.9. *Let $\mathcal{V} \in C^2(\mathbb{R}, [0,\infty))$ with $\min_{x\in\mathbb{R}} \mathcal{V}(x) = 0$. There exists a constant $L > 0$ such that $|\mathcal{V}'(x)| \le L\sqrt{\mathcal{V}(x)}$ for all $x \in \mathbb{R}$. As a consequence, $x \mapsto \sqrt{\mathcal{V}(x)}$ is Lipschitz in $\mathbb{R}$.*

Remark 27. There is an error in the published version of this Lemma in [113] where the author assumes only $\mathcal{V} \in C^2([0,1],[0,\infty))$ with $\min_{x\in\mathbb{R}} \mathcal{V}(x) = 0$ and $\mathcal{V}(0) = \mathcal{V}(1)$. A counter example is $\mathcal{V}(x) = x(1-x)$, which fails to have 0 derivative at 0 and thus $\mathcal{V}'(0) = 1$ while $\mathcal{V}(0) = 0$. The author is grateful to his advisor, H. Tran for pointing out this error and his suggestion on improving the Lemma.

Proof of Lemma 3.2.9. For each $a \in [0,1]$, an δ-neighborhood $\mathcal{N}_{a,\delta}$ of a is defined as $(a - \delta, a+\delta)$ if $a \in (0,1)$ and $[0,\delta) \cup (1-\delta, 1]$ if $a \in \{0,1\}$. It is clear that $\mathcal{N}_{a,\delta}$ is open in $[0,1]$. We claim that there exists $\delta = \delta(a) > 0$ such that

$$\sup_{x\in\mathcal{N}^*_{a,\delta}} \frac{|\mathcal{V}'(x)|}{\sqrt{\mathcal{V}(x)}} \le C_a < \infty \tag{3.2.48}$$

for some constant C_a, where $\mathcal{N}^*_{a,\delta} = \{x \in \mathcal{N}_{a,\delta} : \mathcal{V}(x) \neq 0\}$. Assume that (3.2.48) is false, then there exists a sequence $x_k \to a^+$ such that $\mathcal{V}(x_k) \neq 0$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \frac{|\mathcal{V}'(x_k)|}{\sqrt{\mathcal{V}(x_k)}} = +\infty. \tag{3.2.49}$$

It is clear that $\mathcal{V}'(x_k) \neq 0$ for all $k \in \mathbb{N}$. We can assume that $\mathcal{V}'(x_k) > 0$ for all k. Let $g_k = \sqrt{\mathcal{V}(x_k)}$ and $h_k = \mathcal{V}'(x_k)$, and

$$a_k = \sup\left\{ r > 0 : \mathcal{V}'(x) \ge \frac{h_k}{2} \text{ for all } x \in (x_k - r, x_k) \right\}.$$

Clearly $\mathcal{V}'(x_k - a_k) = \frac{h_k}{2}$. By mean value theorem we have

$$g_k^2 = \mathcal{V}(x_k) \ge \mathcal{V}(x_k) - \mathcal{V}(x_k - a_k) \ge \frac{1}{2} h_k a_k.$$

By mean value theorem again, there exists $\xi_k \in (x_k - a_k, x_k)$ such that

$$\mathcal{V}''(\xi_k) = \frac{\mathcal{V}'(x_k) - \mathcal{V}'(x_k - a_k)}{a_k} = \frac{1}{2}\frac{h_k}{a_k} \geq \frac{1}{4}\left(\frac{h_k}{g_k}\right)^2 \to \infty$$

as $k \to \infty$ due to (3.2.49). It is a contradiction since $\mathcal{V} \in \mathrm{C}^2([0,1])$, thus (3.2.48) must be correct. By compactness of $[0,1]$, we can pick a finite subcover of $[0,1]$ from the open cover $\{\mathcal{N}_{a,\delta} : a \in [0,1]\}$. From (3.2.48) there exists a constant $L > 0$ such that

$$\frac{|\mathcal{V}'(x)|}{\sqrt{\mathcal{V}(x)}} \leq L \qquad \text{whenever} \qquad \mathcal{V}(x) \neq 0. \tag{3.2.50}$$

For $0 < \varepsilon < 1$ let $f_\varepsilon(x) = \sqrt{\mathcal{V}(x) + \varepsilon} \in \mathrm{C}^2([0,1])$. It $\mathcal{V}(x) = 0$ then $\mathcal{V}'(x) = 0$, hence $f'_\varepsilon(x) = 0$ as well, while if $\mathcal{V}(x) \neq 0$ then from (3.2.50) we have

$$|f'_\varepsilon(x)| = \left|\frac{\mathcal{V}'(x)}{2\sqrt{\mathcal{V}(x) + \varepsilon}}\right| \leq \frac{1}{2}\frac{|\mathcal{V}'(x)|}{\sqrt{\mathcal{V}(x)}} \leq \frac{L}{2}$$

Thus $f_\varepsilon(x)$ is Lipschitz on $[0,1]$ with a Lipschitz constant independent of ε. Let $\varepsilon \to 0$ we deduce that $x \mapsto \sqrt{\mathcal{V}(x)}$ is Lipschitz on $[0,1]$. We can repeat this procedure to obtain the result for all $\mathbb{R}$. □

3.2.2 Generalization

We provide a very technical generalization of the previous results where the Hamiltonian is a bit more general.

Theorem 3.2.10. *Let $n = 1$ and $H(x,y,p) = H(p) + V(x,y) + C_0$ where C_0 is a constant. We assume the followings.*

- *$H \in \mathrm{C}^2(\mathbb{R}, [0,\infty))$ is strictly convex with $\min\{H(p) : p \in \mathbb{R}\} = H(0) = 0$.*
- *Assume (H1)–(H4), $V(x,y)$ is continuously differentiable in x variable for each $y \in \mathbb{T}$.*

Define

$$G_1 = \left(H'|_{[0,\infty)}\right) \circ \left(H|_{[0,\infty)}\right)^{-1} \qquad \text{and} \qquad G_2 = \left(H'|_{(-\infty,0]}\right) \circ \left(H|_{(-\infty,0]}\right)^{-1}.$$

We assume also that:

(A0)
$$\limsup_{p \to 0} \left|\frac{H''(p)}{H'(p)}\sqrt{H(p)}\right| < \infty. \tag{3.2.51}$$

(A1) $\max_{\mathbb{R}\times\mathbb{T}} V(x,y) = 0$, *there exists $y_0 \in \mathbb{T}$ such that $V(x,y_0) = 0$ for all $x \in \mathbb{R}$.*

For each compact interval $I \subset \mathbb{R}$ and $i = 1,2$ we have:

(A2)
$$\limsup_{r\to 0^+}\left\{|V_x(x,y)|.\frac{|G_i'(r-V(x,y))|}{|G_i(r-V(x,y))|}\ :\ (x,y)\in I\times\mathbb{T}\right\}<\infty.$$

(A3)
$$\sup_{(x,y)\in I\times\mathbb{T}}\left|\frac{V_x(x,y)}{G_i\left(|V(x,y)|\right)}\right|<\infty.$$

(A4)
$$\limsup_{r\to 0^+}\left(\frac{\displaystyle\max_{x\in I}\int_0^1\frac{dy}{|G_i(r-V(x,y))|}}{\displaystyle\min_{x\in I}\int_0^1\frac{dy}{|G_i(r-V(x,y))|}}\right)<\infty.$$

If $u_0\in\mathrm{Lip}(\mathbb{R})\cap\mathrm{BUC}(\mathbb{R})$ then for any $R,T>0$ we have

$$\|u^\varepsilon-u\|_{L^\infty([-R,R]\times[0,T])}\leq C\varepsilon \tag{3.2.52}$$

where C is a constant depends only on $R,T,\mathrm{Lip}(u_0)$, $H(p)$ and $V(x,y)$.

Remark 28. If $V(x,y)=V(y)$ does not depend on x, then assumptions (A1)–(A4) automatically hold, while (A0) is satisfied after approximating H with uniformly convex Hamiltonians. Indeed, the method can be used to get the result for general convex Hamiltonians. We thus recover Theorem 1.3 in [95] and the convergence is uniform in this case. By Proposition 3.1.1, the rate $\mathcal{O}(\varepsilon)$ is optimal.

Remark 29. Let us give some quick comments on the assumptions of Theorem 3.2.10.

(i) The assumptions (A2)–(A4) are technical assumptions that are needed for the arguments to work. These assumptions are natural in the sense that they are satisfied by a large class of interesting Hamiltonians (cf. Corollary 3.2.11).

(ii) Assumption (A1) plays a key role in establishing the result. Roughly speaking, the rate of convergence of u^ε to u is related to the asymptotic behavior of its corresponding minimizer path via an optimal control formulation as in (3.2.58). Any minimizer path conserves the total energy as in (3.2.59). Assumption (A1) implies that any minimizer with negative total energy is uniformly bounded independent of $\varepsilon>0$.

(iii) Condition (A0) is satisfied for a vast class of strictly convex C^2 Hamiltonians, including those with $H''(0)>0$, $H\in C^3$, or $|p|^\gamma$ with $\gamma\geq 2$ (Lemma 3.2.9).

The following corollary gives some nice examples in which (A1)–(A4) hold, and Theorem 3.2.10 applies.

Corollary 3.2.11. *If $H(x,y,p)=H(p)+V(x,y)$ where $H(p)\geq H(0)=0$ such that:*

- *$H(p)\in C^2(\mathbb{R})$ is strictly convex with $H''(0)>0$, or $H(p)=|p|^\gamma$ where $\gamma\geq 2$.*
- *$\max_{\mathbb{R}\times\mathbb{T}}V(x,y)=0$, there exists $y_0\in\mathbb{T}$ such that $V(x,y_0)=0$ for all $x\in\mathbb{R}$.*

- *For every compact interval $I \subset \mathbb{R}$ then $\alpha_I f_I(y) \le |V(x,y)| \le \beta_I f_I(y)$ for $\alpha_I, \beta_I > 0, f_I \in \mathrm{C}(\mathbb{R}, [0,\infty))$ and*

$$\sup_{(x,y)\in I\times\mathbb{T}} \left| \frac{V_x(x,y)}{V(x,y)} \right| \le C_I < \infty. \tag{3.2.53}$$

If $u_0 \in \mathrm{Lip}(\mathbb{R}) \cap \mathrm{BUC}(\mathbb{R})$ then for any $R, T > 0$ we have

$$\|u^\varepsilon - u\|_{L^\infty([-R,R]\times[0,T])} \le C\varepsilon$$

where C is a constant depends only on $R, T, \mathrm{Lip}(u_0)$, $H(p)$ and $V(x,y)$.

Setting and simplifications

Similarly to the proof of Theorem 3.2.1, we can assume $C_0 = 0$ and $V \in \mathrm{C}^2(\mathbb{R}\times\mathbb{T})$. We have the following estimate ([79]):

$$\|u^\varepsilon_t\|_{L^\infty} + \|Du^\varepsilon\|_{L^\infty} \le M \tag{3.2.54}$$

in the viscosity sense for all $\varepsilon > 0$. Since values of $H(p)$ for $|p| > M$ are irrelevant. This fact together with $H(0) = H'(0) = 0$ allows us to assume that

$$\max\left\{ \frac{|p|^2}{2} - K_0, \frac{|p|^2}{2} - K_0|p| \right\} \le H(p) \le \min\left\{ \frac{|p|^2}{2} + K_0, \frac{|p|^2}{2} + K_0|p| \right\} \tag{3.2.55}$$

for all $p \in \mathbb{R}$ and for some $K_0 > 0$. Let $L(v) = \sup_{p\in\mathbb{R}} \Big(p\cdot v - H(p) \Big)$ for $v \in \mathbb{R}^n$ be the Legendre transform of H, then L is C^2 and strictly convex, $L(v) > L(0) = 0$ for $v \neq 0$ as well as $L(0) = L'(0) = 0$, and

$$\max\left\{ \frac{|v|^2}{2} - K_0, \frac{|v|^2}{2} - K_0|v| \right\} \le L(v) \le \min\left\{ \frac{|v|^2}{2} + K_0, \frac{|v|^2}{2} + K_0|v| \right\} \tag{3.2.56}$$

for $v \in \mathbb{R}^n$. Denote:

$$\begin{cases} H_1^{-1} & := \Big(H|_{[0,\infty)} \Big)^{-1} \\ (L_1')^{-1} & := \Big(L'|_{[0,\infty)} \Big)^{-1} \\ \tilde{G}_1 & := (L_1')^{-1} \circ H_1^{-1} \end{cases} \quad \text{and} \quad \begin{cases} H_2^{-1} & := \Big(H|_{0,+\infty)} \Big)^{-1} \\ (L_2')^{-1} & := \Big(L'|_{(-\infty,0]} \Big)^{-1} \\ \tilde{G}_2 & := (L_2')^{-1} \circ H_2^{-1}. \end{cases}$$

We have $H_i' = (L_i)^{-1}$ and thus $\tilde{G}_i \equiv G_i$ for $i = 1, 2$ where G_i are defined in the statement of Theorem 3.2.10. We see that $x \mapsto |G_i(x)|$ is increasing on $[0,\infty)$, $x \mapsto (L_i')^{-1}(x)$ is increasing for $i = 1, 2$ and for all $x \ge K_0$ then

$$\begin{cases} (1/\sqrt{2})\sqrt{x - K_0} \le G_1(x) \le 2K_0 + 2\sqrt{2(x + K_0)}, \\ -(1/\sqrt{2})\sqrt{x - K_0} \ge G_2(x) \ge 2K_0 - 2\sqrt{2(x + K_0)}. \end{cases} \tag{3.2.57}$$

As a consequence, we have $|G_i(x)| \to +\infty$ as $x \to \infty$ for $i = 1, 2$.

Sketch of the proof of Theorem 3.2.10

For $\varepsilon > 0$ and $R, T > 0$, let us fix $(x_0, t_0) \in [-R, R] \times [0, T]$. Thanks to the optimal control formula we have

$$u^\varepsilon(x_0, t_0) = \inf_{\eta(\cdot)\in\mathcal{T}} \left\{ \varepsilon \int_0^{\varepsilon^{-1}t_0} \Big(L\left(\dot\eta(s)\right) - V\left(\varepsilon\eta(s), \eta(s)\right) \Big) ds + u_0\left(\varepsilon\eta(\varepsilon^{-1}t_0)\right) \right\} \tag{3.2.58}$$

where $\mathcal{T} = \left\{\eta(\cdot) \in \mathrm{AC}\left(\left[0, \varepsilon^{-1}t_0\right]\right), \varepsilon\eta(0) = x_0\right\}$. For each mininminzer $\eta_\varepsilon(\cdot) \in \mathcal{T}$ to (3.2.58), there exists $r = r(\eta_\varepsilon) \in [V(0,0), +\infty)$ such that

$$H\big(L'(\dot\eta_\varepsilon(s))\big) + V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right) = r \tag{3.2.59}$$

for all $s \in (0, \varepsilon^{-1}t_0)$. For $r \in [V(0,0), \infty)$ we have the Euler–Lagrange equation

$$\begin{cases} L''\left(\dot\eta_\varepsilon(s)\right)\ddot\eta_\varepsilon(s) = -\nabla V\left(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\right) \cdot (\varepsilon, 1) \qquad \text{on} \qquad \left(0, \varepsilon^{-1}t_0\right), \\ \dot\eta_\varepsilon(0) = G_i\big(r - V(x_0, \varepsilon^{-1}x_0)\big), \\ \eta_\varepsilon(0) = \varepsilon^{-1}x_0. \end{cases} \tag{E-L}$$

where $i = 1, 2$. For simplicity, let us define the following action functional

$$A^\varepsilon[\eta] = \varepsilon \int_0^{\varepsilon^{-1}t_0} \Big(L\left(\dot\eta(s)\right) - V\left(\varepsilon\eta(s), \eta(s)\right) \Big) ds + u_0\left(\varepsilon\eta(\varepsilon^{-1}t_0)\right)$$

for $\eta(\cdot) \in \mathcal{T}$. Thanks to (3.2.59), the optimization problem (3.2.58) is equivalent to

$$u^\varepsilon(x_0, t_0) = \inf_r \left\{ A^\varepsilon[\eta_\varepsilon] : \text{among all } \eta_\varepsilon(\cdot) \text{ solve (E-L) with energy } r \right\}. \tag{3.2.60}$$

For an interval $I \subset \mathbb{R}$ we denote by $\inf_{r\in I} A^\varepsilon[\eta_\varepsilon]$ the infimum over all solutions $\eta_\varepsilon(\cdot)$ that solve (E-L) and with all energies $r \in I$. We proceed as follows:

1. Estimate for $r \leq 0$ with rate $\mathcal{O}(\varepsilon)$ (Proposition 3.2.12).
2. There is $r_0 > 0$ such that we can ignore $r \geq r_0$ in (3.2.60) (Proposition 3.2.13).
3. For $0 < r < r_0$, $A^\varepsilon[\eta_\varepsilon]$ can be written as in (3.2.67), then we proceed to get estimates for each individual term by using an quantitative ergodic theorem (Propositions 3.2.15, 3.2.14 and 3.2.16).

Proposition 3.2.12. *If $r \leq 0$ then*

$$\left| \inf_{r\leq 0} A^\varepsilon[\eta_\varepsilon] - u_0(x_0) \right| \leq \Big(H_1^{-1}\left(\|V\|_{L^\infty}\right) + \|u_0'\|_{L^\infty} \Big)\varepsilon. \tag{3.2.61}$$

Sketch of the proof. The proof is similar to Proposition 3.2.2 where the crucial Lemma 3.2.9 is replaced with Lemma 3.2.17. ∎

For each $r \in (0,\infty)$, (E-L) has exactly two distinct solutions $\eta_{1,r,\varepsilon}(\cdot)$ and $\eta_{2,r,\varepsilon}(\cdot)$ thanks to the conservation of energy (3.2.59). They are

$$\begin{cases} \dot{\eta}_\varepsilon(s) & = G_i\big(r - V\big(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\big)\big) \qquad \text{on} \qquad (0, \varepsilon^{-1}t_0), \\ \eta_\varepsilon(0) & = \varepsilon^{-1}x_0, \end{cases} \tag{3.2.62}$$

for $i = 1,2$ respectively.

Let us consider the first case $\eta_\varepsilon(\cdot)$ solves (3.2.62) with $i = 1$ since the other case is similar. Since $\dot{\eta}_\varepsilon(s) > 0$ for all $s \geq 0$, we have

$$t_0 = \varepsilon \int_0^{\varepsilon^{-1}t_0} \frac{\dot{\eta}_\varepsilon(s)}{\dot{\eta}_\varepsilon(s)}\, ds = \varepsilon \int_{\eta_\varepsilon(0)}^{\eta_\varepsilon(\varepsilon^{-1}t_0)} \frac{dx}{G_1\big(r - V(\varepsilon x, x)\big)}. \tag{3.2.63}$$

Let $\varepsilon \to 0$ we deduce that $\eta_\varepsilon\left(\varepsilon^{-1}t_0\right) \to +\infty$. It is also clear from (3.2.62) that

$$t_0 G_1(r) \leq \varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) - \varepsilon\eta_\varepsilon(0) \leq t_0 G_1\big(r + \max|V|\big). \tag{3.2.64}$$

Proposition 3.2.13. *There exists $r_0 > 0$ depends on $\mathrm{Lip}(u_0)$ and $H(p)$ such that*

$$\inf_{r \geq r_0} A^\varepsilon[\eta_\varepsilon] = \inf_{r \geq r_0} \Big\{ A^\varepsilon[\eta_{1,r,\varepsilon}], A^\varepsilon[\eta_{2,r,\varepsilon}] \Big\} \geq u^\varepsilon(x_0, t_0) + t_0. \tag{3.2.65}$$

Sketch of the proof. The proof is similar to Proposition 3.2.3 where we utilize the fact that G_1 is increasing and satisfies (3.2.57). $\square$

With (3.2.65), the optimization problem (3.2.60) can be reduced to

$$u^\varepsilon(x_0, t_0) = \min \left\{ \inf_{r \leq 0} A^\varepsilon[\eta_\varepsilon], \inf_{0<r<r_0} A^\varepsilon[\eta_{1,r,\varepsilon}], \inf_{0<r<r_0} A^\varepsilon[\eta_{2,r,\varepsilon}] \right\}. \tag{3.2.66}$$

Let $\eta_\varepsilon = \eta_{1,r,\varepsilon}$, we have $L\left(\dot{\eta}_\varepsilon(s)\right) - V\big(\varepsilon\eta_\varepsilon(s), \eta_\varepsilon(s)\big) = -r + \dot{\eta}_\varepsilon(s) L'\big(\dot{\eta}_\varepsilon(s)\big)$. From that and (3.2.62) we can rewrite the action functional as

$$A^\varepsilon[\eta_\varepsilon] = -rt_0 + \varepsilon \int_{\eta_\varepsilon(0)}^{\eta_\varepsilon(\varepsilon^{-1}t_0)} H_1^{-1}\big(r - V\big(\varepsilon x, x\big)\big) dx + u_0\big(\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)\big). \tag{3.2.67}$$

Define $I_0 = I_0(T,R) = [-R, c_0 + R]$ where $c_0 = TG_1(r_0 + \|V\|_{L^\infty})$. Since (3.2.64) is true for all $0 < r < r_0$ and $(x_0, t_0) \in [-R,R] \times [0,T]$ we have $\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) \in I_0$. Let $c_{1,r}$ and $c_{2,r}$ be unique numbers such that

$$\int_{x_0}^{c_{1,r}} \int_0^1 \frac{dydx}{G_1(r - V(x,y))} = \int_{c_{2,r}}^{x_0} \int_0^1 \frac{dydx}{G_2(r - V(x,y))} = t_0. \tag{3.2.68}$$

Proposition 3.2.14. *For $0 < r < r_0$ we have*

$$\Big| \varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0) - c_{1,r} \Big| \leq C_K \varepsilon. \tag{3.2.69}$$

where $C_K = C_K(R,T,H,V)$ is a constant independent of r.

Sketch of the proof. Let $\mathcal{K}_r(x,y) = \dfrac{1}{G_1(r - V(x,y))}$ for $(x,y) \in \mathbb{R} \times \mathbb{T}$. Similarly to proof of Proposition 3.2.4, we obtain

$$C_K = 2\left(1 + 2c_0\tilde{K}\right) \sup_{0<r<r_0} \left(\frac{\max_{x \in I_0} \int_0^1 \mathcal{K}_r(x,y)\, dy}{\min_{x \in I_0} \int_0^1 \mathcal{K}_r(x,y)\, dy} \right) < \infty$$

by assumption (A4) and

$$\tilde{K} = \sup_{0<r<r_0} \left\{ |V_x(x,y)|. \left| \frac{G_1'(r - V(x,y))}{G_1(r - V(x,y))} \right| : (x,y) \in I_0 \times \mathbb{T} \right\} < \infty$$

by assumption (A2). □

Proposition 3.2.15. *For $0 < r < r_0$, in view of (3.2.67) we have*

$$\left| \int_{x_0}^{\varepsilon\eta_\varepsilon(\varepsilon^{-1}t_0)} H_1^{-1}\left(r - V(x, \varepsilon^{-1}x)\right) dx - \int_{x_0}^{c_{1,r}} \int_0^1 H_1^{-1}\left(r - V(x,y)\right) dy dx \right| \le C_F \varepsilon$$

where C_F is a constant independent of r.

Sketch of the proof. Define $\mathcal{F}_r(x,y) = H_1^{-1}(r - V(x,y))$ for $(x,y) \in \mathbb{R} \times \mathbb{T}$. The proof is similar to Proposition 3.2.5. We use (A3) to get the bound F_2:

$$\begin{aligned}
F_1 &:= H_1^{-1}\left(r_0 + \|V\|_{L^\infty}\right) \ge \max_{x \in I_0} \int_0^1 \mathcal{F}_r(x,y)\, dy \\
F_2 &:= \sup \left\{ \frac{|V_x(x,y)|}{|G_1(-V(x,y))|} \;\middle|\; (x,y) \in I_0 \times \mathbb{T} \right\} \ge \max_{x \in I_0} \int_0^1 \left| \frac{\partial F_r}{\partial x}(x,y) \right| dy.
\end{aligned}$$

Similar to Proposition 3.2.5, we can compute C_F as $C_F = 2F_1 + c_0 F_2 + C_K F_1$. □

Proposition 3.2.16. *We have the following estimate:*

$$\left| \inf_{\substack{0<r<r_0 \\ i=1,2}} A^\varepsilon[\eta_{i,r,\varepsilon}] - \inf_{0<r<r_0} I(r) \right| \le C\varepsilon \tag{3.2.70}$$

where C is a constant independent of r and $I(r) = \min\{I_1(r), I_2(r)\}$ where

$$I_1(r) = -rt_0 + \int_{x_0}^{c_{1,r}} \int_0^1 H_1^{-1}\left(r - V(x,y)\right) dy\, dx + u_0\left(c_{1,r}\right), \tag{3.2.71}$$

$$I_2(r) = -rt_0 + \int_{c_{2,r}}^{x_0} \int_0^1 H_2^{-1}\left(r - V(x,y)\right) dy\, dx + u_0\left(c_{2,r}\right). \tag{3.2.72}$$

The proof is omitted since it is similar to Proposition 3.2.6.
Finally, using (3.2.61) and (3.2.70) in (3.2.66) we obtain the claim of Theorem 3.2.10.

$$|u^\varepsilon(x_0,t_0) - u(x_0,t_0)| \le \left(\max\left\{ H_1^{-1}\left(\|V\|_{L^\infty}\right) + \|u_0'\|_{L^\infty}, C \right\} \right) \varepsilon.$$

Lemma 3.2.17. *Let $\mathcal{V} \in C^2\big([0,1],[0,\infty)\big)$ with $\min_{x\in[0,1]} \mathcal{V}(x) = 0$ and $\mathcal{V}(0) = \mathcal{V}(1)$.*

(i) Let H, G_1, G_2 be defined as in Theorem 3.2.10. If

$$\limsup_{p\to 0} \left| \frac{H''(p)}{H'(p)} \sqrt{H(p)} \right| < \infty, \tag{3.2.73}$$

then $x \mapsto G_i(\mathcal{V}(x))$ is Lipschitz on $[0,1]$ for $i = 1,2$.

(ii) If H, defined in Theorem 3.2.10, satisfies $H''(0) > 0$ then we have something stronger than (3.2.73)

$$\limsup_{p\to 0} \left| \frac{\sqrt{H(p)}}{H'(p)} \right| < \infty. \tag{3.2.74}$$

In this case we have further that $C_{1,I}\sqrt{x} \le |G_i(x)| \le C_{2,I}\sqrt{x}$ on any bounded subset $I \subset \mathbb{R}$, where $i = 1,2$ and $C_{I,1}, C_{I,2} > 0$.

(iii) If H, defined in Theorem 3.2.10, satisfies $H \in C^3(\mathbb{R})$ then

$$\limsup_{p\to 0} \frac{|H''(p)|}{\sqrt{|H'(p)|}} < \infty. \tag{3.2.75}$$

As a consequence, we have something stronger than (3.2.73)

$$\limsup_{p\to 0} \left| \frac{H''(p)}{H'(p)} \sqrt{\frac{H(p)}{|p|}} \right| < \infty. \tag{3.2.76}$$

(iv) If $H(p) = |p|^\gamma$ where $\gamma \ge 2$ then (3.2.73) holds true.

Corollary 3.2.18. *We have the following representation formula*

$$u(x_0,t_0) = \min\left\{ u_0(x_0), \min\left\{ \inf_{0<r<r_0} I_1(r), \inf_{0<r<r_0} I_2(r) \right\} \right\}.$$

where $I_1(r)$ and $I_2(r)$ are defined in (3.2.71) and (3.2.72), respectively.

Proof of Corollary 3.2.11. In order to apply Theorem 3.2.10 we need to check conditions (A0), (A2), (A3), (A4). Let us fix a compact interval $I \subset \mathbb{R}$, in the assumption of V let us denote α, β, f by α_I, β_I, f_I for simplicity.

If $H(p) = |p|^\gamma$ where $\gamma \ge 2$ then $|G_i(p)| = \gamma |p|^{1-\frac{1}{\gamma}}$ and $|G_i'(p)| = (\gamma-1)|p|^{-\frac{1}{\gamma}}$. Therefore conditions (A0),(A2),(A3) follow from direct computation. (A4) follows since $p \mapsto |G_i(p)|$ is increasing and for any compact interval $I \subset \mathbb{R}$ then

$$\max_{x\in I} \int_0^1 \frac{dy}{|G_i(r - V(x,y))|} \le \frac{1}{\gamma}\left(\frac{\beta}{\alpha}\right)^{1-\frac{1}{\gamma}} \min_{x\in I} \int_0^1 \frac{dy}{|G_i(r - V(x,y)|}.$$

In general when $H''(0) > 0$, condition (A0) follows from Lemma 3.2.17. On the bounded set $[0, \|V\|_{L^\infty} + 1]$ by Lemma 3.2.17 we have $C_1\sqrt{x} \leq |G_i(x)| \leq C_2\sqrt{x}$ for $i = 1, 2$ and for some $C_1, C_2 > 0$. For $i = 1, 2$, $0 < r < 1$ and $x \in I$ we have

$$|V_x(x,y)|.\frac{|G_i'(r - V(x,y))|}{|G_i(r - V(x,y))|} \leq \frac{|V_x(x,y)|}{|V(x,y)|}\left(\frac{\sqrt{H(\xi)}}{|G_i(H(\xi))|}\right)\left(\frac{|H''(\xi)|}{|H'(\xi)|}\sqrt{H(\xi)}\right)$$

where $\xi = H_i^{-1}(r - V(x,y))$. The right hand side is bounded as $r \to 0^+$ due to (A0), $C_1\sqrt{x} \leq |G_i(x)| \leq C_2\sqrt{x}$ and (3.2.53), thus (A2) follows. Condition (A3) is true since for $x \in I$ then

$$\left|\frac{V_x(x,y)}{G_i(V(x,y))}\right| \leq \left|\frac{V_x(x,y)}{V(x,y)}\right| . \frac{\sqrt{|V(x,y)|}}{|G_i(|(V(x,y)|)|}.\sqrt{|V(x,y)|}.$$

Finally, for $i = 1, 2$ then $x \mapsto |G_i(x)|$ is increasing, using $C_1\sqrt{x} \leq |G_i(x)| \leq C_2\sqrt{x}$ we deduce that for $0 < r < 1$ then

$$\max_{x\in I}\int_0^1 \frac{dy}{|G_i(r - V(x,y))|} \leq \int_0^1 \frac{dy}{|G_i(r + \alpha f(y))|} \leq \int_0^1 \frac{dy}{C_1\sqrt{r + \alpha f(y)}}$$
$$\min_{x\in I}\int_0^1 \frac{dy}{|G_i(r - V(x,y))|} \geq \int_0^1 \frac{dy}{|G_i(r + \beta f(y))|} \geq \int_0^1 \frac{dy}{C_2\sqrt{r + \beta f(y)}}.$$

Since $\alpha \leq \beta$, we have $\sqrt{r + \alpha f(y)} \geq \sqrt{\frac{\alpha}{\beta}\left(r + \beta f(y)\right)}$ and therefore

$$\int_0^1 \frac{dy}{C_1\sqrt{r + \alpha f(y)}} \leq \left(\frac{C_2}{C_1}\sqrt{\frac{\beta}{\alpha}}\right)\int_0^1 \frac{dy}{C_2\sqrt{r + \beta f(y)}}$$

and thus (A4) follows. □

Proof of Lemma 3.2.17.

(i) It suffices to show for G_1 since the argument is similar for G_2. For simplicity, let us denote G_1, H_1^{-1} by G, H^{-1}. For $0 < \varepsilon < 1$ let $f_\varepsilon(x) = G\left(\mathcal{V}(x) + \varepsilon\right)$ then $f_\varepsilon \in C^2([0,1])$ and

$$f_\varepsilon'(x) = \frac{\mathcal{V}'(x)}{\sqrt{\mathcal{V}(x) + \varepsilon}}\left(\frac{H''\big(H^{-1}\left(\mathcal{V}(x) + \varepsilon\right)\big)}{H'\big(H^{-1}\left(\mathcal{V}(x) + \varepsilon\right)\big)}\sqrt{\mathcal{V}(x) + \varepsilon}\right).$$

For $x \in [0,1]$ such that $\mathcal{V}(x) = \mathcal{V}'(x) = 0$ then obviously $f_\varepsilon'(x) = 0$, while if $x \in [0,1]$ such that $\mathcal{V}(x) \neq 0$ then from (3.2.73) and Lemma 3.2.9 we have

$$|f_\varepsilon'(x)| \leq \left|\frac{\mathcal{V}'(x)}{\sqrt{\mathcal{V}(x)}}\right| . \left|\frac{H''(\xi)}{H'(\xi)}\sqrt{H(\xi)}\right| \leq L\left(\sup_{[0,p^*]}\left|\frac{H''(p)}{H'(p)}\sqrt{H(p)}\right|\right) < \infty$$

where $\xi = H^{-1}(\mathcal{V}(x) + \varepsilon)$ and $p^* = H^{-1}(\|V\|_{L^\infty} + 1)$. Therefore f_ε is Lipschitz on $[0,1]$ with a Lipschitz constant independent of $\varepsilon > 0$. Let $\varepsilon \to 0$ we deduce that $x \mapsto G(\mathcal{V}(x))$ is Lipschitz on $[0,1]$.

(ii) If $H''(0) > 0$ then there exists $\delta > 0$ so that $H''(p) \geq c > 0$ for $p \in (-\delta, \delta)$, thus there are some $m, M > 0$ such that

$$m|p|^2 \leq H(p) \leq M|p|^2 \qquad \text{and} \qquad m|p| \leq |H'(p)| \leq M|p|. \tag{3.2.77}$$

From that (3.2.74) follows. On the other hand, since $G_i(x) = H'(H_i^{-1}(x))$ for $i = 1, 2$ and (3.2.77) we deduce that for all x small then

$$\sqrt{\frac{m}{M}}\sqrt{x} \leq |H'(H_i^{-1}(x))| \leq \sqrt{\frac{M}{m}}\sqrt{x}. \tag{3.2.78}$$

Since $G_i(x) = 0$ if and only if $x = 0$, we have (3.2.78) is true for any bounded set of $\mathbb{R}$ after modifying the upper bound and lower bound.

(iii) Using the convexity of H we have $H(p) \leq pH'(p)$ for all p, hence

$$\left|\frac{H''(p)}{H'(p)}\sqrt{H(p)}\right| \leq \frac{H''(p)}{\sqrt{|H'(p)|}}\sqrt{|p|}. \tag{3.2.79}$$

Let $g(p) = H'(p) \in \mathrm{C}^2(\mathbb{R})$ is strictly increasing on $(0, \infty)$ and is strictly decreasing on $(-\infty, 0)$ with $g(0) = 0$, we claim that indeed

$$\limsup_{p \to 0} \frac{g'(p)}{\sqrt{|g(p)|}} < \infty. \tag{3.2.80}$$

This can be done by a similar argument to Lemma 3.2.9, hence (3.2.76) follows.

(iv) It is clear from direct computation.

$\square$

Chapter 4

State-constraint problems on nested domains

Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of domains in $\mathbb{R}^n$ such that $\Omega_k \subset \Omega_{k+1}$ for all $k \in \mathbb{N}$. We say that $\{\Omega_k\}_{k\in\mathbb{N}}$ is a sequence of nested domains. Then, $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$ is also a domain in $\mathbb{R}^n$. Let $H : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ be a given continuous Hamiltonian. In this paper, we are interested in studying state-constraint solutions to the following static Hamilton-Jacobi equations:

$$u_k(x) + H\big(x, Du_k(x)\big) = 0 \qquad \text{in } \Omega_k, \tag{HJ$_k$}$$

and

$$u(x) + H\big(x, Du(x)\big) = 0 \qquad \text{in } \Omega. \tag{HJ}$$

Under some conditions, (HJ$_k$) has a unique state-constraint viscosity solution $u_k \in \mathrm{C}(\overline{\Omega}_k)$ for each $k \in \mathbb{N}$, and (HJ) has a unique state-constraint viscosity solution $u \in \mathrm{C}(\overline{\Omega})$. Furthermore, by a priori estimates and the stability results of viscosity solutions, we have that $u_k \to u$ locally uniformly on $\overline{\Omega}$. Our main focus here is to study how fast this convergence is in two different types of nested domains.

($\mathcal{P}1$) $\quad \Omega_k = B(0,k)$ and $\Omega = \bigcup_{k\in\mathbb{N}} B(0,k) = \mathbb{R}^n$,

($\mathcal{P}2$) $\quad \Omega_k = B(0, 1-\frac{1}{k})$, and $\Omega = B(0,1)$.

In this chapter, we study the asymptotic behavior of $u_k \to u$ quantitatively as $k \to \infty$. The materials of this chapter are taken from [75]. We summarize the results in this chappter as follows. In the prototype ($\mathcal{P}1$) setting,

- general nonconvex Hamiltonians: $\|u_k - u\|_{L^\infty(B(0,C))} \leq \mathcal{O}(k^{-2})$; nonconvex Hamiltonians of the form $H(x,p) = a(x)K(p)$ with $K(0) = 0$: $\|u_k - u\|_{L^\infty(B(0,C))} \leq \mathcal{O}\left(e^{-k/C}\right)$, and this exponential rate is *optimal*;
- convex Hamiltonians: $\|u_k - u\|_{L^\infty(B(0,C))} \leq \mathcal{O}\left(e^{-k/C}\right)$, and this exponential rate is *optimal*.

In the prototype ($\mathcal{P}2$) setting,

- general nonconvex Hamiltonians: $\|u_k - u\|_{L^\infty(B(0,C))} \leq \mathcal{O}\left(k^{-1}\right)$, and this rate is *optimal*.

Remark 30. In the convex setting, we show the existence of a corresponding minimizer with bounded velocity by using the Lagrangian formulation. This is a highly nontrivial fact, which plays an essential role in the proofs. The two prototypes $(\mathcal{P}1)$ and $(\mathcal{P}2)$ can be generalized to more general set rather than balls (Remark 37).

There have been many works in the literature on the well-posedness of state-constraint Hamilton-Jacobi equations and fully nonlinear elliptic equations. The state-constraint problem for first-order convex Hamilton-Jacobi equations using optimal control frameworks was first studied in [107, 108]. The general nonconvex, coercive first-order equations was then discussed in [30]. For further developments in using optimal control formulation and obtaining optimal paths, we refer the readers to [3, 17, 31, 33, 58, 67, 73, 105, 116] for the finite dimensional cases, and [25, 77] for the infinite dimensional cases. See [23] for discrete numerical schemes, and [90] for large time behavior results. We also refer to the classical books [8, 10] and the references therein.

The state-constraint problem for second-order equations was first studied in [78] for the Laplacian, and in [6] for the general possibly degenerate diffusion matrices. Boundary behavior of blow-up solutions was discussed in [6, 78, 82]. Convex solutions with state-constraint boundary were constructed in [4, 102]. The convergence of solutions to the vanishing discount problems was proved in [70].

In terms of state-constraint problems in nested domains, up to our knowledge, there are only qualitative results in the literature in [6, 30] where certain approximations were needed for the analysis of solutions. We provide here some first quantitative results on the rate of convergence of the solutions to (HJ_k) as k goes to infinity in two different types ($(\mathcal{P}1)$ or $(\mathcal{P}2)$) of nested domains.

4.1 Introduction

We refer the readers to 2.8 for the well-posedness theory and relevant properties we will be using. The statement of the problems is rather clear. We summarize here the statement of the results which we will show in details in this chapter. For clarity we list again assumptions on H and Ω that will be used.

Assumptions on the Hamiltonian

($\mathcal{H}_0$) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$.

($\mathcal{H}_1$) There exists $C_1 > 0$ such that $H(x,p) \geq -C_1$ for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$.

($\mathcal{H}_2$) There exists $C_2 > 0$ such that $|H(x,0)| \leq C_2$ for all $x \in \overline{\Omega}$.

($\mathcal{H}_3$) For each $R > 0$ there exists a constant C_R such that

$$\begin{cases} |H(x,p) - H(y,p)| \leq C_R|x-y|, \\ |H(x,p) - H(x,q)| \leq C_R|p-q|, \end{cases} \tag{4.1.1}$$

for $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$ with $|p|, |q| \leq R$.

($\mathcal{H}_4$) H satisfies the coercivity assumption

$$\lim_{|p|\to\infty} \left(\inf_{x\in\overline{\Omega}} H(x,p) \right) = +\infty. \tag{4.1.2}$$

($\mathcal{H}_5$) $p \mapsto H(x,p)$ is convex for each $x \in \overline{\Omega}$.

($\mathcal{H}_6$) $p \mapsto H(x,p)$ is superlinear uniformly for $x \in \overline{\Omega}$, that is,

$$\lim_{|p|\to\infty} \left(\inf_{x\in\Omega} \frac{H(x,p)}{|p|} \right) = +\infty. \tag{4.1.3}$$

Assumptions on the regularity of the domain

($\mathcal{A}_1$) Ω a bounded star-shaped (with respect to the origin) open subset of $\mathbb{R}^n$ and there exists some $\kappa > 0$ such that $\mathrm{dist}(x, \overline{\Omega}) \geq \kappa r$ for all $x \in (1+r)\partial\Omega$ and $r > 0$.

($\mathcal{A}_2$) There exists a universal pair of positive numbers (r, h) and $\eta \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R}^n)$ such that

$$B(x + t\eta(x), rt) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, h].$$

First of all, we show that the rate of convergence is $\mathcal{O}\left(\frac{1}{k^2}\right)$ for the prototype ($\mathcal{P}1$) for general nonconvex Hamiltonians.

Theorem 4.1.1. *Under the assumptions ($\mathcal{P}1$), ($\mathcal{H}_0$), ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_3$) and ($\mathcal{H}_4$), we have*

(i) $u(x) \leq u_k(x)$ *for every* $x \in \overline{B(0,k)}$,

(ii) there exists a constant $C > 0$ *depending only on* H *such that for all* $k \in \mathbb{N}$ *and* $x \in \overline{B(0,k)}$ *then*

$$0 \leq u_k(x) - u(x) \leq \frac{C(1+|x|^2)}{k^2}.$$

In particular, for any fixed $R > 0$ *and* $|x| \leq R$,

$$0 \leq u_k(x) - u(x) \leq \frac{C(1+R^2)}{k^2}.$$

The condition that $|x| \leq R$ is important since there are examples where the estimate above fails at the boundary of Ω_k. In Proposition 4.4.10, we have, for each $k \in \mathbb{N}$, $|u_k(x) - u(x)| = 1$ for some $x \in \partial\Omega_k$.

Theorem 4.1.2. *Assume* ($\mathcal{P}1$). *Assume further that* $H(x,p) = a(x)K(p)$ *for* $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$. *Here,* $K(0) = 0$, K *is locally Lipschitz and coercive in* $\mathbb{R}^n$, *and* $a \in \mathrm{BUC}(\mathbb{R}^n)$ *satisfies* $\alpha \leq a(\cdot) \leq \beta$ *for some given* $\alpha, \beta > 0$. *Then,* $u \equiv 0$, *and for every* $x \in \overline{B(0,k)}$, *we have*

$$0 \leq u_k(x) \leq \left(Ce^{\frac{|x|}{C}}\right) e^{-\frac{k}{C}},$$

where C *is a constant depending only on* H. *In particular, for any fixed* $R > 0$, *we have*

$$0 \leq u_k(x) \leq \left(Ce^{\frac{R}{C}}\right) e^{-\frac{k}{C}}$$

for every $x \in \overline{B(0,R)}$ *and* $k \geq R$. *In addition to that, this exponential rate is optimal.*

It is quite interesting to observe that we obtain the exponential rate of convergence for this particular class of nonconvex Hamiltonians and the rate is indeed optimal. When $a(x)$ is a positive constant, the assumption $K(0) = 0$ in the theorem above can be removed.

Corollary 4.1.3. *Assume* ($\mathcal{P}1$). *Assume further that* $H(x,p) = H(p)$ *for* $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$. *Here,* H *is locally Lipschitz and coercive in* $\mathbb{R}^n$. *Then,* $u \equiv -H(0)$, *and for every* $x \in \overline{B(0,k)}$, *we have*

$$0 \leq u_k(x) - u(x) \leq \left(Ce^{\frac{|x|}{C}}\right) e^{-\frac{k}{C}},$$

where C *is a constant depending only on* H. *In particular, for any fixed* $R > 0$, *we have*

$$0 \leq u_k(x) - u(x) \leq \left(Ce^{\frac{R}{C}}\right) e^{-\frac{k}{C}}$$

for every $x \in \overline{B(0,R)}$ *and* $k \geq R$. *In addition to that, this rate is optimal.*

When $H(x,p) = K(p) + V(x)$, the analysis becomes much more complicated due to the interaction between K and V. We provide an example where the exponential rate of convergence is obtained in Example 4.

For convex Hamiltonians, we are able to establish the exponential rate of convergence using optimal control theory. Some examples for which the exponential rate is obtained are given in Proposition 4.4.10 and Proposition 4.4.11.

Theorem 4.1.4. *Under the assumptions* ($\mathcal{P}1$), ($\mathcal{H}_0$), ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_3$) *and* ($\mathcal{H}_4$) *and* ($\mathcal{H}_5$), *we have*

(i) $u(x) \leq u_k(x)$ *for every* $x \in \overline{B(0,k)}$,

(ii) for each fixed $x \in B(0,k)$ *we have*

$$u_k(x) \leq u(x) + \left(Ce^{\frac{|x|}{C}}\right) e^{-\frac{k}{C}}, \tag{4.1.4}$$

where C *is a constant depending only on the growth of* H.

In particular, for any fixed $R > 0$, *we have*

$$0 \leq u_k(x) - u(x) \leq \left(Ce^{\frac{R}{C}}\right) e^{-\frac{k}{C}}$$

for all $x \in \overline{B(0,R)}$ *and* $k > R$.

As a byproduct, we prove the existence of a minimizer η with bounded velocity to the minimizing problem (4.4.4) for each given $x \in \mathbb{R}^n$, which is a key element in the proof of Theorem 4.1.4. Moreover, the bound on the velocity of η only depends on the growth of H and not on its smoothness. We believe that this bound (Theorem 4.4.7 and Lemma 4.4.9) is new in the literature. See Remark 35 for further discussions.

For the second prototype $(\mathcal{P}2)$, we establish the rate $\mathcal{O}\left(\frac{1}{k}\right)$ for a quite general class of Hamiltonians. The rate is also optimal, as pointed out in Remark 38.

Theorem 4.1.5. *Under assumptions $(\mathcal{P}2)$,$(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, for any $k \geq 2$,*

$$0 \leq u_k(x) - u(x) \leq \frac{C}{k}$$

for every $x \in \overline{B\left(0, 1 - \frac{1}{k}\right)}$ where C is a constant depending only on H. Moreover, this rate is optimal.

Although we only deal with two prototype cases $(\mathcal{P}1)$ and $(\mathcal{P}2)$ in this paper, the obtained results can be extended to more general domains in a similar fashion under some appropriate conditions. See Remarks 37 and 39 for example.

4.2 A rate of convergence for general Hamiltonians in unbounded domain

In this section, we consider the first prototype $(\mathcal{P}1)$. The assumptions $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ are enforced throughout the section. From Perron's method and Theorem 2.8.6, there exists $u_k \in \mathrm{Lip}(\overline{B(0,k)})$ which is the unique solution to

$$\begin{cases} u_k(x) + H(x, Du_k(x)) \leq 0 & \text{in } B(0,k), \\ u_k(x) + H(x, Du_k(x)) \geq 0 & \text{on } \overline{B(0,k)} \end{cases} \tag{4.2.1}$$

in the viscosity sense. Based on the construction of solutions via Perron's method together with the coercivity of H, we have the following a priori estimate:

$$|u_k(x)| + |Du_k(x)| \leq C_H$$

for all $x \in B(0,k)$ in the viscosity sense. Here, C_H is a positive constant depending only on H (one can take $C_H = \max\{C_1, C_2, C_3\}$. By the Arzelà–Ascoli theorem, there is a subsequence $\{k_m\} \to \infty$, and a function $u \in \mathrm{Lip}(\mathbb{R}^n)$ such that

$$u_{k_m} \to u \qquad \text{locally uniformly in } \mathbb{R}^n. \tag{4.2.2}$$

Theorem 4.2.1. *The function u defined in (4.2.2) is a viscosity solution to*

$$u(x) + H(x, Du(x)) = 0 \qquad \text{in } \mathbb{R}^n. \tag{4.2.3}$$

Moreover, $u_k \to u$ locally uniformly in $\mathbb{R}^n$ as k grows to infinity.

Proof. It is clear from the stability of viscosity solutions that u is a solution to (4.2.3). The fact that $u_k \to u$ locally uniformly in $\mathbb{R}^n$ follows from the uniqueness of solutions to (4.2.3). $\square$

Now we are ready to give a proof for Theorem 4.1.1 using the doubling variables method.

Proof of Theorem 4.1.1. We first note that u_k solves $u_k(x) + H(x, Du_k(x)) \geq 0$ on $\overline{B(0,k)}$, and u solves $u(x) + H(x, Du(x)) \leq 0$ in $\overline{B(0,k)}$ in viscosity sense. By the comparison principle, we get $u_k(x) \geq u(x)$ for all $x \in \overline{B(0,k)}$.

For the upper bound of $u_k - u$, we define the following auxiliary function

$$\Phi^k(x,y) = u_k(x) - u(y) - 2C_H k^2 |x-y|^2 - \frac{8C_H}{k^2}|y|^2$$

for $(x,y) \in \overline{B(0,k)} \times \mathbb{R}^n$. It is clear that Φ^k is bounded above by $2C_H$ independent of $k \in \mathbb{N}$. If $|y| > \frac{k}{2}$, then we have

$$\Phi^k(0,0) - \Phi^k(x,y) \geq -u_k(x) + u_k(0) - u(0) + u(y) + 2C_H k^2 |x-y|^2 + \frac{8C_H}{k^2}|y|^2 > 0,$$

which implies that for each $k \in \mathbb{N}$, $\Phi^k(x,y)$ achieves a global maximum over $\overline{B(0,k)} \times \mathbb{R}^n$ at $(x_k, y_k) \in \overline{B(0,k)} \times \overline{B\left(0, \frac{k}{2}\right)}$. Of course, $|y_k| \leq \frac{k}{2}$. Now we use $\Phi^k(x_k,y_k) \geq \Phi^k(y_k,y_k)$ to get

$$2C_H k^2 |x_k - y_k|^2 \leq u_k(x_k) - u_k(y_k) \leq C_H |x_k - y_k|.$$

Therefore, we deduce that

$$|x_k| \leq |y_k| + \frac{1}{2k^2} < k \tag{4.2.4}$$

for all $k \geq 1$ since $|y_k| \leq \frac{k}{2}$. Observing that $x \mapsto \Phi^k(x, y_k)$ obtains a maximum at x_k with $|x_k| < k$, we have

$$u_k(x_k) + H(x_k, p_k) \leq 0, \tag{4.2.5}$$

where $p_k = 4C_H k^2 (x_k - y_k)$ by the definition of viscosity subsolutions. We also observe that $y \mapsto \Phi^k(x_k, y)$ obtains a maximum at y_k, which implies that

$$u(y) - \left(-2C_H k^2 |x_k - y_k|^2 - \frac{8C_H}{k^2}|y|^2 \right)$$

has a minimum at y_k. By the definition of viscosity supersolutions, we get

$$u(y_k) + H(y_k, p_k + q_k) \geq 0 \tag{4.2.6}$$

where $q_k = -\frac{16C_H}{k^2} y_k$. Here, it needs to be noted that

$$|p_k|, |p_k + q_k| \leq C_H,$$

which comes from Lipschitz continuity of u_k. Using (4.2.5), (4.2.6) and assumption $(\mathcal{H}_3)$, there exists a constant $\tilde{C}_H$ such that

$$\begin{aligned} u_k(x_k) - u(y_k) &\le H(y_k, p_k + q_k) - H(x_k, p_k) \\ &= H(y_k, p_k + q_k) - H(y_k, p_k) + H(y_k, p_k) - H(x_k, p_k) \\ &\le \tilde{C}_H |q_k| + \tilde{C}_H |x_k - y_k| \\ &\le \frac{16\tilde{C}_H C_H}{k^2}|y_k| + \frac{\tilde{C}_H}{k^2} \le \frac{8\tilde{C}_H C_H}{k} + \frac{\tilde{C}_H}{k^2}. \end{aligned} \tag{4.2.7}$$

If we stop here, the fact that $\Phi^k(x_k, y_k) \ge \Phi^k(x, x)$ for $x \in B(0,k)$ gives

$$u_k(x) - u(x) \le u_k(x_k) - u(y_k) + \frac{8C_H}{k^2}|x|^2 \le \frac{C}{k} + \frac{C(1+|x|^2)}{k^2}$$

for all $k \ge 2$. This gives us the rate of convergence of u_k to u is $\mathcal{O}(\frac{1}{k})$ for $x \in B(0,R)$, which is typically the case in light of the doubling variables method.

Nevertheless, a key new point here is to bootstrap once more to improve this rate. The monotonicity of $\{u_k\}$ allows us to bound $|y_k|$ better. We use that $\Phi^k(x_k, y_k) \ge \Phi^k(0,0)$ together with (4.2.7) and $u_k \ge u$ to yield

$$\begin{aligned} 2C_H k^2 |x_k - y_k|^2 + \frac{8C_H}{k^2}|y_k|^2 &\le u_k(x_k) - u_k(0) + u(0) - u(y_k) \\ &\le u_k(x_k) - u(y_k) \\ &\le \frac{16\tilde{C}_H C_H}{k^2}|y_k| + \frac{\tilde{C}_H}{k^2}. \end{aligned}$$

Therefore,

$$|y_k|^2 \le 2\tilde{C}_H |y_k| + \frac{\tilde{C}_H}{8C_H} \le \frac{1}{2}|y_k|^2 + 2\tilde{C}_H^2 + \frac{\tilde{C}_H}{8C_H} = \frac{1}{2}|y_k|^2 + C.$$

In particular, $|y_k| \le C$. This bound is much better than the earlier bound that $|y_k| \le \frac{k}{2}$.

Now for any $x \in \overline{B(0,k)}$, clearly we have that $\Phi^k(x_k, y_k) \ge \Phi^k(x,x)$. This, together with (4.2.7) and $|y_k| \le C$, implies

$$u_k(x) - u(x) \le u_k(x_k) - u(y_k) + \frac{8C_H}{k^2}|x|^2 \le \frac{C(1+|x|^2)}{k^2}$$

for all $k \ge 2$. If $|x| \le R$, then

$$0 \le u_k(x) - u(x) \le \frac{C(1+R^2)}{k^2},$$

which gives the desired result. □

Remark 31. In the general setting, one only has that $\Phi^k(x,y)$ achieves a global maximum over $\overline{B(0,k)} \times \mathbb{R}^n$ at (x_k, y_k) where $|y_k| \le \frac{k}{2}$ and $|x_k| < k$. In our current situation, the monotonicity of $\{u_k\}$ allows us to bootstrap once more to deduce further that $|y_k| \le C$, which helps to obtain $\mathcal{O}\left(\frac{1}{k^2}\right)$ rate of convergence. This seems to be the best convergence rate that one is able to get through the doubling variables method here as it is unlikely that $|y_k|$ vanishes as $k \to \infty$.

We do not know yet whether the $\mathcal{O}\left(\frac{1}{k^2}\right)$ rate of convergence is optimal or not in the general nonconvex setting. See Questions 1 and 2 in Section 4.6 below.

4.3 An optimal rate for a class of nonconvex Hamiltonians on unbounded domain

In this section, we show that the rate of convergence $u_k \to u$ is of order $\mathcal{O}(e^{-Ck})$ for a class of possibly nonconvex Hamiltonians which are written as $H(x,p) = a(x)K(p)$ with $K(0) = 0$ and $0 < \alpha \leq a(x) \leq \beta$. The aforementioned rate is indeed optimal.

A brief idea for the proof is that we construct a supersolution to (4.2.1) by finding a symmetric Hamiltonian $\tilde{H}$ such that $\tilde{H}(0) = 0$ and $\tilde{H} \leq H$. The following proposition is needed as a building block.

Proposition 4.3.1. *Let $H : \mathbb{R}^n \to \mathbb{R}$ be defined by*

$$H(p) = \begin{cases} -\alpha|p| & \text{for } |p| \leq \beta, \\ f(p) & \text{for } |p| \geq \beta, \end{cases}$$

where $\alpha, \beta > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a coercive continuous function such that $f(p) = -\alpha\beta$ for $|p| = \beta$ and $\min_{\mathbb{R}^n} f = -\alpha\beta$. Then,

$$u_k(x) = \alpha\beta e^{\frac{|x|-k}{\alpha}}$$

for $x \in \overline{B(0,k)}$ is the unique solution to the state-constraint problem (4.2.1).

Proof. It is clear that $u_k(x) + H(Du_k(x)) = 0$ in $B(0,k)\backslash\{0\}$ in classical sense. For $x \in \partial B(0,k)$ and $\varphi \in C^1(\overline{B(0,k)})$ such that $u_k - \varphi$ has a local minimum over $\overline{B(0,k)}$ at x, we have $u_k(x) + H(D\varphi(x)) \geq 0$ since $u_k(x) = \alpha\beta = -\min H$. We only need to check if u_k is a viscosity supersolution at $x = 0$.

Let $\varphi \in C^1(\mathbb{R}^n)$ such that $\varphi(0) = u_k(0)$ and $u_k - \varphi$ has a local minimum over $\overline{B(0,k)}$ at $x = 0$. Since u_k is convex, we can replace φ by an affine function $\varphi(x) = \xi \cdot x + u_k(0)$ for some $\xi \in \mathbb{R}^n$. Without loss of generality, it suffices to consider $\xi \neq 0$. For $|x|$ sufficiently small, we have $u_k(x) - \varphi(x) \geq u_k(0) - \varphi(0)$, which implies that

$$\alpha\beta e^{-\frac{k}{\alpha}}\left(e^{\frac{|x|}{\alpha}} - 1\right) \geq \xi \cdot x. \tag{4.3.1}$$

Now we choose $x = t\frac{\xi}{|\xi|}$ for $t > 0$ small, then (4.3.1) implies that $\alpha\beta e^{-\frac{k}{\alpha}}\left(e^{\frac{t}{\alpha}} - 1\right) \geq t|\xi|$ for all $t > 0$ sufficiently small. Dividing both sides by t and sending t to 0, we deduce that $|D\varphi(0)| = |\xi| \leq \beta e^{-\frac{k}{\alpha}}$. Therefore,

$$u_k(0) + H(D\varphi(0)) = \alpha\beta e^{-\frac{k}{\alpha}} - \alpha|D\varphi(0)| \geq 0.$$

Consequently, u_k is the unique viscosity solution to (4.2.1). □

Proof of Theorem 4.1.2. Since $K(0) = 0$, $u \equiv 0$ is the unique solution to (4.2.3). Recalling the a priori estimate $\|u_k\|_{L^\infty(B(0,k))} + \|Du_k\|_{L^\infty(B(0,k))} \leq C_H$, condition $(\mathcal{H}_3)$ gives

$$|K(p) - K(q)| \leq L|p - q|$$

for all $p, q \in \overline{B(0, C_H)}$. Let $K(p_0) = \min K \le 0$ for some $p_0 \in \mathbb{R}^n$. Let $f(p)$ be a coercive, continuous function such that $f(p) = -L|p_0|$ for $|p| \le |p_0|$, $\min_{\mathbb{R}^n} f = -L|p_0|$, and $f(p) \le K(p)$ for $|p| \ge |p_0|$. Now we consider

$$\tilde{H}(p) = \begin{cases} -L|p| & \text{for } |p| \le |p_0|, \\ f(p) & \text{for } |p| \ge |p_0|. \end{cases}$$

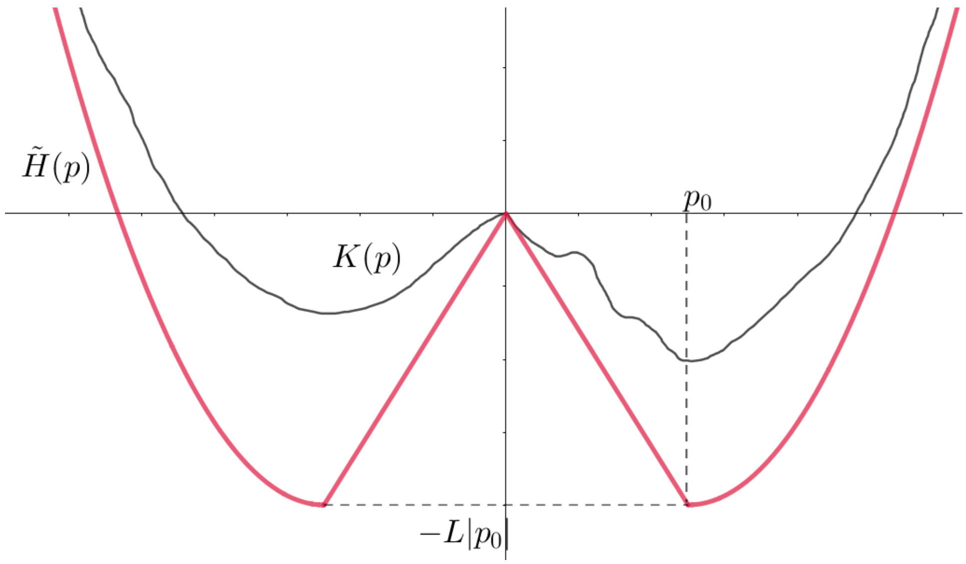

Figure 4.1: The graph of $\tilde{H}(p)$ and $K(p)$.

The graph of $\tilde{H}$ is described in Figure 4.1. It is clear that $\tilde{H}(p) \le K(p)$ for all $p \in \mathbb{R}^n$. Moreover, using Proposition 4.3.1, the unique viscosity solution to the state-constraint problem $\tilde{u}_k(x) + \beta\tilde{H}(D\tilde{u}_k(x)) = 0$ in $B(0,k)$ is given by $\tilde{u}_k(x) = \beta L|p_0|e^{\frac{|x|-k}{\beta L}}$ for $x \in \overline{B(0,k)}$.

It is clear that $\tilde{u}_k$ is also the unique viscosity solution to $\frac{1}{\beta}\tilde{u}_k(x) + \tilde{H}(D\tilde{u}_k(x)) = 0$ in $B(0,k)$. Since $\beta \ge a(x) \ge \alpha > 0$ and $\tilde{H} \le K$, we deduce that

$$\tfrac{1}{\beta}\tilde{u}_k(x) + \tilde{H}(D\tilde{u}_k(x)) \le \tfrac{1}{a(x)}\tilde{u}_k(x) + K(D\tilde{u}_k(x))$$

on $\overline{B(0,k)}$. Therefore, $\tilde{u}_k(x) + a(x)K(D\tilde{u}_k(x)) \ge 0$ on $\overline{B(0,k)}$. By the comparison principle, one gets

$$0 \le u_k(x) \le \beta L|p_0|e^{\frac{|x|-k}{\beta L}}$$

for all $x \in \overline{B(0,k)}$. The conclusion for $|x| \le R$ follows immediately. □

In case that $H(x,p) = K(p)$ for $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, where K is locally Lipschitz continuous and coercive in $\mathbb{R}^n$, we have the unique viscosity solution to (4.2.3) is $u \equiv -K(0)$. Therefore, we can assume that $K(0) = 0$, and Corollary 4.1.3 follows without assuming that $K(0) = 0$.

It should be noted that the local Lipschitz continuity of Hamiltonians is important when it comes to getting an exponential rate of convergence. If a Hamiltonian is only Hölder continuous around 0, we get a slower rate of convergence depending on the regularity of H as described in the following proposition.

Proposition 4.3.2. *Let $H : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$H(p) = \begin{cases} -|p|^\gamma & \text{if } |p| \leq 1, \\ f(p) & \text{if } |p| \geq 1, \end{cases}$$

where $\gamma \in (0,1)$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous, coercive function with $f(x) = -1$ for $|x| \leq 1$, and $\min_{\mathbb{R}^n} f = -1$. Then, the solution to (4.2.1) is given by

$$u_k(x) = \left[\tfrac{1-\gamma}{\gamma}\left(k + \tfrac{\gamma}{1-\gamma} - |x|\right)\right]^{\frac{\gamma}{\gamma-1}}, \qquad x \in \overline{B(0,k)}. \tag{4.3.2}$$

As a consequence, $u_k \to 0$ with the rate $\mathcal{O}\left(\frac{1}{k^{\frac{\gamma}{1-\gamma}}}\right)$.

Proof. Let us first consider the one dimensional case. The higher dimensional setting can be done in a same manner. Let $\mu = \gamma^{-1}$, we look for a nonnegative solution to $u(x)^\mu = u'(x)$ where $x \in (0,k)$. We have

$$u(x)^{1-\mu} = (1-\mu)x - C_k \qquad \Longrightarrow \qquad u(x) = (\mu-1)^{\frac{1}{1-\mu}}(C_k - x)^{\frac{1}{1-\mu}}.$$

We want to choose C_k such that $u'(x) \in [0,1]$ for $x \in (0,k)$. Equivalently,

$$u'(x) = u(x)^\mu = (\mu-1)^{\frac{\mu}{1-\mu}}(C_k - x)^{\frac{\mu}{1-\mu}} \in [0,1]$$

for $x \in (0,k)$. Since it is an increasing function, $C_k = k + \frac{1}{\mu-1}$. Using symmetry, we guess that u_k is written as

$$u_k(x) = (\mu-1)^{\frac{1}{1-\mu}}\left(k + \tfrac{1}{\mu-1} - |x|\right)^{\frac{1}{1-\mu}}.$$

It is straightforward to see that u_k satisfies the equation in the classical sense $u_k(x) - |u_k'(x)|^\gamma = 0$ in $(-k,k)\backslash\{0\}$. Since $|u_k'(x)| \leq 1$ on $(-k,k)\backslash\{0\}$, we have $u_k(x) + H(u_k'(x)) = 0$ in the classical sense in $(-k,k)\backslash\{0\}$. At $|x| = k$, we have $u_k(x) = 1 \geq -\min H$. Therefore, the supersolution test at these points are satisfied. Finally, at $x = 0$ we only need to verify the supersolution test, which is simple since if $p \in D^-u_k(0)$ then

$$|p| \leq (\mu-1)^{\frac{\mu}{1-\mu}}\left(k + \tfrac{\mu}{\mu-1}\right)^{\frac{\mu}{1-\mu}} \qquad \Longrightarrow \qquad u_k(0) + H(p) \geq u_k(0) - |p|^\mu \geq 0.$$

Thus, u_k defined above is the unique viscosity solution to the constraint problem (4.2.1). Using a similar argument as in the proof of Proposition 4.3.1, this formula of u_k can be extended naturally to the n-dimensional case, as given in (4.3.2) and the conclusion follows. ∎

***Remark* 32.** From Proposition 4.3.2 we see that the optimal rate of convergence can be as slow as we wish as the Hölder exponent $\gamma \to 0^+$. This shows that the required condition $(\mathcal{H}_3)$ is really essential in this section.

When Hamiltonians are of the form $H(x,p) = K(p) + V(x)$, the situation becomes much more complicated. See Example 4 for a situation where we get the optimal exponential rate of convergence with nonconvex K.

4.4 An optimal exponential rate for convex Hamiltonians

In this section, the assumptions $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_5)$ are always in force. The state-constraint problem was studied in the context of optimal control for convex Hamiltonians (see [8, 30, 107]). When H is convex, we are able to obtain a representation formula for the viscosity solution based on the optimal control theory. Finding a minimizer with bouned velocity to that optimal control problem is crucial. We provide proofs to some lemmas establishing such a minimizer in Section 4.6.

Let us assume $(\mathcal{H}_6)$, the superlinear property to make things easier (see Remark 33 where we can remove this assumption), which we recall here for convenient

$(\mathcal{H}_6)$ $p \mapsto H(x,p)$ is superlinear uniformly for $x \in \overline{\Omega}$, that is,

$$\lim_{|p|\to\infty} \left(\inf_{x\in\Omega} \frac{H(x,p)}{|p|} \right) = +\infty. \tag{4.4.1}$$

If $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$ hold, then the Legendre transform $L : \overline{\Omega} \times \mathbb{R}^n$ of H is defined as

$$L(x,v) := \sup_{p\in\mathbb{R}^n} \{p \cdot v - H(x,p)\}, \qquad (x,v) \in \overline{\Omega} \times \mathbb{R}^n.$$

Lemma 4.4.1. *Assume $(\mathcal{H}_0)$, $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$. Then, $L : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is continuous satisfying:*

$(\mathcal{L}_1)$ *If $(\mathcal{H}_1)$ holds, then $L(x,0) \le C_1$ for all $x \in \overline{\Omega}$;*

$(\mathcal{L}_2)$ *If $(\mathcal{H}_2)$ holds, then $L(x,v) \ge -C_1$ for all $(x,v) \in \overline{\Omega} \times \mathbb{R}^n$;*

$(\mathcal{L}_3)$ *If$(\mathcal{H}_3)$ holds, then for each $R > 0$ there exists a modulus $\tilde{\omega}_R(\cdot)$ such that*

$$|L(x,v) - L(y,v)| \le \tilde{\omega}_R(|x-y|) \qquad \text{for all } x,y \in \overline{\Omega}, |v| \le R.$$

$(\mathcal{L}_5)$ *$p \mapsto H(x,p)$ is convex for each $x \in \overline{\Omega}$.*

$(\mathcal{L}_6)$ *$p \mapsto H(x,p)$ is superlinear uniformly for $x \in \overline{\Omega}$, that is,*

$$\lim_{|p|\to\infty} \left(\inf_{x\in\Omega} \frac{H(x,p)}{|p|} \right) = +\infty. \tag{4.4.2}$$

We omit the proof of this lemma and refer the interested readers to [27].

For each $x \in \overline{\Omega}$, we define the admissible set of paths as

$$\mathcal{A}_x = \Big\{ \eta \in \mathrm{AC}\big([0,\infty);\mathbb{R}^n\big) : \eta(0) = x \text{ and } \eta(s) \in \overline{\Omega} \text{ for all } s \ge 0 \Big\}$$

where $\mathrm{AC}\big([0,\infty);\mathbb{R}^n\big)$ denotes the set of absolutely continuous curves from $[0,\infty)$ to $\mathbb{R}^n$. Note that $\mathcal{A}_x \neq \emptyset$ since $\eta(s) \equiv x$ for all $s \in [0,\infty)$ is an admissible path. From this, define the value function as

$$u(x) := \inf_{\eta\in\mathcal{A}_x} J[x,\eta] \tag{4.4.3}$$

where the cost functional is defined as

$$J[x,\eta] = \int_0^\infty e^{-s} L\big(\eta(s), -\dot{\eta}(s)\big)\, ds$$

for $(x,\eta) \in \overline{\Omega} \times \mathcal{A}_x$. Now we have the following classical dynamic programming principle.

Theorem 4.4.2 (Dynamic Programming Principle)**.** *For any $t > 0$, we have*

$$u(x) = \inf_{\eta \in \mathcal{A}_x} \left\{ \int_0^t e^{-s} L\big(\eta(s), -\dot{\eta}(s)\big)\, ds + e^{-t} u\big(\eta(t)\big) \right\}.$$

Using the Dynamic Programming Principle, one can prove that $u \in \mathrm{BUC}(\overline{\Omega})$ and indeed a viscosity solution to (HJ_δ) as stated in the following theorems.

Theorem 4.4.3. *Assume $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$. The function $u(x)$ defined by (4.4.3) is bounded and is uniformly continuous up to the boundary, which is $u \in \mathrm{BUC}(\overline{\Omega})$.*

Theorem 4.4.4. *The value function $u \in \mathrm{BUC}(\overline{\Omega})$ defined in (4.4.3) is a viscosity solution to the state-constraint Hamilton-Jacobi equation $u(x) + H(x, Du(x)) = 0$ in Ω, i.e.,*

$$\begin{cases} u(x) + H(x, Du(x)) \le 0 & \text{in } \Omega, \\ u(x) + H(x, Du(x)) \ge 0 & \text{on } \overline{\Omega}. \end{cases}$$

We omit the proofs of Theorems 4.4.2, 4.4.3 and 4.4.4. We refer to [8, 30, 107] for those who are interested.

On the other hand, when $\Omega = \mathbb{R}^n$, it is known that the function $u(x)$ defined in (4.4.3) satisfies the Hamilton-Jacobi equation (4.2.3) in viscosity sense (see [8, 79]).

Theorem 4.4.5. *For each $x \in \mathbb{R}^n$, we define*

$$u(x) = \inf_{\eta \in \mathcal{A}_x} \int_0^\infty e^{-s} L\left(\eta(s), -\dot{\eta}(s)\right)\, ds \tag{4.4.4}$$

subject to $\mathcal{A}_x = \big\{\eta \in \mathrm{AC}\big([0,\infty);\mathbb{R}^n\big) : \eta(0) = x\big\}$. Then, $u \in \mathrm{BUC}(\mathbb{R}^n)$ is a viscosity solution to (4.2.3) and we have the following priori estimate:

$$\|u\|_{L^\infty(\mathbb{R}^n)} + |Du\|_{L^\infty(\mathbb{R}^n)} \le C_H. \tag{4.4.5}$$

Remark 33. We may assume that H is just coercive rather than superlinear. When a Hamiltonian is coercive, we still have that (4.4.5) holds for some $C = C_H > 0$. Therefore, for $|p| \ge C$, we can modify H so that $(\mathcal{H}_6)$ holds. Furthermore, we can impose a quadratic growth rate on H as following.

$(\mathcal{H}_7)$ There exist some positive constants A, B such that

$$A^{-1}|v|^2 - B \le H(x,p) \le A|v|^2 + B \qquad \text{for } (x,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.4.6}$$

It is easy to see from $(\mathcal{H}_7)$ that we have $(4A)^{-1}|v|^2 - B \le L(x,v) \le 4A|v|^2 + B$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$. By making A bigger, we can assume the following.

($\mathcal{L}_7$) There exist some positive constants A, B such that

$$A^{-1}|v|^2 - B \leq H(x,p) \leq A|v|^2 + B \qquad \text{for } (x,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.4.7}$$

We give a proof for the existence of a minimizer with bounded velocity to (4.4.4) for the sake of readers' convenience in Appendix. This is an extremely important fact in our analysis and is a key element in the proof of Theorem 4.1.4 (see Remark 35 for further discussions). To establish this point, the following lemma on the subdifferentials of $L(x,v)$ in v is needed. For continuously differentiable Lagrangians, it is obvious, but we state here a slightly more general version.

Lemma 4.4.6. *Let $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be continuous and satisfy ($\mathcal{L}_5$) and ($\mathcal{L}_7$). There exists $C_L > 0$ such that for all $v \in \mathbb{R}^n$, we have*

$$|\xi| \leq C_L(1+|v|) \qquad \textit{whenever } \xi \in D_v^- L(x,v). \tag{4.4.8}$$

For simplicity, let us assume further that

($\mathcal{L}_8$) $\mathrm{T}(x,v) \mapsto L(x,v)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$.

This assumption can be removed in the proof of Theorem 4.1.4 due to the fact that the estimate (4.1.4) does not depend on the regularity of H, hence, we can approximate H by convex, smooth Hamiltonians.

Theorem 4.4.7 (Existence of a minimizer)**.** *Let $L(x,v)$ be a continuous Lagrangian satisfying ($\mathcal{L}_5$), ($\mathcal{L}_7$) and ($\mathcal{L}_8$). Then, for each $x \in \mathbb{R}^n$, there exists $\eta \in \mathcal{A}_x$ such that $J[x,\eta] = u(x)$ and also*

$$\|e^{-s/2}\dot{\eta}(s)\|_{L^2(0,\infty)} \leq C_4$$

where C_4 depends only on C_H, A, B.

The existence of minimizers of smooth Lagrangian is sufficient for our proof of Theorem 4.1.4 since the last estimate does not depend on the smoothness of L or H. Clearly, a minimizer for a general continuous Lagrangian can be obtained via approximation of smooth Lagrangians (see Section 4.6).

A minimizer to (4.4.4) satisfies the following properties.

Lemma 4.4.8. *Let $x \in \mathbb{R}^n$ and η be a corresponding minimizer. For any $t > 0$, we have*

$$u(x) = \int_0^t e^{-s} L\big(\eta(s), -\dot{\eta}(s)\big)\, ds + e^{-t} u\left(\eta(t)\right). \tag{4.4.9}$$

Furthermore, for every $t, h > 0$, we have

$$u(\eta(t)) = e^t \int_t^\infty e^{-s} L\big(\eta(s), -\dot{\eta}(s)\big)\, ds \tag{4.4.10}$$

and

$$e^{-t} u\left(\eta(t)\right) = \int_t^{t+h} e^{-s} L\big(\eta(s), -\dot{\eta}(s)\big)\, ds + e^{-(t+h)} u\left(\eta(t+h)\right). \tag{4.4.11}$$

Lemma 4.4.9. *Let $x \in \mathbb{R}^n$ and η be a minimizer to (4.4.4) associated with it. Then, there exists a constant $C_5 > 0$ depending only on C_H, A, B such that $|\dot{\eta}(s)| \leq C_5$ for a.e. $s \in (0, \infty)$.*

Remark 34. We provide here a connection between a minimizer η of $u(x) = J[x, \eta]$ and some properties in the view of the method of characteristics. If H is assumed to be C^2, then $L \in \mathrm{C}^2$ and η is a weak solution to the Euler-Lagrange equation

$$D_x L(\eta(s), -\dot{\eta}(s)) - D_v L(\eta(s), -\dot{\eta}(s)) + \frac{d}{ds}\Big(D_v L(\eta(s), -\dot{\eta}(s))\Big) = 0. \tag{4.4.12}$$

Assume that $\eta \in \mathrm{C}^2$ (it holds if, for instance $L \in \mathrm{C}^{2,\alpha}$ for some $\alpha \in (0,1)$). Then, one can define the momentum $\mathbf{p}(s) = D_v L(\eta(s), -\dot{\eta}(s))$ and show that

$$u(\eta(t)) + H\big(\eta(t), \mathbf{p}(t)\big) = 0 \tag{4.4.13}$$

for $t > 0$. Indeed, for every fixed $x \in \mathbb{R}^n$, we recall that

$$v \in D_p^- H(x,p) \quad \Leftrightarrow \quad p \in D_v^- L(x,v) \quad \Leftrightarrow \quad H(x,p) + L(x,v) = p \cdot v. \tag{4.4.14}$$

Using (4.4.14) we can deduce that

$$\begin{cases} \frac{d}{ds}\left(e^{-s}\mathbf{p}(s)\right) &= e^{-s} D_x L\left(\eta(s), \dot{\eta}(s)\right), \\ \frac{d}{ds}\left(H\left(\eta(s), \mathbf{p}(s)\right)\right) &= -\dot{\eta}(s) \cdot \mathbf{p}(s), \\ \frac{d}{ds}\left(e^{-s} H\left(\eta(s), \mathbf{p}(s)\right)\right) &= e^{-s} L\left(\eta(s), -\dot{\eta}(s)\right). \end{cases}$$

From that, we can derive the characteristic ODEs for $s > 0$, which are

$$\begin{cases} -\dot{\eta}(s) &= D_p H(\eta(s), \mathbf{p}(s)), \\ \dot{\mathbf{p}}(s) &= \mathbf{p}(s) - D_x L\big(\eta(s), -\dot{\eta}(s)\big). \end{cases}$$

This together with (4.4.10) yields that

$$u(\eta(t)) + H\big(\eta(t), \mathbf{p}(t)\big) = Ce^t \qquad \text{where} \qquad C = \lim_{a \to \infty} e^{-a} H\big(\eta(a), \mathbf{p}(a)\big).$$

Lemma 4.4.6 together with Lemma 4.4.9 gives us a uniform bound on $\mathbf{p}$, thus $C = 0$. Hence, (4.4.13) follows.

Now we give a proof for Theorem 4.1.4. Recall that we have the value function

$$u_k(x) = \inf_{\eta \in \mathcal{A}_x^k} \int_0^\infty e^{-s} L\left(\eta(s), -\dot{\eta}(s)\right)\, ds, \tag{4.4.15}$$

where $\mathcal{A}_x^k = \Big\{\eta \in \mathrm{AC}\big([0,\infty); \mathbb{R}^n\big) : \eta(0) = x \text{ and } \eta(s) \in \overline{B(0,k)} \text{ for } s \geq 0\Big\}$. Then, u_k solves the state-constraint problem (4.2.1).

Proof of Theorem 4.1.4. Let $k \in \mathbb{N}$ be given. We may assume that H satisfies $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ up to modification for $|p|$ large enough. Also, since the final estimate does not depend on the smoothness of L, we can assume H is smooth and thus L is smooth without any loss of generality. Clearly, $\mathcal{A}_x^k \subset \mathcal{A}_x$ for any $x \in \overline{B(0,k)}$, which implies that $u_k(x) \geq u(x)$.

For $x \in B(0,k)$, let $\eta \in \mathcal{A}_x$ be a minimizer to (4.4.4), if $\eta(s) \in \overline{B(0,k)}$ for all $s > 0$, then $\eta \in \mathcal{A}_x^k$ as well, hence $u(x) = u_k(x)$. Otherwise, there exists $t > 0$ such that $\eta(t) \in \partial B(0,k)$ and $\eta(s) \in B(0,k)$ for all $s \in (0,t)$. By Lemma 4.4.9, we have

$$k = |\eta(t)| \leq |\eta(0)| + \int_0^t |\dot{\eta}(s)|\, ds \leq |x| + C_5 t,$$

which implies that $t \geq \frac{k-|x|}{C_5}$. Let us define

$$\gamma(s) = \begin{cases} \eta(s) & \text{if } s \in [0,t], \\ \eta(t) & \text{if } s \in [t,\infty), \end{cases}$$

so that $\gamma \in \mathcal{A}_x^k$. Using Lemma 4.4.8, we have

$$\begin{aligned} u(x) &= \int_0^t e^{-s} L\left(\eta(s), -\dot{\eta}(s)\right)\, ds + e^{-t} u(\eta(t)) \\ &\geq \int_0^t e^{-s} L\left(\gamma(s), -\dot{\gamma}(s)\right)\, ds - C_H e^{-t} \\ &\geq \int_0^\infty e^{-s} L\left(\gamma(s), -\dot{\gamma}(s)\right)\, ds - C_1 e^{-t} - C_H e^{-t} \\ &\geq u_k(x) - \left((C_1 + C_H) e^{\frac{|x|}{C_5}}\right) e^{-\frac{k}{C_5}}. \end{aligned}$$

Consequently, we obtain (4.1.4). The conclusion for $|x| \leq R$ follows immediately. □

Remark 35. Here, we note that the constants in the proof above do not depend on the regularity of the Lagrangian. As long as a minimizer exists, we get the same exponential rate of convergence. See Appendix for a discussion on the existence of minimizers. It is worth noting here that, for each $x \in \overline{B(0,k)}$, the existence of a minimizer $\eta \in \mathcal{A}_x$ to (4.4.4) with bounded velocity is a nontrivial fact and plays an essential role in the proof above. Moreover, the bound on the velocity of η only depends on C_H, A, B.

In the rest of this section, we provide two explicit examples to show that the rate $\mathcal{O}\left(e^{-\frac{k}{C}}\right)$ is indeed optimal.

4.4.1 Examples with exponential rate of convergence

Proposition 4.4.10. *Let $H(p) : \mathbb{R} \to \mathbb{R}$ be defined by $H(p) = |p-1| - 1$ for $p \in [0,2]$ and $H(p) \geq 0$ elsewhere such that H is continuous and coercive. Let u_k be the solution to (4.2.1) on $[-k,k]$, then $u_k \to 0$ locally uniformly on $\mathbb{R}$ as $k \to \infty$. Here, $u \equiv 0$ is the unique solution to (4.2.3). Furthermore, we have $u_k(k) = 1$ for all $k \in \mathbb{N}$ and*

$$u_k(x) \geq e^{-2k} \text{ on } [-k,k].$$

Proof. It is clear that $u_k(x) = e^{x-k}$ solves $v(x) + H(v'(x)) = 0$ in $(-k,k)$ in the classical sense, and indeed, in viscosity sense. We need to verify that u_k is a viscosity supersolution on $[-k,k]$. Let $u_k - \varphi$ has a local minimum at $x = -k$ for $\varphi(x) \in \mathrm{C}^1(\mathbb{R})$. Clearly, we can see that

$$\varphi'(-k) \leq u_k'(-k) = e^{-2k},$$

which implies $e^{-2k} + H(\varphi'(-k)) \geq 0$. On the other hand, at $x = k$, one has

$$u_k(k) + H(\varphi'(k)) = 1 + H(\varphi'(k)) \geq 0$$

since by definition of H, it is bounded below by -1. Therefore, $u_k(x) = e^{x-k}$ is the unique viscosity solution to (4.2.1), and furthermore $e^{-2k} \leq u_k(x) \leq \left(e^{|x|}\right) e^{-k}$ for all $x \in [-k, k]$. In addition to that, we have $u_k(k) = 1$ for all $k \in \mathbb{N}$, hence, the convergence fails when $x = k$. □

4.4.2 Optimal control formulations

We give another example from the optimal control theory point of view (see [107]). Let us recall briefly the setting of optimal control as follows. Let U be a compact metric space. We regard a control as a Borel measurable map $\alpha : [0, \infty) \mapsto U$. Let Ω be an open subset of $\mathbb{R}^n$ with the connected boundary satisfying (A2). We also assume that $b = b(x, a) : \overline{\Omega} \times U \to \mathbb{R}^n$, $f = f(x, a) : \overline{\Omega} \times U \to \mathbb{R}$ satisfy

$$\begin{aligned}
&\sup_{a\in U} |b(x,a) - b(y,a)| \leq L(b)|x-y| && \text{for all } x, y \in \overline{\Omega}, \\
&\sup_{a\in U} |b(x,a)| \leq K(b) && \text{for all } x \in \overline{\Omega}, \\
&\sup_{a\in U} |f(x,a) - f(y,a)| \leq \omega_f(|x-y|) && \text{for all } x, y \in \overline{\Omega}, \\
&\sup_{a\in U} |f(x,a)| \leq K(f) && \text{for all } x \in \overline{\Omega},
\end{aligned}$$

where $K(b), L(b), K(f)$ are positive constants and ω_f is a nondecreasing continuous function with $\omega_f(0^+) = 0$.

For each $x \in \overline{\Omega}$ and a given control $\alpha(\cdot) : [0, \infty) \to U$, let $y^{x,\alpha}(t)$ be a controlled process (we will write α instead of $\alpha(\cdot)$ as a control for simplicity), which is a solution to

$$\begin{cases} \frac{d}{dt} y^{x,\alpha}(t) = b\left(y^{x,\alpha}(t), \alpha(t)\right) & \text{for } t > 0, \\ y^{x,\alpha}(0) = x. \end{cases}$$

We denote the set of controls (strategies) α where $y^{x,\alpha}(t) \in \overline{\Omega}$ for all $t \geq 0$ and $y^{x,\alpha}$ solves the ODE above by $\mathcal{A}_x$. The value function is defined by

$$u(x) = \inf_{\alpha \in \mathcal{A}_x} \int_0^\infty e^{-t} f\left(y^{x,\alpha}(t), \alpha(t)\right) dt.$$

Here, one can define the Hamiltonian associated with b and f as

$$H(x,p) := \sup_{a\in U} \{-b(x,a)\cdot p - f(x,a)\} \in \mathrm{C}\left(\overline{\Omega} \times \mathbb{R}^n; \mathbb{R}\right).$$

It was proved in [107] that u is a viscosity solution to (HJ_δ).

Proposition 4.4.11. *Let $n = 1$ and $U = [-1,1]$. Let us consider the following Hamiltonian defined as*

$$H(x,p) = \sup_{a\in[-1,1]} \left\{ -a\cdot p - e^{-|x|} \right\} = |p| - e^{-|x|}, \qquad (x,p) \in \mathbb{R}\times\mathbb{R}.$$

Then, the solution to (4.2.1) is given by $u_k(x) = \frac{e^{-|x|}}{2} + \frac{e^{|x|-2k}}{2}$ for $x \in [-k,k]$, while the solution to (4.2.3) is $u(x) = \frac{e^{-|x|}}{2}$. Hence, the exponential rate of convergence is obtained.

Proof. In the optimal control setting, the Hamiltonian above is obtained by considering $U = [-1,1]$, $b(x,a) = a$ and $f(x,a) = e^{-|x|}$. To find $u_k(x_0)$ and $u(x_0)$, one needs to find a control $\alpha(t)$ that minimizes

$$\int_0^\infty e^{-s-|y(s)|}\, ds \qquad \text{subject to} \qquad \begin{cases} \dot{y}(t) & = \alpha(t) \in [-1,1], \\ y(0) & = x_0. \end{cases}$$

It is easy to see the following points:

(i) An optimal control for the unconstrained problem with $x_0 \geq 0$ ($x_0 < 0$) is $\alpha(t) \equiv 1$ ($\alpha(t) \equiv -1$, respectively).

(ii) An optimal control for the constrained problem on $[-k,k]$ with $x_0 \geq 0$ ($x_0 < 0$) is $\alpha(t) \equiv 1$ on $[0, k-x_0]$ and 0 elsewhere ($\alpha(t) \equiv -1$ on $[0, k+x_0]$ and 0 elsewhere, respectively).

Once we have the optimal controls, we can easily compute the value function and the result follows. In conclusion, for all $x \in [-k,k]$ we have

$$0 \leq u_k(x) - u(x) = \left(\frac{e^{|x|}}{2}\right) e^{-2k}.$$

In this example, the convergence holds everywhere in $[-k,k]$ with the rate $\mathcal{O}\left(e^{-k}\right)$. □

***Remark* 36.** One interesting fact to point out here is that the optimal path starting from x_0 for the state-constraint problem on $[-k,k]$ in Proposition 4.4.11 stays on the boundary $\pm k$ for all $t \geq k - |x_0|$.

4.5 The case of bounded domain

The second prototype case is considered in this section. Let us assume that $(\mathcal{P}2)$, $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, are enforced. Recall that $\Omega_k = B\left(0, 1-\frac{1}{k}\right)$ for $k \in \mathbb{N}$, and $\Omega = B(0,1)$. Let $u_k \in \mathrm{Lip}\left(\overline{\Omega}_k\right)$ be the unique viscosity solution to

$$\begin{cases} u_k(x) + H(x, Du_k(x)) & \leq 0 \qquad \text{in } \Omega_k, \\ u_k(x) + H(x, Du_k(x)) & \geq 0 \qquad \text{on } \overline{\Omega}_k. \end{cases} \tag{4.5.1}$$

It is clear that we still have the following priori estimate

$$\|u_k\|_{L^\infty(\overline{\Omega}_k)} + \|Du_k\|_{L^\infty(\overline{\Omega}_k)} \leq C_H. \tag{4.5.2}$$

Proposition 4.5.1. *For each $k \in \mathbb{N}$, let u_k be the unique solution to* (4.5.1)*. Then, there exists $u \in BUC(\overline{\Omega})$ such that $u_k \to u$ locally uniformly on $\overline{\Omega}$ as k grows to infinity. Moreover, u has the same bounds as in* (4.5.2) *and solves*

$$\begin{cases} u(x) + H(x, Du(x)) & \leq 0 \qquad \text{in } \Omega, \\ u(x) + H(x, Du(x)) & \geq 0 \qquad \text{on } \overline{\Omega} \end{cases} \tag{4.5.3}$$

in viscosity sense.

Proof. From a priori estimate (4.5.2), by Arzelà-Ascoli's theorem and a diagonal argument we can extract a subsequence such that $u_{k_m} \to u$ uniformly on compact subsets of Ω. By the stability of viscosity solutions we obtain that $u \in C(\Omega)$ is a viscosity solution to

$$u(x) + H(x, Du(x)) = 0 \qquad \text{in } \Omega. \tag{4.5.4}$$

We deduce that $|u(x)| \leq C_B$ and $|u(x) - u(y)| \leq C_H|x - y|$ for $x, y \in \Omega$. We can extend $u \in \mathrm{Lip}(\overline{\Omega})$ with the same priori bound as in (4.5.2). We need to show that u is a viscosity supersolution to $u(x) + H(x, Du(x)) = 0$ on $\overline{\Omega}$.

We can verify it using Corollary 2.8.4. Indeed, let $v \in C(\overline{\Omega})$ be a viscosity subsolution to (4.5.4) in Ω. Applying the comparison principle to $u_k(x) + H(x, Du_k(x)) \geq 0$ on $\overline{\Omega}_k$, we have that $v(x) \leq u_k(x)$ for $x \in \overline{\Omega}_k$. Now fixing $r \in (0, 1)$, we have $v(x) \leq u_k(x)$ for all $x \in \overline{B(0, r)}$ and $r \leq 1 - \frac{1}{k}$ if k is large enough. Letting $k \to \infty$, we deduce that $v(x) \leq u(x)$ for $x \in \overline{B(0, r)}$. Since we have $u, v \in C(\overline{B(0, 1)})$, the inequality $v \leq u$ on $\overline{B(0, 1)}$ follows. Hence, u is a viscosity supersolution to (4.5.4) by Corollary 2.8.4. $\square$

Now we are ready to give a proof for Theorem 4.1.5. We note that star-shaped and scaling properties of $\{\Omega_k\}$ play an important role.

Proof of Theorem 4.1.5. The fact that $u_k(x) \geq u(x)$ on $\overline{\Omega}_k$ is clear by the comparison principle. For $k \geq 2$, let us define

$$\tilde{u}_k(x) := \frac{k}{k-1} u_k \left(\frac{k-1}{k} x \right) \qquad \text{for } x \in \overline{B(0, 1)}.$$

It is clear that $\tilde{u}_k$ is a viscosity subsolution to

$$\frac{k-1}{k} \tilde{u}_k(x) + H\left(\frac{k-1}{k} x, D\tilde{u}_k(x) \right) = 0 \qquad \text{in } B(0, 1). \tag{4.5.5}$$

From (4.5.2) and $(\mathcal{H}_3)$, there exists $\tilde{C}_H$ such that $|H(x, p) - H(x, p)| \leq \tilde{C}_H |x - y|$ for all $x, y \in \overline{\Omega}$ and $|p| \leq C_H$. Therefore, by using (4.5.5) we have

$$\tilde{u}_k(x) + H(x, D\tilde{u}_k(x)) \leq \frac{1}{k} \tilde{u}_k(x) + H(x, D\tilde{u}_k(x)) - H\left(\frac{k-1}{k} x, D\tilde{u}_k(x) \right) \leq \frac{C_H + \tilde{C}_H}{k}$$

for all $x \in B(0, 1)$. By the comparison principle and the fact that u solves (4.5.3) in the viscosity sense, we deduce that

$$\tilde{u}_k(x) - \frac{C_H + \tilde{C}_H}{k} \leq u(x) \qquad \text{for all } x \in \overline{B(0, 1)}.$$

Consequently, we obtain the conclusion $u_k(x) \leq u(x) + \frac{C}{k}$ for $x \in \overline{\Omega}_k$ where the constant C can be chosen as $C = 2C_H + \tilde{C}_H$. $\square$

Remark 37.

(i) It is clear from the proof that prototype condition $(\mathcal{P}2)$ can be relaxed as following.

$(\mathcal{P}2')$ Assume $0 \in \Omega_k$ for all $k \in \mathbb{N}$, $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$ is bounded, and the comparison principle for the state-constraint problem holds on Ω_k, Ω. Assume further that, for $k \in \mathbb{N}$,

$$\left(1 - \frac{1}{k}\right) \Omega \subset \Omega_k.$$

(ii) Theorem 4.1.5 can also be proved using the doubling variable method with the following auxiliary function (see [30])

$$\Phi^k(x,y) := \tfrac{k+1}{k-1} u_k \left(\tfrac{k-1}{k+1} x\right) - u(y) - C_H k^2 |x-y|^2$$

for $(x,y) \in \left(1 + \frac{1}{k}\right) \overline{\Omega} \times \overline{\Omega}$.

The following remark shows that the rate $\mathcal{O}\left(\frac{1}{k}\right)$ is indeed optimal.

Remark 38. Let H be defined as in Proposition 4.4.10, we see that $u_k(x) = e^{x - \left(1 - \frac{1}{k}\right)}$ solves (4.5.1) and $u(x) = e^{x-1}$ solves (4.5.3), therefore

$$0 \le u_k(x) - u(x) = e^{x-1}\left(e^{\frac{1}{k}} - 1\right) \le \frac{2}{k}$$

for $x \in \left[-\left(1 - \frac{1}{k}\right), 1 - \frac{1}{k}\right]$. Besides, $e^{\frac{1}{k}} - 1 \ge \frac{1}{k}$, and so, $\mathcal{O}\left(\frac{1}{k}\right)$ is optimal.

4.6 Discussions

4.6.1 Examples and open questions

We give here some further discussions along the line with the topics considered in the paper. Firstly, when our Hamiltonian is given as $H(x,p) = a(x)K(p)$ in the first prototype $(\mathcal{P}1)$, we get an exponential rate of convergence provided that the assumption $(\mathcal{H}_1)$ is enforced (Theorem 4.1.2). Without this assumption, we have an example with a polynomial rate of convergence whose power can be increased or decreased as much as we want.

Example 3. *Let us consider $n = 1$, $H(x,p) = \left(\frac{1+|x|}{m}\right) K(p)$ for $m > 1$ and $K : \mathbb{R} \to \mathbb{R}$ defined by*

$$K(p) = \begin{cases} -|p| & \text{for } |p| \le 1, \\ |p| - 2 & \text{for } |p| \ge 1. \end{cases} \tag{4.6.1}$$

The unique viscosity solution to (4.2.1) is

$$u_k(x) = \frac{(1+|x|)^m}{m(1+k)^{m-1}} \qquad \text{for } x \in [-k,k].$$

Clearly, $u_k(x) \to 0$ locally uniformly with rate $\mathcal{O}\left(\frac{1}{k^{m-1}}\right)$ for any given $m > 1$. We should note that the limit 0 is not a unique solution to (4.2.3). Another solution to (4.2.3) is $u(x) = m^{-1}(1+|x|)^m$, but it does not belong to $\mathrm{BUC}(\mathbb{R})$.

Example 4. *Assume $n = 1$, $H(x,p) = K(p) + V(x)$ where $V(x) = e^{-|x|}$ and $K : \mathbb{R} \to \mathbb{R}$ defined by*

$$K(p) = \begin{cases} -|p| & \text{for } |p| \leq 1, \\ |p| - 2 & \text{for } |p| \geq 1. \end{cases}$$

The unique state-constraint viscosity solution to (4.2.1) is

$$u_k(x) = -\frac{1}{2}e^{-|x|} + \left(e^{-k} - \frac{1}{2}e^{-2k}\right) e^{|x|}, \qquad x \in [-k,k],$$

and the unique viscosity solution to (4.2.3) is

$$u(x) = -\frac{1}{2}e^{-|x|}, \qquad x \in \mathbb{R}.$$

We have $u_k \to u$ locally uniformly in $\mathbb{R}$ with rate $\mathcal{O}(e^{-k})$.

Secondly, prototype condition $(\mathcal{P}1)$ can be relaxed as follows.

***Remark* 39.** It is clear from the proofs of our main results (Theorems 4.1.1, 4.1.2, 4.1.4, and Corollary 4.1.3) that prototype condition $(\mathcal{P}1)$ can be relaxed as following.

$(\mathcal{P}1')$ Assume Ω_k is bounded, $B(0,k) \subset \Omega_k$, and the comparison principle for the state-constraint problem holds on Ω_k for all $k \in \mathbb{N}$. Of course, $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k = \mathbb{R}^n$ here.

Thirdly, there are some open questions we are not able to answer yet.

Question 1. *In the first prototype $(\mathcal{P}1)$ case, what is the optimal rate of convergence of u_k to u in the general nonconvex setting?*

A more specific question is as follows.

Question 2. *Assume $(\mathcal{P}1)$, and $H(x,p) = K(p) + V(x)$, where $K \in \mathrm{Lip}(\mathbb{R}^n)$ is coercive and nonconvex, and $V \in \mathrm{BUC}(\mathbb{R}^n)$. Is it true that we always have an exponential rate of convergence of u_k to u?*

4.6.2 Existence of minimizers in the general case

We show that one can remove the smoothness of L in Theorem 4.4.7 under the assumption $(\mathcal{L}_3)$.

Let us consider mollifiers in $\mathbb{R}^{2n}$ defined as $\{\eta_\varepsilon\}_{\varepsilon>0}$ such that $\eta_\varepsilon(x) = \frac{1}{\varepsilon^{2n}}\eta\left(\frac{x}{\varepsilon}\right)$ for $x \in \mathbb{R}^{2n}$ where $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^{2n})$ satisfying $0 \leq \eta \leq 1$, supp $(\eta) \subset B_{\mathbb{R}^{2n}}(0,1)$ and $\int_{\mathbb{R}^{2n}} \eta(x)\,dx = 1$.

For each $\varepsilon > 0$ we define the convolution $L^\varepsilon = \eta_\varepsilon * L \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. It is easy to see that L^ε is bounded below, $(\mathcal{L}_5)$, $(\mathcal{L}_6)$ are preserved to L^ε and $(\mathcal{L}_7)$ now becomes:

$(\mathcal{L}_7^\varepsilon)$ There exist $A_\varepsilon, B_\varepsilon > 0$ such that $A_\varepsilon^{-1}|v|^2 - B_\varepsilon^{-1} \leq L^\varepsilon(x,v) \leq A_\varepsilon|v|^2 + B_\varepsilon$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$, and $A_\varepsilon \to A, B_\varepsilon \to B$ as $\varepsilon \to 0$.

By Theorem 4.4.7, there exists a minimizer γ_ε in $\mathcal{A}_x$ such that

$$u^\varepsilon(x) := \inf_{\zeta\in\mathcal{A}_x} \int_0^\infty e^{-s} L^\varepsilon\big(\zeta(s), -\dot{\zeta}(s)\big)\, ds = \int_0^\infty e^{-s} L^\varepsilon\big(\gamma_\varepsilon(s), -\dot{\gamma}_\varepsilon(s)\big)\, ds.$$

Let H^ε be the Legendre transform of L^ε. Then, we can show that u^ε is the unique solution to $u^\varepsilon(x) + H^\varepsilon(x, Du^\varepsilon(x)) = 0$ in $\mathbb{R}^n$. It is easy to see that $H^\varepsilon \to H$ locally uniformly in $\mathbb{R}^n \times \mathbb{R}^n$, therefore by stability of viscosity solutions, $u^\varepsilon \to u$ locally uniformly in $\mathbb{R}^n$ as $\varepsilon \to 0$.

We indeed have that γ^ε is smooth according to Remark 34. Furthermore, Theorem 4.4.7 yields that $\|e^{-\frac{s}{2}}\dot{\gamma}_\varepsilon(s)\|_{L^2} \le C$ and $|\dot{\gamma}_\varepsilon| \le C$ pointwise in $(0,\infty)$. Therefore, we can define $\gamma \in \mathcal{A}_x$ such that (up to subsequence) $\gamma_\varepsilon \to \gamma$ locally uniformly on $[0,\infty)$ and $e^{-\frac{s}{2}}\dot{\gamma}_\varepsilon \rightharpoonup e^{-\frac{s}{2}}\dot{\gamma}$ weakly in L^2. Since $L^\varepsilon \to L$ uniformly on a compact set and $\{-\dot{\gamma}_\varepsilon(s)\}_{\varepsilon>0}$ is bounded, we obtain that

$$L^\varepsilon\big(\gamma_\varepsilon(s), -\dot{\gamma}_\varepsilon(s)\big) = L\big(\gamma(s), -\dot{\gamma}_\varepsilon(s)\big) + \tilde{\omega}_{C_5}(\varepsilon) + \tilde{\omega}_{C_5}(|\gamma_\varepsilon(s) - \gamma(s)|)$$

using $(\mathcal{L}_3)$. Therefore, it suffices to show that

$$\int_0^\infty e^{-s} L(\gamma(s), -\dot{\gamma}(s))\, ds \le \liminf_{\varepsilon\to 0} \int_0^\infty e^{-s} L\big(\gamma(s), -\dot{\gamma}_\varepsilon(s)\big)\, ds. \tag{4.6.2}$$

For simplicity, let $d\mu = e^{-s}ds$ be a probability measure on $[0,\infty)$. It is easy to see that the functional $I : L^2(\mu) \to \mathbb{R}$ that maps

$$f \mapsto \int_0^\infty L(\gamma(s), f(s)) d\mu(s)$$

is convex and lower semicontinuous, thus it is also weakly lower semicontinuous. Now since $\dot{\gamma}_\varepsilon \rightharpoonup \dot{\gamma}$ weakly in $L^2(d\mu)$, we obtain (4.6.2) and thus γ is a minimizer for $u(x)$.

Remark 40. Inequality (4.6.2) for the Cauchy problem (finite time horizon) is proved using a different argument by H. Ishii in [65] under more general assumptions. Such inequalities are crucial for the analysis of large time behavior of solutions to the time-dependent problems.

4.6.3 Supplemental results on the existence of minimizers

In this section we provide the detailed proofs for all lemmas in Section 4.4.

Proof of Lemma 4.4.6. We first prove the result for all (x,v) with $|v| \le 1$, then by scaling we get the result for all (x,v). Using $(\mathcal{L}_7)$ we have $-B \le L(x,v) \le 4A + B$ for all (x,v) with $|v| \le 2$. For $u, v \in B(0,1)$ with $u \ne v$, let $w = v + |v-u|^{-1}(v-u)$. Then, $|w| < |v| + 1 < 2$. Let $\lambda = (1 + |u-v|)^{-1} \in (0,1)$, we have $v = \lambda u + (1-\lambda)w$. By the convexity, one obtains

$$L(x,v) - L(x,u) \le (1-\lambda)\big(L(x,w) - L(x,u)\big) \le (4A + 2B)|u - v|.$$

By symmetry, we deduce that $|L(x,u) - L(x,v)| \le (4A + 2B)|u - v|$ for all (x,v) with $|v| \le 1$. In other words, we have that $|\xi| \le 4A + 2B$ whenever $\xi \in D_v^- L(x,v)$ for $(x,v) \in$

$\mathbb{R}^n \times \overline{B(0,1)}$. Now for $r > 1$, we define $L_r(x,v) = r^{-2}L(x,rv)$ for $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$. We observe that

$$A^{-1}|v|^2 - B \le A^{-1}|v|^2 - Br^{-2} \le L_r(x,v) \le A|v|^2 + Br^{-2} \le A|v|^2 + B$$

for all (x,v). For $v \in \mathbb{R}^n$ with $|v| \ge 1$, let $r = 2|v| > 1$ and $u = \frac{v}{2|v|} \in B(0,1)$ so that $v = ru$. Since $\xi \in D_v^- L(x,v)$ implies $\frac{\xi}{r} \in D_u^- L_r(x,u)$, we have

$$|\xi| \le (4A + 2B)|r| = (8A + 4B)|v|.$$

□

Proof of Theorem 4.4.7. Let $\{\eta_k\}_{k=1}^{\infty} \subset \mathcal{A}_x$ be a minimizing sequence in $\mathrm{AC}([0,\infty))$ such that $\lim_{k\to\infty} J[x,\eta_k] = u(x)$. From the uniform boundedness of u and the quadratic bounds of $L(x,v)$, we have

$$\left\| e^{-\frac{s}{2}} \dot{\eta}_k(s) \right\|_{L^2((0,\infty);\mathbb{R}^n)} \le C_4.$$

Here, C_4 can be chosen as $(A(2C_H + B))^{\frac{1}{2}}$. By the weak compactness of L^2, there exists g such that $e^{-\frac{s}{2}} g(s) \in L^2\big((0,\infty);\mathbb{R}^n\big)$ and a subsequence $\{k_j\} \to \infty$ such that $e^{-\frac{s}{2}}\dot{\eta}_{k_j} \rightharpoonup e^{-\frac{s}{2}} g$ weakly in $L^2\big((0,\infty);\mathbb{R}^n\big)$ as $j \to \infty$.

Writing g as $e^{\frac{s}{2}} g \cdot e^{-\frac{s}{2}}$ and using the Cauchy-Schwartz inequality, we get

$$g \in L^1_{\mathrm{loc}}\big((0,\infty);\mathbb{R}^n\big)$$

For $t > 0$, we let $\eta(t) = x + \int_0^t g(s)\, ds$. Clearly, $\eta \in \mathcal{A}_x$ and one obtains that $\eta_{k_j} \to \eta$ pointwise with $\dot{\eta} = g$ almost everywhere. On the other hand, the convexity of L implies

$$L\left(\eta_{k_j}(s), -\dot{\eta}_{k_j}(s)\right) \ge L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right) - D_v L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right) \cdot \left(\dot{\eta}_{k_j}(s) - \dot{\eta}(s)\right).$$

Therefore,

$$\int_0^\infty e^{-s} L\left(\eta_{k_j}(s), -\dot{\eta}_{k_j}(s)\right) ds \ge \int_0^\infty e^{-s} L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right) ds + \int_0^\infty e^{-s/2} D_v L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right) \cdot e^{-s/2}\left(\dot{\eta}_{k_j}(s) - \dot{\eta}(s)\right) ds.$$

Since $|D_v L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right)| \le C_L(1 + |\dot{\eta}(s)|)$ for a.e. $s \in (0,\infty)$, it is clear that

$$e^{-s/2} D_v L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right) \to e^{-s/2} D_v L\left(\eta(s), -\dot{\eta}(s)\right)$$

in $L^2((0,\infty);\mathbb{R}^n)$ and thus

$$\int_0^\infty e^{-s} D_v L\left(\eta_{k_j}(s), -\dot{\eta}(s)\right) \cdot \left(\dot{\eta}_{k_j}(s) - \dot{\eta}(s)\right) ds$$

converges to 0 as k goes to infinity, which yields that $J[x,\eta] \le u(x)$. Hence $J[x,\eta] = u(x)$. □

Proof of Lemma 4.4.8. By the definition of u in (4.4.3), we have

$$u(\eta(t)) \leq \int_0^\infty e^{-s} L(\gamma(s), -\dot{\gamma}(s))\, ds = e^t \int_t^\infty e^{-\xi} L\left(\eta(\xi), -\dot{\eta}(\xi)\right)\, d\xi,$$

where $\gamma(s) = \eta(t+s)$ for $s \geq 0$. Thus,

$$e^{-t} u(\eta(t)) \leq \int_t^\infty e^{-\xi} L\left(\eta(\xi), -\dot{\eta}(\xi)\right)\, d\xi. \tag{4.6.3}$$

By the dynamic programming principle and (4.6.3), we have

$$u(\eta(0)) \leq \int_0^t e^{-s} L(\eta(s), -\dot{\eta}(s))\, ds + e^{-t} u(\eta(t)) \leq \int_0^\infty e^{-s} L(\eta(s), -\dot{\eta}(s))\, ds = u(\eta(0)).$$

Therefore, (4.4.9), (4.4.10) and (4.4.11) follow. □

Proof of Lemma 4.4.9. For every $t, h > 0$, by Lemma 4.4.8 we have that

$$\frac{e^{-t} u(\eta(t)) - e^{-(t+h)} u(\eta(t+h))}{h} = \frac{1}{h} \int_t^{t+h} e^{-s} L(\eta(s), -\dot{\eta}(s))\, ds.$$

Let $\varphi \in C^1(\mathbb{R})$ such that $u - \varphi$ has a local min at $\eta(t)$ and $u(\eta(t)) = \varphi(\eta(t))$, then

$$\frac{e^{-t} u(\eta(t)) - e^{-(t+h)} u(\eta(t+h))}{h} \leq \frac{e^{-t} \varphi(\eta(t)) - e^{-(t+h)} \varphi(\eta(t+h))}{h}.$$

Therefore,

$$\frac{1}{h} \int_t^{t+h} e^{-s} L\Big(\eta(s), -\dot{\eta}(s)\Big)\, ds \leq \frac{e^{-t} \varphi(\eta(t)) - e^{-(t+h)} \varphi(\eta(t+h))}{h}.$$

Since $\eta(t)$ is differentiable a.e. in $(0, \infty)$, at those t where $\eta(t)$ is differentiable, let $h \to 0^+$ we deduce that

$$e^{-t} L(\eta(t), -\dot{\eta}(t)) \leq -\frac{d}{dt}\Big(e^{-t} \varphi(\eta(t))\Big) = e^{-t} \varphi(\eta(t)) - e^{-t} D\varphi(\eta(t)) \cdot \dot{\eta}(t).$$

Thus, for a.e. $t > 0$ where η is differentiable, we have

$$L(\eta(t), -\dot{\eta}(t)) \leq \varphi(\eta(t)) - D\varphi(\eta(t)) \cdot \dot{\eta}(t).$$

By $(\mathcal{L}_7)$ and a priori estimate (4.4.5) for a.e. $t \in (0, \infty)$ we have that

$$A^{-1} |\dot{\eta}(t)|^2 - B \leq \varphi(\eta(t)) - D\varphi(\eta(t)) \cdot \dot{\eta}(t) \leq C_H + C_H |\dot{\eta}(t)|.$$

This shows that $|\dot{\eta}(t)| \leq C_5$ for a.e. $t \in (0, \infty)$, and C_5 only depends on C_H, A, B. □

It is worth emphasizing again here that the bound C_5 on the velocity of η only depends on C_H, A, B, which can be seen clearly from the last chain of inequalities in the above proof. In fact, one can choose explicitly that $C_5 = (2AB + 2AC_H + A^2 C_H^2)^{1/2}$.

Chapter 5

State-constraint problems with vanishing discount and eigenvalues on changing domains

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $H(x,p) : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a continuous Hamiltonian that is convex in the second argument. We study the asymptotic behavior, as $\lambda \to 0^+$, of the state-constraint Hamilton–Jacobi equation

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \le 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \ge 0 & \text{on } (1+r(\lambda))\overline{\Omega}, \end{cases} \tag{S_λ}$$

and the corresponding additive eigenvalues, or ergodic constant

$$\begin{cases} H(x, Dv_\lambda(x)) \le c(\lambda) & \text{in } (1+r(\lambda))\Omega, \\ H(x, Dv_\lambda(x)) \ge c(\lambda) & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases} \tag{E_λ}$$

Here, $\phi(\lambda), r(\lambda) : (0,\infty) \to \mathbb{R}$ are continuous functions such that ϕ is nonnegative and

$$\lim_{\lambda\to 0^+} \phi(\lambda) = \lim_{\lambda\to 0^+} r(\lambda) = 0.$$

In this chapter we study the behavior of u_λ and c_λ as $\lambda \to 0^+$. We obtain both convergence and non-convergence results for the convex Hamilton–Jacobi equations. Moreover, we provide a very first result on the asymptotic expansion of the additive eigenvalue $c(\lambda)$ as $\lambda \to 0^+$. The main tool we use is a duality representation of solution with viscosity Mather measures. The materials of this chapter are mainly taken from [114] with some additional background on the state-constraint problems added.

5.1 Introduction

Let $\phi(\lambda) : (0,\infty) \to (0,\infty)$ be continuous nondecreasing and $r(\lambda) : (0,\infty) \to \mathbb{R}$ be continuous such that $\lim_{\lambda\to 0^+} \phi(\lambda) = \lim_{\lambda\to 0^+} r(\lambda) = 0$. We study the asymptotic behavior, as the discount factor $\phi(\lambda)$ goes to 0, of the viscosity solutions to the following state-constraint Hamilton–Jacobi equation

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \le 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \ge 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases} \tag{S_λ}$$

Here, Ω is a bounded domain of $\mathbb{R}^n$. For simplicity, we will write $\Omega_\lambda = (1+r(\lambda))\Omega$ for $\lambda > 0$. Roughly speaking, along some subsequence $\lambda_j \to 0^+$, we obtain the limiting equation as a state-constraint ergodic problem:

$$\begin{cases} H(x, Du(x)) \le c(0) & \text{in } \Omega, \\ H(x, Du(x)) \ge c(0) & \text{on } \overline{\Omega}. \end{cases} \tag{S_0}$$

Here $c(0)$ is the so-called critical value (additive eigenvalue) defined as

$$c(0) = \inf\Big\{c \in \mathbb{R} : H(x, Du(x)) \le c \ \text{ in } \Omega \text{ has a solution}\Big\}. \tag{5.1.1}$$

This quantity is finite and indeed the infimum in (5.1.1) can be replaced by minimum under our assumptions. We want to study the convergence of u_λ, solution to (S_λ), under some normalization, to solution of (S_0). It turns out this problem is interesting and challenging as it concerns both the vanishing discount and the rate of changing domains at the same time.

The selection problem for the vanishing discount problems on fixed domains was studied extensively in the literature recently. The first-order equations on the torus was obtained in [44], and the second-order equations on the torus were studied in [69, 92]. The problems in bounded domains with boundary conditions were proved in [1, 70]. The problem in $\mathbb{R}^n$ under additional assumptions that lead to the compactness of the Aubry set was studied in [71]. For the selection problems with state-constraint boundary conditions, so far, there is only [70] that deals with a fixed domain, and there is not yet any result studying the situation of the changing domains. It turns out that the problem is much more subtle as we have to take into account the changing domain factor appropriately. Surprisingly, we can obtain both convergence results and non-convergence results in this setting.

This result is an extension of the selection principle in the setting of changing domains. Generally speaking, known results assert that in the convex setting the whole family of solutions of the discounted problems, which are uniquely solved if the ambient space is compact, converges to a distinguished solution of the ergodic limit equation

$$H(x, Du(x)) = c(0). \tag{5.1.2}$$

We emphasize that (5.1.2) has multiple solutions, therefore it is a non-trivial problem to characterize the limiting solution.

We show the convergence for some natural normalization of solutions to (S_λ) together with characterizing their limits, related characterizations are done in [71] for the case the domain is $\mathbb{R}^n$ and in [44, 69, 111] for the case the domain is torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We also discuss other related results concerning the asymptotic behavior of the additive eigenvalue of H in Ω_λ as $\lambda \to 0^+$.

5.1.1 Assumptions

In this chapter, by a domain, we mean an open, bounded, connected subset of $\mathbb{R}^n$. Without loss of generality, we will always assume $0 \in \Omega$. To have well-posedness for (S_λ), one needs to have a comparison principle. Throughout the chapter, we will assume that $H : \overline{U} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous Hamiltonian where $U = B(0, R_0)$ such that $2\Omega \subseteq U$. We recall here the assumptions on H and Ω for clarity.

Assumptions on the Hamiltonian

($\mathcal{H}_0$) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$.

($\mathcal{H}_1$) There exists $C_1 > 0$ such that $H(x,p) \geq -C_1$ for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$.

($\mathcal{H}_2$) There exists $C_2 > 0$ such that $|H(x,0)| \leq C_2$ for all $x \in \overline{\Omega}$.

($\mathcal{H}_3$) For each $R > 0$ there exists a constant C_R such that

$$\begin{cases} |H(x,p) - H(y,p)| \leq C_R|x-y|, \\ |H(x,p) - H(x,q)| \leq C_R|p-q|, \end{cases} \tag{5.1.3}$$

for $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$ with $|p|, |q| \leq R$.

($\mathcal{H}_4$) H satisfies the coercivity assumption

$$\lim_{|p|\to\infty} \left(\inf_{x\in\overline{\Omega}} H(x,p) \right) = +\infty. \tag{5.1.4}$$

($\mathcal{H}_5$) $p \mapsto H(x,p)$ is convex for each $x \in \overline{\Omega}$.

($\mathcal{H}_8$) For $v \in \mathbb{R}^n$, $x \mapsto L(x,v)$ is continuously differentiable on $\overline{U}$, where the Lagrangian L of H is defined as

$$L(x,v) = \sup_{p\in\mathbb{R}^n} \Big(p \cdot v - H(x,p) \Big), \qquad (x,v) \in \overline{U} \times \mathbb{R}^n.$$

The regularity assumption ($\mathcal{H}_8$) is needed for technical reason when we deal with changing domains, it satisfies for a vast class of Hamiltonians, for example $H(x,p) = H(p) + V(x)$ or $H(x,p) = V(x)H(p)$ with $V \in \mathrm{C}^1$.

Remark *41.* In fact we only need that $\lambda \mapsto L((1+\lambda)x, v)$ is continuously differentiable at $\lambda = 0$ but we assume ($\mathcal{H}_8$) for simplicity.

Assumptions on the regularity of the domain

($\mathcal{A}_1$) Ω a bounded star-shaped (with respect to the origin) open subset of $\mathbb{R}^n$ and there exists some $\kappa > 0$ such that $\mathrm{dist}(x, \overline{\Omega}) \geq \kappa r$ for all $x \in (1+r)\partial\Omega$ and $r > 0$.

($\mathcal{A}_2$) There exists a universal pair of positive numbers (r, h) and $\eta \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R}^n)$ such that

$$B(x + t\eta(x), rt) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, h].$$

We consider the following case in our paper about the vanishing and changing domain rates:

$$\lim_{\lambda \to 0^+} \left(\frac{r(\lambda)}{\phi(\lambda)} \right) = \gamma \in [-\infty, +\infty]. \tag{5.1.5}$$

Remark *42.* Under the assumption (5.1.5), there are only three possible cases:

1. (Inner approximation) $r(\lambda)$ is negative for $\lambda \ll 1$, consequently $\gamma \leq 0$.
2. (Outer approximation) $r(\lambda)$ is positive for $\lambda \ll 1$, consequently $\gamma \geq 0$.
3. $r(\lambda)$ is oscillating around 0 when $\lambda \to 0^+$, consequently $\gamma = 0$. An example for this case is $r(\lambda) = \lambda \sin\left(\lambda^{-1}\right)$.

We note that assumption (5.1.5) does not cover the case where $r(\lambda)/\phi(\lambda)$ is bounded but the limit at $\lambda \to 0^+$ does not exist, for example $r(\lambda) = \lambda \sin\left(\lambda^{-1}\right)$ and $\phi(\lambda) = \lambda$. Nevertheless, when the limit (5.1.5) exists and $r(\lambda)$ is oscillating near 0, the limit must be $\gamma = 0$ and it turns out that the case $\gamma = 0$ is substantially simpler to analyze, as solutions of (5.2.7) converge to the maximal solution of (S_0) (Theorem 5.1.1).

5.1.2 Literature on state-constraint and vanishing discount problems

There is a vast amount of works in the literature on the well-posedness of state-constraint Hamilton-Jacobi equations and fully nonlinear elliptic equations. The state-constraint problem for first-order convex Hamilton-Jacobi equations using optimal control frameworks was first studied in [107, 108]. The general nonconvex, coercive first-order equations were then discussed in [30]. For the nested domain setting, a rate of convergence for the discount problem is studied in [75]. We also refer to the classical books [8, 10], and the references therein.

There are also many works in the spirit of looking at a general framework of the vanishing discount problem. The convergence of solutions to the vanishing discount problems is proved in [64]. A problem with a similar spirit to ours is considered in [32], in which the authors study the asymptotic behavior of solution on compact domain with respect to the Hamiltonian. In this work we take advantage of the clear from and structure of (S_λ) to obtain more explicit properties on solutions and furthermore the asymptotic expansion of the additive eigenvalues. We also remark that the continuity of the additive eigenvalue for general increasing domains for second-order equation is concerned in [12]. See [66] for a recent work on vanishing discount for weakly coupled system and [97] for second-order equation with Neumann boundary condition. We also mention the recent work in [45, 115, 117] where the representation of solution using Mather measures is used.

5.1.3 Main results

There are two natural normalizations for solutions of (S_λ). The first one is similar to what has been considered in [64, 71, 111] as

$$\left\{ u_\lambda + \frac{c(0)}{\phi(\lambda)} \right\}_{\lambda>0}, \tag{5.1.6}$$

and the second one is given by

$$\left\{ u_\lambda + \frac{c(\lambda)}{\phi(\lambda)} \right\}_{\lambda>0}, \tag{5.1.7}$$

where $c(\lambda)$ is the additive eigenvalues of H in Ω_λ, i.e., the unique constant such that the following ergodic problem can be solved

$$\begin{cases} H(x, Dv_\lambda(x)) \leq c(\lambda) & \text{in } \Omega_\lambda \\ H(x, Dv_\lambda(x)) \geq c(\lambda) & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{E_λ}$$

Let u^0 be the limiting solution of the vanishing discount problem on fixed bounded domain (see [44, 64, 69, 111] and Theorem 5.2.8), our first result is as follows.

Theorem 5.1.1. *Assume* $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$ *and* $(\mathcal{A}_1)$*. If* $\gamma = 0$ *then both families* (5.1.6) *and* (5.1.7) *converge to* u^0 *locally uniformly as* $\lambda \to 0^+$*.*

We note that Theorem 5.1.1 includes the case where $r(\lambda)$ is oscillating, as long as the limit (5.1.5) exists. For example $r(\lambda) = \lambda \sin\left(\lambda^{-1}\right)$ and $\phi(\lambda) = \lambda^p$ with $p \in (0, 1)$.

If γ is finite then (5.1.6) is bounded and convergent. Its limit can be characterized in terms of probability minimizing measures $\mathcal{M}_0$ (or viscosity Mather measures, see Section 2). For $h > 0$ we denote by B_h the open ball $B_h = \{x \in \mathbb{R}^n : |x| < h\}$. For a ball $\overline{B}_h \subset \mathbb{R}^n$ and a measure μ defined on $\overline{\Omega} \times \overline{B}_h$, we define

$$\langle \mu, f \rangle := \int_{\overline{\Omega} \times \overline{B}_h} f(x, v)\, d\mu(x, v), \qquad \text{for } f \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h). \tag{5.1.8}$$

Theorem 5.1.2. *Assume* $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$, $(\mathcal{H}_8)$ *and* $(\mathcal{A}_1)$*. If* $\gamma \in \mathbb{R}$ *then the family* (5.1.6) *converge to* u^γ *locally uniformly in* Ω *as* $\lambda \to 0^+$*. Furthermore*

$$u^\gamma = \sup_{w \in \mathcal{E}^\gamma} w, \tag{5.1.9}$$

where $\mathcal{E}^\gamma$ *denotes the family of subsolutions* w *to the ergodic problem* (S_0) *such that*

$$\gamma \big\langle \mu, (-x) \cdot D_x L(x, v) \big\rangle + \langle \mu, w \rangle \leq 0 \qquad \textit{for all } \mu \in \mathcal{M}_0.$$

Remark 43. The factor $\gamma \langle \mu, (-x) \cdot D_x L(x, v) \rangle$ here captures the scaling property of the problem, which is where u^γ and u^0 are different from each other. Also, if $\gamma = \infty$ then the family (5.1.6) could be unbounded (Example 5). We note that Theorem 5.1.2 includes the conclusion of Theorem 5.1.1 for the family (5.1.6) but we do not need the technical assumption $(\mathcal{H}_8)$ for Theorem 5.1.1.

Corollary 5.1.3. *The mapping $\gamma \mapsto u^\gamma(\cdot)$ from $\mathbb{R}$ to $C(\overline{\Omega})$ is concave and decreasing. Precisely, if $\alpha, \beta \in \mathbb{R}$ with $\alpha \le \beta$ then $u^\beta \le u^\alpha$ and*

$$(1-\lambda)u^\alpha + \lambda u^\beta \le u^{(1-\lambda)\alpha+\lambda\beta} \qquad \text{for every } \lambda \in [0,1].$$

For the second family (5.1.7), we observe that it is bounded even if $\gamma = \infty$, and the difference between the two normalization (5.1.7) and (5.1.6) is given by

$$\left\{ \frac{c(\lambda)-c(0)}{\phi(\lambda)} \right\}_{\lambda>0}. \tag{5.1.10}$$

If $\gamma < \infty$ then the two families (5.1.6) and (5.1.7) are convergent if and only if the limit of (5.1.10) as $\lambda \to 0^+$ exists. In that case we have

$$\lim_{\lambda\to 0^+} \left(\frac{c(\lambda)-c(0)}{\phi(\lambda)} \right) = \gamma \lim_{\lambda\to 0^+} \left(\frac{c(\lambda)-c(0)}{r(\lambda)} \right). \tag{5.1.11}$$

The limit on the right-hand side should be understood as taking along sequences where $r(\lambda) \neq 0$. In other words we only concern those functions $r(\cdot)$ that are not identically zero near 0, since otherwise $c(\lambda) = c(0)$ for $\lambda \ll 1$ and the problem is not interesting. It leads naturally to the question of the asymptotic expansion of the critical value

$$c(\lambda) = c(0) + c^{(1)} r(\lambda) + \wr(r(\lambda)) \qquad \text{as } \lambda \to 0^+. \tag{5.1.12}$$

To our knowledge, this kind of question is new in the literature. We prove that the limit in (5.1.11) always exists if $r(\lambda)$ does not oscillate its sign near 0, and as a consequence it provides a necessary and sufficient condition under which the limit (5.1.11) exists for a general oscillating $r(\lambda)$. Of course this oscillating behavior is excluded when we only concern about the convergence of (5.1.6) and (5.1.7) (since $\gamma = 0$). We also give a characterization for the limit in (5.1.11) in terms of $\mathcal{M}_0$.

Theorem 5.1.4. *Assume $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$, $(\mathcal{H}_8)$ and $(\mathcal{A}_1)$, we have*

$$\lim_{\substack{\lambda\to 0^+ \\ r(\lambda)>0}} \left(\frac{c(\lambda)-c(0)}{r(\lambda)} \right) = \max_{\mu\in\mathcal{M}_0} \langle \mu, (-x)\cdot D_x L(x,v) \rangle,$$

$$\lim_{\substack{\lambda\to 0^+ \\ r(\lambda)<0}} \left(\frac{c(\lambda)-c(0)}{r(\lambda)} \right) = \min_{\mu\in\mathcal{M}_0} \langle \mu, (-x)\cdot D_x L(x,v) \rangle.$$

Thus $(c(\lambda)-c(0))/r(\lambda)$ converges as $\lambda \to 0^+$ if and only if the following invariant holds

$$\langle \mu, (-x)\cdot D_x L(x,v) \rangle = c^{(1)} \qquad \text{for all } \mu \in \mathcal{M}_0$$

where $c^{(1)}$ is a positive constant.

Corollary 5.1.5. *If $\langle \mu, (-x)\cdot D_x L(x,v)\rangle = c^{(1)}$ for all $\mu \in \mathcal{M}_0$ then $u^\gamma(\cdot) + \gamma c_{(1)} = u^0(\cdot)$.*

Corollary 5.1.6. *If $u^0(z) = u^\gamma(z)$ for some $z \in \Omega$ and $\gamma > 0$ then $c^{(1)}_- = 0$.*

Theorem 5.1.4 gives us the convergence of the second normalization (5.1.7) for finite γ. We recall that the case $\gamma = 0$ is already considered in Theorems 5.1.1 and 5.1.2.

Corollary 5.1.7. *Assume* $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$, $(\mathcal{H}_8)$ *and* $(\mathcal{A}_1)$ *and* $\gamma \in \mathbb{R}\backslash\{0\}$, *then*

$$\lim_{\lambda\to 0^+}\left(u_\lambda(x) + \frac{c(\lambda)}{\phi(\lambda)}\right) = u^\gamma(x) + \gamma \lim_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right)$$

locally uniformly in Ω.

Even though the second normalization (5.1.7) remains uniformly bounded when $\gamma = \pm\infty$, it is rather surprising that we have a divergent result in this convex setting. Using tools from weak KAM theory, we can construct an example where divergence happens when approximating from the inside. To our knowledge, this kind of example is new in the literature.

Theorem 5.1.8. *There exists a Hamiltonian where given any* $r(\lambda) \le 0$ *we can construct* $\phi(\lambda)$ *such that along a subsequence* $\lambda_j \to 0^+$ *we have* (5.1.7) *diverges, i.e.,*

$$\lim_{j\to\infty}\frac{r(\lambda_j)}{\phi(\lambda_j)} = -\infty \qquad \textit{and} \qquad \left\{u_\lambda + \frac{c(\lambda_j)}{\phi(\lambda_j)}\right\}_{j\in\mathbb{N}} \quad \textit{is divergent}.$$

5.2 Preliminaries on vanishing discount and duality representation

We refer the readers to Section 2.8 for some preliminaries on state-constraint solutions. For $\delta \ge 0$, we consider the problem

$$\delta u(x) + H(x, Du(x)) = 0 \qquad \text{in } \Omega. \tag{HJ}$$

and the state-constraint problem

$$\begin{cases} \delta v(x) + H(x, Dv(x)) \le 0 & \text{in } \Omega, \\ \delta v(x) + H(x, Dv(x)) \ge 0 & \text{on } \overline{\Omega}, \end{cases} \tag{HJ$_\delta$}$$

5.2.1 Duality representation of solutions

The duality representation is well-known in the literature (see [64] or [111, Theorem 5.3]). We present here a variation of that result. For $\delta > 0$, let u_δ be the unique solution to (HJ$_\delta$), and we have the following bound:

$$\delta|u_\delta(x)| + |Du_\delta(x)| \le C_H \qquad \text{for all } x \in \Omega. \tag{5.2.1}$$

That means the value of $H(x,p)$ for large $|p|$ is irrelevant, therefore without loss of generality we can assume that there exists $h > 0$ such that, the Legendre transform L of H will satisfy:

$$\begin{cases} H(x,p) = \sup\limits_{|v|\le h}\Big(p\cdot v - L(x,v)\Big), & (x,p) \in \overline{\Omega}\times\mathbb{R}^n \\ L(x,v) = \sup\limits_{p\in\mathbb{R}^n}\Big(p\cdot v - H(x,p)\Big), & (x,v) \in \overline{\Omega}\times\overline{B}_h. \end{cases} \tag{5.2.2}$$

This simplification allows us to work with the compact subset $\overline{\Omega} \times \overline{B}_h$ rather than $\overline{\Omega} \times \mathbb{R}^n$, as will be utilized to obtain the duality representation. Let us define for each $f \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h)$ the function

$$H_f(x,p) = \max_{|v|\le h} \Big(p\cdot v - f(x,v) \Big), \qquad (x,p) \in \overline{\Omega} \times \overline{B}_h.$$

Recall the definition of the action $\langle \cdot, \cdot \rangle$. The underlying domain of the integral will be implicitly understood. Let $\mathcal{R}(\overline{\Omega} \times \overline{B}_h)$ be the space of Radon measures on $\overline{\Omega} \times \overline{B}_h$. For $\delta > 0$, $z \in \overline{\Omega}$ we define

$$\begin{aligned}
\mathcal{F}_{\delta,\Omega} &= \Big\{ (f,u) \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h) \times \mathrm{C}(\overline{\Omega}) : \delta u + H_f(x, Du) \le 0 \text{ in } \Omega \Big\} \\
\mathcal{G}_{z,\delta,\Omega} &= \Big\{ f - \delta u(z) : (f,u) \in \mathcal{F}_{\delta,\Omega} \Big\} \\
\mathcal{G}'_{z,\delta,\Omega} &= \Big\{ \mu \in \mathcal{R}(\overline{\Omega} \times \overline{B}_h) : \langle \mu, f \rangle \ge 0 \text{ for all } f \in \mathcal{G}_{z,\delta,\Omega} \Big\}.
\end{aligned}$$

Here $\mathcal{G}_{z,\delta,\Omega} \subset \mathrm{C}(\overline{\Omega} \times \overline{B}_h)$ is the evaluation cone of $\mathcal{F}_{\delta,\Omega}$, and its dual cone consists of Radon measures with non-negative actions against elements in $\mathcal{G}_{z,\delta,\Omega}$. Note that $\mathcal{R}(\overline{\Omega} \times \overline{B}_h)$ is the dual space of $\mathrm{C}(\overline{\Omega} \times \overline{B}_h)$. We also denote by $\mathcal{P}$ the set of probability measures on $\overline{\Omega} \times \overline{B}_h$.

Lemma 5.2.1. *$\mathcal{F}_{\delta,\Omega}$ is a convex set, $\mathcal{G}_{z,\delta,\Omega}$ is a convex cone with vertex at the origin, and $\mathcal{G}'_{z,\delta,\Omega}$ consists of only non-negative measures.*

Theorem 5.2.2. *For $(z,\delta) \in \overline{\Omega} \times (0,\infty)$ and u is the viscosity solution to (HJ$_\delta$), we have*

$$\delta u(z) = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{z,\delta,\Omega}} \langle \mu, L \rangle = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{z,\delta,\Omega}} \int_{\overline{\Omega} \times \overline{B}_h} L(x,v)\, d\mu(x,v).$$

As $\delta \to 0^+$, we also have a representation for the erogdic problem (S_0) in the same manner. Let us define

$$\begin{aligned}
\mathcal{F}_{0,\Omega} &= \Big\{ (f,u) \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h) \times \mathrm{C}(\overline{\Omega}) : H_f(x, Du(x)) \le 0 \text{ in } \Omega \Big\} \\
\mathcal{G}_{0,\Omega} &= \Big\{ f : (f,u) \in \mathcal{F}_{0,\Omega} \text{ for some } u \in \mathrm{C}(\overline{\Omega}) \Big\} \\
\mathcal{G}'_{0,\Omega} &= \Big\{ \mu \in \mathcal{R}(\overline{\Omega} \times \overline{B}_h) : \langle \mu, f \rangle \ge 0 \text{ for all } f \in \mathcal{G}_{0,\Omega} \Big\}.
\end{aligned}$$

Here the notion of viscosity subsolution is equivalent to a.e. subsolution in Ω. We also have $\mathcal{G}_{0,\Omega} \subset \mathrm{C}(\overline{\Omega} \times \overline{B}_h)$ is the evaluation cone of $\mathcal{F}_{0,\Omega}$ and $\mathcal{G}'_{0,\Omega}$ is the dual cone of $\mathcal{G}_{0,\Omega}$ in $\mathcal{R}(\overline{\Omega} \times \overline{B}_h)$.

Definition 16. *A measure μ defined on $\overline{\Omega} \times \overline{B}_h$ is called a* holonomic measures *if*

$$\langle \mu, v \cdot D\psi(x) \rangle = 0 \qquad \textit{for all } \psi \in \mathrm{C}^1(\overline{\Omega}).$$

Lemma 5.2.3. *Measures in $\mathcal{G}'_{0,\Omega}$ are holonomic.*

Proof. If $\psi \in \mathrm{C}^1(\Omega)$ then $\pm(v \cdot D\psi(x), \psi) \in \mathcal{F}_{0,\Omega}$, therefore $\pm v \cdot D\psi(x) \in \mathcal{G}_{0,\Omega}$ and thus $\langle \mu, v \cdot D\psi(x) \rangle = 0$. □

Lemma 5.2.4. *Fix $z \in \overline{\Omega}$ and $\delta_j \to 0$. Assume $u_j \in \mathcal{G}'_{z,\delta_j,\Omega}$ and $\mu_j \rightharpoonup \mu$ weakly in the sense of measures, then $\mu \in \mathcal{G}'_{0,\Omega}$.*

Lemma 5.2.5. *$\mathcal{F}_{0,\Omega}$ is a convex set, $\mathcal{G}_{0,\Omega}$ is a convex cone with vertex at the origin, and $\mathcal{G}'_{0,\Omega}$ consists of only nonnegative measures.*

Theorem 5.2.6. *We have*

$$-c(0) = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega}} \langle \mu, L \rangle = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega}} \int_{\overline{\Omega} \times \overline{B}_h} L(x,v)\, d\mu(x,v). \tag{5.2.3}$$

The set of all measures in $\mathcal{P} \cap \mathcal{G}'_{0,\Omega}$ that minimizing (5.2.3) is denoted $\mathcal{M}_0$. We call them viscosity Mather measures ([64]). We omit the proofs of Lemmas 5.2.1, 5.2.4, 5.2.5 and Theorems 5.2.2, 5.2.6 as they are slight modifications of those in the periodic setting, which we refer the interested readers to [64, 111].

5.2.2 Vanishing discount for fixed bounded domains

In this section we use the representation formulas in Theorem 5.2.2 and Theorem 5.2.6 to show the convergence of solution of (S_λ) to solution of (S_0). See also [64, 111] where the similar technique is used.

Theorem 5.2.7. *Assume $(\mathcal{H}_0)$ – $(\mathcal{H}_4)$ and $(\mathcal{A}_2)$. Let $u_\delta \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{Lip}(\Omega)$ be the unique solution to (HJ_δ). Then $\delta u_\delta(\cdot) \to -c(0)$ uniformly on $\overline{\Omega}$ as $\delta \to 0$. Indeed, there exists $C > 0$ depends on H and $\mathrm{diam}(\Omega)$ such that for all $x \in \overline{\Omega}$ there holds*

$$|\delta u_\delta(x) + c(0)| \leq C\delta. \tag{5.2.4}$$

Also, for each $x_0 \in \overline{\Omega}$ there exist a subsequence λ_j and $u \in \mathrm{C}(\overline{\Omega})$ solving (S_0) such that:

$$\begin{cases} u_{\delta_j}(x) - u_{\delta_j}(x_0) \to u(x) \\ u_{\delta_j}(x) + c(0)/\delta_j \to w(x) \end{cases}$$

uniformly on $\overline{\Omega}$ as $\delta_j \to 0$ and the difference between the two limits are $w(x) - u(x) = w(x_0)$.

Theorem 5.2.8. *Assume $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$ and $(\mathcal{A}_2)$. Let $u_\delta \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{Lip}(\Omega)$ be the unique solution to (HJ_δ). Then, $u_\delta + \delta^{-1}c(0) \to u^0$ uniformly on $\mathrm{C}(\overline{\Omega})$ as $\delta \to 0^+$ and u^0 solves (S_0). Furthermore, the limiting solution can be characterized as*

$$u^0 = \sup_{v \in \mathcal{E}} v \tag{5.2.5}$$

where $\mathcal{E}$ is the set of all subsolutions $v \in \mathrm{C}(\overline{\Omega})$ to $H(x, Dv(x)) \leq c(0)$ in Ω such that $\langle \mu, v \rangle \leq 0$ for all $\mu \in \mathcal{M}_0$, the set of all minimizing measures $\mu \in \mathcal{P} \cap \mathcal{G}'_0$ such that $-c(0) = \langle \mu, L \rangle$.

We provide the proof of Theorem 5.2.7 in the Section 5.6. The proof of Theorem 5.2.8 is omitted as it is a slight modification of the one in [111]. The characterization (5.2.5) also appears in [44, 69, 70, 71] under different settings.

5.2.3 Maximal subsolutions and the Aubry set

For any domain (with nice boundary) $\Omega \subset U$, we recall that the additive eigenvalue of H in Ω is defined as

$$c_\Omega = \inf \Big\{ c \in \mathbb{R} : H(x, Dv(x)) \leq c \text{ has a viscosity subsolution in } \Omega \Big\}.$$

We consider the following equation

$$H(x, Dv(x)) \leq c_\Omega \qquad \text{in } \Omega. \tag{S_Ω}$$

We note that viscosity subsolutions of (S_Ω) in U are Lipschitz, and therefore they are equivalent to a.e. subsolutions (see [14, 111]). Also it is clear that $c_\Omega \leq c_U$, where c_U is the additive eigenvalue of H in U.

Definition 17. *For a fixed $z \in \Omega$ as a vertex, we define*

$$S_\Omega(x,z) = \sup \Big\{ v(x) - v(z) : v \textit{ solves } (S_\Omega) \Big\}, \qquad x \in \Omega.$$

There is a unique (continuous) extension $S_\Omega : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, we call $x \mapsto S_\Omega(x,z)$ the maximal subsolution *to (S_Ω) with vertex z.*

Theorem 5.2.9.

(i) For each fixed $z \in \overline{\Omega}$ then $x \mapsto S_\Omega(x,z)$ solves

$$\begin{cases} H(x, Du(x)) \leq c_\Omega & \text{in } \Omega, \\ H(x, Du(x)) \geq c_\Omega & \text{on } \overline{\Omega} \backslash \{z\}. \end{cases} \tag{5.2.6}$$

(ii) We have the triangle inequality $S_\Omega(x,z) \leq S_\Omega(x,y) + S_\Omega(y,z)$ for all $x, y, z \in \overline{\Omega}$.

We call $S_\Omega : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ an intrinsic semi-distance on Ω (see also [8, 10, 65]).

Inequality (5.2.6) means $x \mapsto u(x)$ is a subsolution to $H(x, Du(x)) = c_\Omega$ in Ω, and $x \mapsto u(x)$ is a supersolution to $H(x, Du(x)) = c_\Omega$ in $\overline{\Omega} \backslash \{z\}$. We omit the proof of Theorem 5.2.9 as it is a simple variation of Perron's method.

Definition 18. *Let us define the ergodic problem in Ω as*

$$\begin{cases} H(x, Du(x)) \leq c_\Omega & \text{in } \Omega, \\ H(x, Du(x)) \geq c_\Omega & \text{on } \overline{\Omega}. \end{cases} \tag{E}$$

The Aubry set $\mathcal{A}$ in Ω is defined as

$$\mathcal{A}_\Omega = \Big\{ z \in \overline{\Omega} : x \mapsto S_\Omega(x,z) \textit{ is a solution to } (\mathrm{E}) \Big\}.$$

Theorem 5.2.10. *Assuming $H(x,p) = |p| - V(x)$ where $V \in \mathrm{C}(\overline{\Omega})$ is nonnegative.*

(i) The additive eigenvalue of H in Ω is $c_\Omega = -\min_{\overline{\Omega}} V$.

(ii) The Aubry set of H in Ω is $\mathcal{A}_\Omega = \left\{z \in \overline{\Omega} : V(z) = -c_\Omega = \min_{\overline{\Omega}} V\right\}$.

Definition 19. *He say $u \in \mathrm{C}(\overline{\Omega})$ is a* strict subsolution *to $H(x, Du(x)) = c_\Omega$ in $B(x_0, r) \subset \Omega$ if there exists some $\varepsilon > 0$ such that $H(x, p) \leq c_\Omega - \varepsilon$ for all $p \in D^+u(x)$ and $x \in B(x_0, r)$.*

Theorem 5.2.11. *Given $z \in \Omega$, then $z \notin \mathcal{A}_\Omega$ if and only if there is a subsolution of $H(x, Du(x)) \leq c_\Omega$ in Ω which is strict in some neighborhood of z.*

Theorem 5.2.12. *If $\mathcal{A}_U \subset\subset \Omega \subset U$ then the additive eigenvalue of H in Ω is $c_\Omega = c_U$.*

We give proofs for 5.2.10 and 5.2.12 in Section 5.6. A proof of Theorem 5.2.10 for the case $\Omega = \mathbb{R}^n$ can be found in [111]. Theorem 5.2.12 is taken from [71, Proposition 5.1]. Proof of Theorem 5.2.11 can be found in [71].

The maximal solution $S_\Omega(x, y)$ also has an optimal control formulation (minimal exists time) as follows (see [55, 71]).

Theorem 5.2.13 (Optimal control formula)**.** *Let us define for $\beta \geq \alpha \geq 0$ the following set:*

$$\mathcal{F}_\Omega(x, y; \alpha, \beta) = \left\{\xi \in \mathrm{AC}\left([0, T], \overline{\Omega}\right); T > 0, \xi(\alpha) = y, \xi(\beta) = x\right\}.$$

Then

$$S_\Omega(x, y) = \inf\left\{\int_0^T \left(c(0) + L(\xi(s), \dot{\xi}(s))\right) ds : \xi \in \mathcal{F}_\Omega(x, y; 0, T)\right\}.$$

5.2.4 The vanishing discount problem on changing domains

Let $\Omega_\lambda = (1 + r(\lambda))\Omega$. For each $\lambda \in (0, 1)$ let $u_\lambda \in \mathrm{BUC}(\overline{\Omega}_\lambda) \cap \mathrm{Lip}(\Omega_\lambda)$ be the unique viscosity state-constraint solutions to

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } \Omega_\lambda, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{5.2.7}$$

The additive eigenvalue $c(\lambda)$ of H in Ω_λ is the unique constant defined as in (E_λ).

By comparison principle, it is clear that if $r(\lambda) \geq 0$ then $c(0) \leq c(\lambda)$ and if $\lambda \mapsto r(\lambda)$ is increasing (decreasing) then $\lambda \mapsto c(\lambda)$ increasing (decreasing) as well.

Theorem 5.2.14. *Considering the problem (5.2.7) with ($\mathcal{H}_3$), ($\mathcal{H}_4$) and ($\mathcal{A}_1$).*

(i) We have the priori estimate $\phi(\lambda)|u_\lambda(x)| + |Du_\lambda(x)| \leq C_H$ for $x \in \Omega_\lambda$.

(ii) We have $\phi(\lambda)u_\lambda(\cdot) \to -c(0)$ locally uniformly as $\lambda \to 0^+$. Furthermore for all $x \in \overline{\Omega}_\lambda$ and $\lambda > 0$ we have

$$\begin{cases} |\phi(\lambda)u_\lambda(x) + c(0)| \leq C\left(\phi(\lambda) + |r(\lambda)|\right) \\ |\phi(\lambda)u_\lambda(x) + c(\lambda)| \leq C\phi(\lambda). \end{cases} \tag{5.2.8}$$

As a consequence, whenever $r(\lambda) \neq 0$ there holds

$$\left|\frac{c(\lambda) - c(0)}{r(\lambda)}\right| \leq C. \tag{5.2.9}$$

(iii) For $x_0 \in \overline{\Omega}$ there exists a subsequence λ_j and $u, w \in \mathrm{BUC}(\overline{\Omega}) \cap \mathrm{Lip}(\Omega)$ such that $u_{\lambda_j}(\cdot) - u_{\lambda_j}(x_0) \to u(\cdot)$ and $u_{\lambda_j}(\cdot) + \phi(\lambda_j)^{-1}c(0) \to w(\cdot)$ locally uniformly as $\lambda_j \to 0$ and u, w solve (S_0) with $w(x) - u(x) = w(x_0)$.

Proof of Theorem 5.2.14. The priori estimate is clear from the coercivity assumption $(\mathcal{H}_4)$. Fix $x_0 \in \Omega$, by the Arzelà–Ascoli theorem there exists a subsequence $\lambda_j \to 0^+$, $c \in \mathbb{R}$ and u defined in Ω such that $\phi(\lambda_j)u_{\lambda_j}(x_0) \to -c$ and $u_{\lambda_j}(\cdot) - u_\lambda(x_0) \to u(\cdot)$ locally uniformly as $\lambda_j \to 0^+$. The case for $w(\cdot)$ can be done in the same manner as well as the relation between u and w. It follows that $u \in \mathrm{BUC}(\overline{\Omega})$ and by stability of viscosity solution we have $H(x, Du(x)) = c$ in Ω. Since $u_\lambda(\cdot)$ is Lipschitz, we deduce also that $\phi(\lambda_j)u_{\lambda_j}(x) \to -c$ for any $x \in \overline{\Omega}$.

We show that $H(x, Du(x)) \geq c$ on $\overline{\Omega}$. Let $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $u - \varphi$ has a strict minimum over $\overline{\Omega}$ at $\tilde{x} \in \partial\Omega$, we aim to show that $H(\tilde{x}, D\varphi(\tilde{x})) \geq c$. Let us define

$$\tilde{u}_\lambda(x) = (1 + r(\lambda))^{-1} u_\lambda\left((1 + r(\lambda))x\right), \qquad x \in \overline{\Omega} \tag{5.2.10}$$

then

$$\begin{cases} \phi(\lambda)(1 + r(\lambda))\tilde{u}_\lambda(x) + H((1 + r(\lambda))x, D\tilde{u}_\lambda(x)) \leq 0 & \text{in } \Omega, \\ \phi(\lambda)(1 + r(\lambda))\tilde{u}_\lambda(x) + H((1 + r(\lambda))x, D\tilde{u}_\lambda(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{5.2.11}$$

Let us define

$$\varphi_\lambda(x) = (1 + |r(\lambda)|)\varphi\left(\frac{x}{1 + |r(\lambda)|}\right), \qquad x \in (1 + |r(\lambda)|)\overline{\Omega}.$$

Note that $D\varphi_\lambda(x) = D\varphi\left((1 + |r(\lambda)|)^{-1}x\right)$ for $x \in (1 + |r(\lambda)|\Omega$. We us define

$$\Phi^\lambda(x, y) = \varphi_\lambda(x) - \tilde{u}_\lambda(y) - \frac{|x - y|^2}{2r(\lambda)^2}, \qquad (x, y) \in (1 + |r(\lambda)|)\,\overline{\Omega} \times \overline{\Omega}.$$

Assume $\Phi^\lambda(x, y)$ has a maximum over $(1 + |r(\lambda)|)\,\overline{\Omega} \times \overline{\Omega}$ at (x_λ, y_λ). By definition we have $\Phi^\lambda(x_\lambda, y_\lambda) \geq \Phi^\lambda(y_\lambda, y_\lambda)$, therefore

$$\varphi_\lambda(x_\lambda) - \frac{|x_\lambda - y_\lambda|^2}{2r(\lambda)^2} \geq \varphi_\lambda(y_\lambda)$$

and thus

$$|x_\lambda - y_\lambda| \leq 2r(\lambda)\|\varphi\|^{1/2}_{L^\infty(\overline{\Omega})}.$$

From that we can assume that $(x_\lambda, y_\lambda) \to (\overline{x}, \overline{x})$ for some $\overline{x} \in \overline{\Omega}$ as $\lambda \to 0^+$, then

$$\limsup_{\lambda \to 0^+} \left(\frac{|x_\lambda - y_\lambda|^2}{2r(\lambda)^2}\right) \leq \limsup_{\lambda \to 0^+} \left(\varphi_\lambda(x_\lambda) - \varphi_\lambda(y_\lambda)\right) = 0. \tag{5.2.12}$$

In other words, $|x_\lambda - y_\lambda| = o\left(|r(\lambda)|\right)$. Now $\Phi^\lambda(x_\lambda, y_\lambda) \geq \Phi^\lambda(\tilde{x}, \tilde{x})$ gives us

$$\varphi_\lambda(x_\lambda) - \tilde{u}_\lambda(y_\lambda) - \frac{|x_\lambda - y_\lambda|^2}{2r(\lambda)^2} \geq \varphi_\lambda(\tilde{x}) - \tilde{u}_\lambda(\tilde{x}).$$

Take $\lambda \to 0^+$, by (5.2.12) we obtain that $u(\tilde{x}) - \varphi(\tilde{x}) \geq u(\overline{x}) - \varphi(\overline{x})$, which implies that $\tilde{x} = \overline{x}$ as $u - \varphi$ has a strict minimum over $\overline{\Omega}$. From $(\mathcal{A}_1)$ and $|x_\lambda - y_\lambda| = o(|r(\lambda)|)$, we deduce that $x_\lambda \in (1 + |r(\lambda)|)\Omega$. As $y \mapsto \Phi^\lambda(x_\lambda, y)$ has a max at y_λ, we deduce that

$$\tilde{u}_\lambda(y) - \left(- \frac{|x_\lambda - y|^2}{2r(\lambda)^2} \right)$$

has a minimum at y_λ, therefore as $\tilde{u}_\lambda$ is Lipschitz with constant C_H we deduce that

$$\left| \frac{x_\lambda - y_\lambda}{r(\lambda)^2} \right| \leq C_H, \tag{5.2.13}$$

and we can apply the supersolution test for (5.2.11) to obtain

$$\phi(\lambda)(1 + r(\lambda))\tilde{u}_\lambda(y_\lambda) + H\left((1 + r(\lambda))y_\lambda, \frac{x_\lambda - y_\lambda}{r(\lambda)^2} \right) \geq 0. \tag{5.2.14}$$

On the other hand, since $x_\lambda \in (1 + |r(\lambda)|)\Omega$ as an interior point and $x \mapsto \Phi^\lambda(x, y_\lambda)$ has a max at x_λ, we deduce that

$$D\varphi_\lambda(x_\lambda) = \frac{x_\lambda - y_\lambda}{r(\lambda)^2} \qquad \Longrightarrow \qquad D\varphi\left(\frac{x_\lambda}{1 + r(\lambda)} \right) = \frac{x_\lambda - y_\lambda}{r(\lambda)^2}. \tag{5.2.15}$$

From (5.2.10), (5.2.14) and (5.2.15) we obtain

$$\phi(\lambda)u_\lambda\left((1 + r(\lambda))y_\lambda\right) + H\left((1 + r(\lambda))y_\lambda, D\varphi_\lambda(x_\lambda)\right) \geq 0. \tag{5.2.16}$$

Recall that $\phi(\lambda_j)u_{\lambda_j}(x) \to -c$ uniformly as $\lambda_j \to 0^+$ for any $x \in \overline{\Omega}$, we observe that

$$\begin{aligned}
|\phi(\lambda)u_\lambda\left((1 + r(\lambda))y_\lambda\right) + c| &\leq \left|\phi(\lambda)u_\lambda(\tilde{x}) + c\right| + \phi(\lambda)\left|u_\lambda\left((1 + r(\lambda))y_\lambda\right) - u_\lambda(\tilde{x})\right| \\
&\leq \left|\phi(\lambda)u_\lambda(\tilde{x}) + c\right| + \phi(\lambda)C_H\left|(y_\lambda - \tilde{x}) + r(\lambda)y_\lambda\right| \\
&\leq \left|\phi(\lambda)u_\lambda(\tilde{x}) + c\right| + \phi(\lambda)C_H\left|y_\lambda - \tilde{x}\right| + C_H\phi(\lambda)|r(\lambda)|\mathrm{diam}\Omega.
\end{aligned}$$

Let $\lambda \to 0^+$ along λ_j we obtain

$$\lim_{\lambda_j \to 0^+} \phi(\lambda_j)u_{\lambda_j}\left(\left(1 + r(\lambda_j)\right) y_{\lambda_j} \right) = c. \tag{5.2.17}$$

From (5.2.13) and $(\mathcal{H}_3)$ we have (up to subsequences)

$$\lim_{\lambda_j \to 0^+} H\left(\left(1 + r(\lambda_j)\right) y_{\lambda_j}, D\varphi_{\lambda_j}(x_{\lambda_j}) \right) = H(\tilde{x}, D\varphi(\tilde{x})). \tag{5.2.18}$$

From (5.2.16), (5.2.17) and (5.2.18) we deduce that $H(\tilde{x}, D\varphi(\tilde{x})) \geq c$. The comparison principle for state-constraint problem gives us the uniqueness of c and furthermore that $c = c(0)$.

The estimate (5.2.8) can be established using comparison principle. We see that

$$u(x) - \phi(\lambda)^{-1}c(0) - C, u(x) - \phi(\lambda)^{-1}c(0) + C,$$

are subsolution and supersolution, respectively, to

$$\begin{cases} \phi(\lambda)w(x) + H(x, Dw(x)) \leq 0 & \text{in } \Omega, \\ \phi(\lambda)w(x) + H(x, Dw(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{5.2.19}$$

On the other hand, from (5.2.11), the priori estimate $|\phi(\lambda)u_\lambda| \leq C$ and $(\mathcal{H}_3)$ we have $\tilde{u}_\lambda(x) - C\phi(\lambda)^{-1}|r(\lambda)|$, $\tilde{u}_\lambda(x) + C\phi(\lambda)^{-1}|r(\lambda)|$ are subsolution and supersolution, respectively, to (5.2.19). Therefore by comparison principle for (5.2.19) we have

$$\begin{cases} u(x) - \phi(\lambda)^{-1}c(0) - C \leq \tilde{u}_\lambda(x) + C|r(\lambda)|\phi(\lambda)^{-1}c(0), \\ u(x) - \phi(\lambda)^{-1}c(0) + C \geq \tilde{u}_\lambda(x) - C|r(\lambda)|\phi(\lambda)^{-1}c(0). \end{cases}$$

Therefore $|\phi(\lambda)\tilde{u}_\lambda(x) + c(0)| \leq C\left(\phi(\lambda) + |r(\lambda)|\right)$. The other estimate in (5.2.8) is a direct consequence of (5.2.4). □

Remark 44. We note that $\tilde{u}_\lambda$ defined as in (5.2.10) is not necessarily close to u_λ. In fact, for $x \in \overline{\Omega}$ we have

$$\tilde{u}_\lambda(x) - u_\lambda(x) = \frac{u_\lambda((1+r(\lambda))x) - u_\lambda(x)}{1+r(\lambda)} - \frac{r(\lambda)}{1+r(\lambda)}\left(u_\lambda(x) + \frac{c(0)}{\phi(\lambda)}\right) + \frac{r(\lambda)c(0)}{\phi(\lambda)(1+r(\lambda))}.$$

Using (5.2.8) and u_λ is Lipschitz, we obtain that

$$|\tilde{u}_\lambda(x) - u_\lambda(x)| \leq 2C(|r(\lambda)||x|) + 2C|r(\lambda)|\left(1 + \left|\frac{r(\lambda)}{\varphi(\lambda)}\right|\right) + 2\left|\frac{r(\lambda)}{\phi(\lambda)}c(0)\right|. \tag{5.2.20}$$

Therefore $\tilde{u}_\lambda$ and u_λ are close if $\gamma = 0$, and $\left\{\tilde{u}_\lambda + \phi(\lambda)^{-1}c(0)\right\}_{\lambda>0}$ is uniformly bounded in $\lambda > 0$ if γ is finite (or more generally if $|r(\lambda)/\phi(\lambda)|$) is bounded, in which case

$$\lim_{\lambda\to 0^+}\left(\tilde{u}_\lambda(x) - u_\lambda(x)\right) = \gamma c(0). \tag{5.2.21}$$

As we are working with domains that are smaller or bigger than Ω, we introduce the scaling of measures for convenience.

Definition 20. *For a measure σ defined on $(1+r)\overline{\Omega} \times \overline{B}_h$, we define its scaling $\tilde{\sigma}$ as a measure on $\overline{\Omega} \times \overline{B}_h$ by*

$$\int_{\overline{\Omega}\times\overline{B}_h} f(x,v)\, d\tilde{\sigma}(x,v) = \int_{(1+r)\overline{\Omega}\times\overline{B}_h} f\left(\frac{x}{1+r}, v\right)\, d\sigma(x,v). \tag{5.2.22}$$

We introduce the following definition for simplicity, as we will deal with mainly approximation from the inside and outside of Ω.

Definition 21. *For $r(\lambda) \geq 0$, we define $\Omega_\lambda^\pm = (1 \pm r(\lambda))\Omega$. We denote by $c(\lambda)^\pm$ and $u_\lambda^\pm$, respectively, the additive eigenvalues of H in $(1 \pm r(\lambda))\Omega$ and the solutions to the discounted problem (HJ_δ) on $(1 \pm r(\lambda))\Omega$ with discount factor $\delta = \phi(\lambda)$. We let u_λ^- and u_λ^+ be solutions to*

$$\begin{cases} \phi(\lambda)v(x) + H(x, Dv(x)) \leq 0 & \text{in } \Omega_\lambda, \\ \phi(\lambda)v(x) + H(x, Dv(x)) \geq 0 & \text{on } \overline{\Omega}_\lambda; \end{cases}$$

with Ω_λ being replaced by $(1 - r(\lambda))\Omega$ and $(1 + r(\lambda))\Omega$, respectively.

5.3 The first normalization: convergence and a counter example

In view of Theorems 5.2.14, it is natural to ask the question if the convergence of $u_\lambda(x) - u_\lambda(x_0)$ holds for the whole sequence as $\lambda \to 0^+$. The two natural normalization one can study are

$$\left\{ u_\lambda(x) + \frac{c(0)}{\phi(\lambda)} \right\}_{\lambda>0} \qquad \text{and} \qquad \left\{ u_\lambda(x) + \frac{c(\lambda)}{\phi(\lambda)} \right\}_{\lambda>0}. \tag{5.3.1}$$

We observe that from Theorem 5.2.14 we have

$$\left| u_\lambda(x) + \frac{c(0)}{\phi(\lambda)} \right| \leq C\left(1 + \frac{r(\lambda)}{\phi(\lambda)}\right) \qquad \text{and} \qquad \left| u_\lambda(x) + \frac{c(\lambda)}{\phi(\lambda)} \right| \leq C. \tag{5.3.2}$$

We observe that $u_\lambda(x) + \phi(\lambda)^{-1}c(0)$ is bounded if γ defined in (5.1.5) is finite, or more generally if $|r(\lambda)| = \mathcal{O}(\phi(\lambda))$ as $\lambda \to 0^+$, while $u_\lambda(x) + \phi(\lambda)^{-1}c(\lambda)$ is bounded even if γ is infinite. The following example show a divergence for $u_\lambda(x) + \phi(\lambda)^{-1}c(0)$ when $\gamma = \infty$.

Example 5. *Let us consider $H(x,p) = |p| + x$, $\Omega = (-1,1)$, $\phi(\lambda) = \lambda$ and $r(\lambda) = \lambda^m$ for $\lambda > 0$. Using the optimal control formula we obtain*

$$u_\lambda(x) = \inf_{\alpha(\cdot)} \left(-\int_0^\infty e^{-\lambda s} y(s)\, ds \right) \qquad \textit{where} \qquad \begin{cases} \dot{y}(s) & = \alpha(s) \in [-1,1] \\ y(0) & = x. \end{cases}$$

Regarding Definition 21, we have $c(0) = 1$, $c(\lambda)^\pm = 1 \pm \lambda^m$ and

$$u_\lambda(x)^\pm + \frac{c(0)}{\lambda} = \frac{1-x}{\lambda} + \frac{e^{-\lambda(1\pm\lambda^m - x)} - 1}{\lambda^2} = \mp\lambda^{m-1} + \frac{(1-x\pm\lambda^m)^2}{2} + \mathcal{O}(\lambda)$$

as $\lambda \to 0^+$, which are convergent only if $m \geq 1$. On the other hand, we have

$$u_\lambda(x)^\pm + \frac{c(\lambda)^\pm}{\lambda} = \frac{(1-x\pm\lambda^m)^2}{2} + \mathcal{O}(\lambda)$$

as $\lambda \to 0^+$, which converge to the same limit for all $m \geq 0$. In this example the family $\{u_\lambda + \phi(\lambda)^{-1}c(\lambda)\}_{\lambda>0}$ still converges even if $\gamma = \infty$. However it is not true in general, as we will prove an example in Section 5.

We give a simple proof for the convergence of both families in (5.3.1) when $\gamma = 0$.

Proof of Theorem 5.1.1. Let $v_\lambda \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{Lip}(\Omega)$ solving

$$\begin{cases} \phi(\lambda)v_\lambda(x) + H(x, Dv_\lambda(x)) \leq 0 & \text{in } \Omega, \\ \phi(\lambda)v_\lambda(x) + H(x, Dv_\lambda(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{5.3.3}$$

By Theorem 5.2.8, there exists u^0 solves (S_0) such that $v_\lambda(x) + \phi(\lambda)^{-1}c(0) \to u^0(x)$ uniformly on $\overline{\Omega}$ as $\lambda \to 0^+$. Define $\tilde{u}_\lambda(x)$ as in (5.2.10) then $\tilde{u}_\lambda$ solves (5.2.11). Similarly to Theorem 5.2.14 we obtain that $\tilde{u}_\lambda(x) - C\phi(\lambda)^{-1}|r(\lambda)|, \tilde{u}_\lambda(x) + C\phi(\lambda)^{-1}|r(\lambda)|$ are subsolution and supersolution, respectively, to (5.3.3), therefore

$$\left| \left(\tilde{u}_\lambda(x) + \frac{c(0)}{\phi(\lambda)} \right) - \left(v_\lambda(x) + \frac{c(0)}{\phi(\lambda)} \right) \right| \leq C\frac{|r(\lambda)|}{\phi(\lambda)}.$$

Recall (5.2.20), as $\gamma = 0$ we have $|\tilde{u}_\lambda - u_\lambda| \to 0$ as $\lambda \to 0^+$, therefore we deduce that $u_\lambda(x) + \phi(\lambda)^{-1}c(0) \to u^0(x)$ locally uniformly as $\lambda \to 0^+$. From (5.2.9) in Theorem 5.2.14 and $\gamma = 0$ we obtain

$$\lim_{\lambda \to 0^+} \left(\frac{c(\lambda) - c(0)}{\phi(\lambda)} \right) = 0.$$

and thus $u_\lambda(x) + \phi(\lambda)^{-1}c(\lambda) \to u^0(x)$ locally uniformly as $\lambda \to 0^+$. □

Remark 45. If $\gamma \neq 0$ then in general the solution v_λ to (5.3.3) and the solution u_λ to (5.2.7) are not close to each other, as we will see in example 6.

Example 6. *Let $H(x,p) = |p| - e^{-|x|}$ on $\Omega = (-1,1)$ and $\phi(\lambda) = r(\lambda) = \lambda$. Using the optimal control formula, solutions to* (5.2.7) *(regarding Definition 21) are*

$$u_\lambda^-(x) = \frac{e^{-|x|}}{1+\lambda} + \frac{e^{-(1-\lambda^2)+\lambda|x|}}{\lambda(1+\lambda)}, \qquad x \in [-(1-\lambda),(1-\lambda)],$$
$$u_\lambda^+(x) = \frac{e^{-|x|}}{1+\lambda} + \frac{e^{-(1+\lambda)^2+\lambda|x|}}{\lambda(1+\lambda)}, \qquad x \in [-(1+\lambda),(1+\lambda)].$$

On fixed bounded domain, the solution v_λ to (5.3.3) *is given by*

$$v_\lambda(x) = \frac{e^{-|x|}}{1+\lambda} + \frac{e^{-1-\lambda+\lambda|x|}}{\lambda(1+\lambda)}, \qquad x \in [-1,1].$$

We have $c(0) = -e^{-1}$ and $c(\lambda)^\pm = -e^{-1\mp\lambda}$, thus $c_-^{(1)} = c_+^{(1)} = e^{-1}$ and

$$\lim_{\lambda \to 0^+} \left(\frac{c(\lambda)_- - c(0)}{-\lambda} \right) = \lim_{\lambda \to 0^+} \left(\frac{c(\lambda)_+ - c(0)}{\lambda} \right) = e^{-1}.$$

The maximal solution (in the sense of Theorem 5.2.8) on Ω is given by

$$u^0(x) = \lim_{\lambda \to 0^+} \left(v_\lambda(x) + \frac{c(0)}{\lambda} \right) = e^{-|x|} + e^{-1}|x| - e^{-1}, \qquad x \in [-1,1].$$

On the other hand, using notation as in Theorem 5.1.2 with $\gamma = 1$ we have

$$u^{-1} = \lim_{\lambda \to 0^+} \left(u_\lambda^-(x) + \frac{c(0)}{\lambda} \right) = e^{-|x|} + e^{-1}|x|, \qquad x \in [-1,1],$$
$$u^{+1} = \lim_{\lambda \to 0^+} \left(u_\lambda^+(x) + \frac{c(0)}{\lambda} \right) = e^{-|x|} + e^{-1}|x| - 2e^{-1}, \qquad x \in [-1,1]$$

and

$$\lim_{\lambda \to 0^+} \left(u_\lambda(x)^\pm + \frac{c(\lambda)_\pm}{\lambda} \right) = u^0(x), \qquad x \in [-1,1].$$

In this example $u^{+1}(\cdot) + u^{-1}(\cdot) = 2u^0(\cdot)$ and u_λ and v_λ are not close to each other.

Using the representation formula as in Theorem 5.2.8, we show the convergence of $\{u_\lambda + \phi(\lambda)^{-1}c(0)\}_{\lambda > 0}$ when γ is finite. This method also recovers the result of Theorem 5.1.1. The following technical lemma is a consequence from $(\mathcal{H}_8)$, we give a proof for it in Appendix.

Lemma 5.3.1. *Assume L satisfies $(\mathcal{H}_8)$ then*

$$\frac{L(x,v)-L((1\pm\delta)x,v)}{\delta}\to(\mp x)\cdot D_xL(x,v)\quad \textit{uniformly on } \overline{\Omega}\times\overline{B}_h \textit{ as } \delta\to 0^+$$

Proof of Theorem 5.1.2. By the reduction step earlier, we may assume that H satisfies (5.2.2) for some $h>0$ and $L\in \mathrm{C}(\overline{\Omega}\times\overline{B}_h)$. By (5.3.2) and $\gamma<\infty$ we have the boundedness of $\left\{u_\lambda(x)+\phi(\lambda)^{-1}c(0)\right\}_{\lambda>0}$.

Recall Remark 44, let $\tilde{u}_\lambda$ be defined as in (5.2.10) and $\tilde{\mathcal{U}}$ be the set of accumulation points of $\left\{\tilde{u}_\lambda+\phi(\lambda)^{-1}c(0)\right\}_{\lambda>0}$ in $\mathrm{C}(\overline{U})$ as $\lambda\to 0^+$. By Theorem 5.2.14 we have $\tilde{\mathcal{U}}$ is nonempty. To show that $\tilde{\mathcal{U}}$ is singleton, we show that if $u,w\in\tilde{\mathcal{U}}$ then $u\equiv w$.

Assume that there exist $\lambda_j\to 0$ and $\delta_j\to 0$ such that $\tilde{u}_{\lambda_j}+\phi(\lambda_j)^{-1}c(0)\to u$ and $\tilde{u}_{\delta_j}+\phi(\delta_j)^{-1}c(0)\to w$ locally uniformly as $j\to\infty$. Let us fix $z\in\Omega$, by Theorem 5.2.2 there exists $\mu_\lambda\in\mathcal{P}\cap\mathcal{G}'_{z,\phi(\lambda),\Omega_\lambda}$ such that

$$\phi(\lambda)u_\lambda(z)=\int_{\overline{\Omega}_\lambda\times\overline{B}_h}L(x,v)\,d\mu_\lambda(x,v)=\min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{z,\phi(\lambda),\Omega_\lambda}}\int_{\overline{\Omega}_\lambda\times\overline{B}_h}L(x,v)\,d\mu(x,v). \tag{5.3.4}$$

Let $\tilde{\mu}_\lambda$ be the measure obtained from μ_λ defined as in Definition 20, it is clear that $\tilde{\mu}_\lambda$ is a probability measures on $\overline{\Omega}$, therefore the set

$$\mathcal{U}_*(z)=\left\{\mu\in\mathcal{P}(\overline{\Omega}\times\overline{B}_h):\tilde{\mu}_\lambda\rightharpoonup\mu \text{ in measure along some subsequences}\right\} \tag{5.3.5}$$

is nonempty. By Lemma 5.3.4 we can assume that (up to subsequence) there exists $\mu_0\in\mathcal{M}_0$ such that $\tilde{\mu}_\lambda\rightharpoonup\mu_0$ in measure. We have $H(x,Dw(x))\le c(0)$ in Ω, let $w_\lambda(x)=(1+r(\lambda))w\left((1+r(\lambda))^{-1}x\right)$ in $x\in(1+r(\lambda))\overline{\Omega}$ then $w_\lambda(x)\to w(x)$ pointwise s $\lambda\to 0^+$ and

$$\phi(\lambda)w_\lambda(x)+H_{L\left(\frac{x}{1+r(\lambda)},v\right)+\phi(\lambda)w_\lambda(x)+c(0)}(x,Dw_\lambda(x))\le 0\qquad \text{in } (1+r(\lambda))\Omega.$$

By definition we obtain

$$\left(L\left(\frac{x}{1+r(\lambda)},v\right)+\phi(\lambda)w_\lambda(x)+c(0),w_\lambda(x)\right)\in\mathcal{F}_{\phi(\lambda),\Omega_\lambda}$$

and therefore

$$\left\langle\mu_\lambda,L\left(\frac{x}{1+r(\lambda)},v\right)+\phi(\lambda)w_\lambda(x)-\phi(\lambda)w_\lambda(z)+c(0)\right\rangle\ge 0.$$

In other words, we have

$$\left\langle\mu_\lambda,L\left(\frac{x}{1+r(\lambda)},v\right)\right\rangle+\phi(\lambda)(1+r(\lambda))\left\langle\mu_\lambda,w\left(\frac{x}{1+r(\lambda)}\right)\right\rangle+c(0)\ge\phi(\lambda)w_\lambda(z).$$

Combine with $-\langle\mu_\lambda,L(x,v)\rangle+\phi(\lambda)u_\lambda(z)=0$ from (5.3.4) we obtain

$$\begin{aligned}\left\langle\mu_\lambda,L\left(\frac{x}{1+r(\lambda)},v\right)-L(x,v)\right\rangle&+\phi(\lambda)(1+r(\lambda))\langle\tilde{\mu}_\lambda,w(x)\rangle\\&+\phi(\lambda)u_\lambda(z)+c(0)\ge\phi(\lambda)w_\lambda(z).\end{aligned}$$

Dividing both sides by $\phi(\lambda)$ we deduce that

$$\frac{r(\lambda)}{\phi(\lambda)}\left\langle \tilde{\mu}_\lambda, \frac{L(x,v)-L((1+r(\lambda))x,v)}{r(\lambda)}\right\rangle + (1+r(\lambda))\langle\tilde{\mu}_\lambda, w\rangle + \left(u_\lambda(z)+\frac{c(0)}{\phi(\lambda)}\right) \geq w_\lambda(z).$$

Since $\tilde{\mu}_{\lambda_j} \rightharpoonup \mu_0$ in measure, using Lemma 5.3.1 we deduce that

$$\gamma\langle\mu_0, (-x)\cdot D_xL(x,v)\rangle + \langle\mu_0, w\rangle + \big(u(z)-\gamma c(0)\big) \geq w(z), \tag{5.3.6}$$

where $u_\lambda(z)+\phi(\lambda_j)^{-1}c(0) \to (u(z)-\gamma c(0))$ comes from $\tilde{u}_\lambda(z)+\phi(\lambda_j)^{-1}c(0) \to u(z)$ and (5.2.21) in Remark 44. On the other hand, from (5.2.11) we have

$$\phi(\lambda)\tilde{u}_\lambda(x) + H\big((1+r(\lambda))x, D\tilde{u}_\lambda(x)\big) \leq 0 \qquad \text{in } \Omega$$

In other words, we have

$$L\big((1+r(\lambda))x,v\big) - \phi(\lambda)(1+r(\lambda))\tilde{u}_\lambda(x) \in \mathcal{F}_{0,\Omega}$$

and thus

$$\big\langle \mu, L\big((1+r(\lambda))x,v\big) - \phi(\lambda)(1+r(\lambda))\tilde{u}_\lambda(x)\big\rangle \geq 0 \qquad \text{for all } \mu \in \mathcal{M}_0.$$

Recall that $-\langle\mu, L(x,v)\rangle = c(0)$ for all $\mu\in\mathcal{M}_0$, we have

$$\frac{r(\lambda)}{\phi(\lambda)}\left\langle \mu, \frac{L\big((1+r(\lambda))x,v\big)-L(x,v)}{r(\lambda)}\right\rangle \geq (1+r(\lambda))\left\langle \mu, \tilde{u}_\lambda(x)+\frac{c(0)}{\phi(\lambda)}\right\rangle - \frac{r(\lambda)}{\phi(\lambda)}c(0)$$

for all $\mu\in\mathcal{M}_0$. Let $\lambda=\delta_j$ then as $j\to\infty$ we have $\gamma\langle\mu, x\cdot D_xL(x,v)\rangle \geq \langle\mu, w\rangle - \gamma c(0)$, i.e.,

$$\gamma\langle\mu, (-x)\cdot D_xL(x,v)\rangle + \langle\mu, w\rangle - \gamma c(0) \leq 0, \qquad \text{for all } \mu\in\mathcal{M}_0. \tag{5.3.7}$$

From (5.3.6) and (5.3.7) we deduce that $u(z)\geq w(z)$. Since $z\in\Omega$ arbitrarily we have $u\geq w$ and similarly $u\leq w$, thus $u\equiv w$ and we have the uniform convergence for the full sequence

$$\lim_{\lambda\to 0}\left(\tilde{u}_\lambda(x)+\frac{c(0)}{\phi(\lambda)}\right)$$

Denote this limit as $\tilde{u}^\gamma$, then from Remark 44 we have

$$u_\lambda + \phi(\lambda)^{-1}c(0) \to u^\gamma = \tilde{u}^\gamma - \gamma c(0)$$

locally uniformly in Ω as $\lambda\to 0^+$. Clearly $u^\gamma\in\mathcal{E}^\gamma$ thanks to (5.3.7). If $v\in\mathcal{E}^\gamma$ then since $\mu_0\in\mathcal{M}_0$, we can establish (5.3.6) with w being replaced by v to obtain $u^\gamma\geq v$, hence $u^\gamma = \sup\mathcal{E}^\gamma$. □

Corollary 5.3.2. *For any $\mu\in\mathcal{U}_*(z)$ there holds $\gamma\big\langle\mu, (-x)\cdot D_xL(x,v)\big\rangle + \big\langle\mu, u^\gamma\big\rangle = 0$.*

Lemma 5.3.3. *For any $\mu\in\mathcal{M}_0$ there holds $\langle\mu, (-x)\cdot D_xL(x,v)\rangle \geq 0$.*

Proof of Lemma 5.3.3. For $\mu \in \mathcal{M}_0$ and $0 < \lambda \ll 1$ we define μ_λ by

$$\langle \mu_\lambda, f(x,v)\rangle := \langle \mu, f((1-\lambda)x, v)\rangle, \qquad \text{for } f \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h).$$

It is easy to see that μ_λ is a probability measure on $\overline{\Omega} \times \overline{B}_h$. Furthermore $\mu_\lambda \in \mathcal{G}'_{0,\Omega}$ as well. In fact, if $f \in \mathcal{G}_{0,\Omega}$ then there exists $u \in \mathrm{C}(\overline{\Omega})$ such that $H_f(x, Du(x)) \leq 0$ in Ω. It is clear that $H_{f((1-\lambda)x,v)}(x, D\tilde{u}(x)) \leq 0$ in Ω as well where $\tilde{u}(x) = (1-\lambda)^{-1}u((1-\lambda)x)$, therefore $f((1-\lambda)x, v) \in \mathcal{G}_{0,\Omega}$, hence

$$\langle \mu_\lambda, f(x,v)\rangle = \langle \mu, f((1-\lambda)x, v)\rangle \geq 0.$$

As $\mu_\lambda \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega}$, we deduce that

$$\langle \mu, L((1-\lambda)x, v)\rangle = \langle \mu_\lambda, L(x,v)\rangle \geq \langle \mu, L(x,v)\rangle.$$

Let $\lambda \to 0^+$ we deduce that $\langle \mu, (-x)\cdot D_x L(x,v)\rangle \geq 0$. □

Remark 46. If the domain is periodic then by translation invariant, we can get an invariant for Mather measures as

$$\langle \mu, (-x)\cdot D_x L(x,v)\rangle = 0 \qquad \text{for all } \mu \in \mathcal{M}_0.$$

We refer the reader's to [16] for more properties like this in the case of periodic domain. In our setting, it is natural to expect a similar invariant holds. Indeed, it is interesting that

$$\begin{cases} \langle \mu, (-x)\cdot D_x L(x,v)\rangle = \text{constant} \\ \text{for all } \mu \in \mathcal{M}_0 \end{cases} \qquad \Longleftrightarrow \qquad \lambda \mapsto c(\lambda) \text{ is differentiable at } \lambda = 0$$

and if that is the case, the constant in the above is $c'(0)$, the derivative of the map $\lambda \to c(\lambda)$ at $\lambda = 0$. For instance, in Example 6 we have $c'(0) = e^{-1}$.

Lemma 5.3.4. *We have $\mathcal{U}_*(z) \subseteq \mathcal{M}_0$ where $\mathcal{U}_*(z)$ is defined as in (5.3.5).*

Proof. Assume $\tilde{\mu}_{\lambda_j} \rightharpoonup \mu_0$. From (5.3.4) it is clear that $-c(0) = \langle \mu_0, L\rangle$. For $f \in \mathcal{G}_{0,\Omega}$ there exists $u \in \mathrm{C}(\overline{\Omega})$ such that $H_f(x, Du(x)) \leq 0$ in Ω. Let us define $\tilde{u}$ as in (5.2.10), we have

$$\phi(\lambda)\tilde{u}(x) + H_{f\left(\frac{x}{1+r(\lambda)}, v\right) + \phi(\lambda)\tilde{u}(x)}(x, D\tilde{u}(x)(x)) \leq 0 \quad \text{in } (1+r(\lambda))\Omega.$$

By definition we deduce that

$$\left\langle \tilde{\mu}_\lambda, f(x,v) + \phi(\lambda)(1+r(\lambda))\left(u - u(z)\right)\right\rangle = \left\langle \mu_\lambda, f\left(\frac{x}{1+r(\lambda)}, v\right) + \phi(\lambda)\left(\tilde{u} - \tilde{u}(z)\right)\right\rangle \geq 0$$

Let $\lambda \to 0^+$ along λ_j we deduce that $\langle \mu_0, f\rangle \geq 0$, hence $\mu_0 \in \mathcal{M}_0$. □

Proof of Corollary 5.1.3. From (5.3.7) we have

$$\begin{cases} \alpha\langle \mu, (-x)\cdot D_x L(x,v)\rangle + \langle \mu, u^\alpha\rangle \leq 0 \\ \beta\langle \mu, (-x)\cdot D_x L(x,v)\rangle + \langle \mu, u^\beta\rangle \leq 0 \end{cases}$$

for all $\mu \in \mathcal{M}_0$. If $\theta = \beta - \alpha \geq 0$ then

$$\theta \langle \mu, (-x) \cdot D_x L(x,v) \rangle + \alpha \langle \mu, (-x) \cdot D_x L(x,v) \rangle + \left\langle \mu, u^\beta \right\rangle \leq 0$$

for all $\mu \in \mathcal{M}_0$. Since $\theta \langle \mu, (-x) \cdot D_x L(x,v) \rangle \geq 0$, we have $u^\beta \in \mathcal{E}^\alpha$, therefore $u^\beta \leq u^\alpha$. Denote $\gamma = (1-\lambda)\alpha + \lambda\beta$ for $\lambda \in (0,1)$, we have

$$\gamma \langle \mu, (-x) \cdot D_x L(x,v) \rangle + \left\langle \mu, (1-\lambda)u^\alpha + \lambda u^\beta \right\rangle \leq 0, \qquad \text{for all } \mu \in \mathcal{M}_0.$$

By the convexity of H we see that $u = (1-\lambda)u^\alpha + \lambda u^\beta$ belongs to $\mathcal{E}^{(1-\lambda)\alpha+\lambda\beta}$, therefore $(1-\lambda)u^\alpha + \lambda u^\beta \leq u^{(1-\lambda)\alpha+\lambda\beta}$. □

5.4 The asymptotic expansion of the eigenvalue

5.4.1 The expansion at zero

In this section, we want to study the asymptotic expansion of $c(\lambda)$ as $\lambda \to 0^+$. If the following limit exist

$$c^{(1)} = \lim_{\lambda \to 0^+} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) \tag{5.4.1}$$

then heuristically we have $c(\lambda) = c(0) + c^{(1)} r(\lambda) + o(r(\lambda))$ as $\lambda \to 0^+$.

Remark 47. The dependence of λ and the eigenvalue should be $c(r(\lambda))$ but in fact $c^{(1)}$ is independent of $r(\lambda)$ if it exists. Indeed, assume $\lambda \mapsto c(\lambda) = c(r(\lambda))$ is differentiable at $\lambda = 0$, for any $\mu \in \mathcal{M}_0$ by scaling into a measure μ_λ on $(1 + r(\lambda))\Omega$, we can show that $-c(r(\lambda)) \leq \langle \mu_\lambda, L \rangle$, hence

$$\left\langle \mu, L((1 + r(\lambda))x, v) \right\rangle + c(r(\lambda)) \geq 0$$

for all λ with an equality at $\lambda = r(\lambda) = 0$, therefore

$$c^{(1)} = c'(0) = \langle \mu, (-x) \cdot D_x L \rangle.$$

Thus we can simply write $c(\lambda)$ for simplicity and we can simply choose $r(\lambda) = \pm\lambda$.

We will show that (5.4.1) holds when $\lambda \mapsto r(\lambda)$ does not change its sign around 0, provided that $(\mathcal{H}_3)$ is satisfied. Example 7 shows that it can be divergent if $(\mathcal{H}_3)$ is violated and Example 6 shows that in general it is not zero.

Example 7. *Let $H(x,p) = |p| - \sqrt{1-|x|}$ for $(x,p) \in [-1,1] \times \mathbb{R}$. Let $r(\lambda) = \lambda \in (0,1)$, the $c(\lambda) = -\sqrt{\lambda}$ and the limit* (5.4.1) *does not exist.*

Let ν_λ be measures in $\mathcal{P} \cap \mathcal{G}'_{0,\Omega_\lambda}$ such that

$$-c(\lambda) = \int_{\overline{\Omega}_\lambda \times \overline{B}_h} L(x,v) \; d\nu_\lambda(x,v) = \min_{\nu \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega_\lambda}} \int_{\overline{\Omega}_\lambda \times \overline{B}_h} L(x,v) \; d\nu(x,v). \tag{5.4.2}$$

Let $\tilde{\nu}_\lambda$ be the corresponding measures on $\overline{\Omega}$ after scaling from ν_λ as in Definitions 20, it is easy to see that $\tilde{\nu}_\lambda$ is still a probability measure on $\overline{\Omega}$, thus by compactness the set of weak limit points $\mathcal{V}_*$ of $\{\tilde{\nu}_\lambda\}_{\lambda>0}$ is nonempty.

Lemma 5.4.1. *We have $\mathcal{V}_* \subseteq \mathcal{M}_0$.*

Proof of Lemma 5.4.1. From (5.4.2) we have

$$-c(\lambda) = \int_{\overline{\Omega}\times\overline{B}_h} L\big(\,(1+r(\lambda))\,x,v\big)\,d\tilde{\nu}_\lambda(x,v).$$

Assume $\tilde{\nu}_{\lambda_j} \rightharpoonup \nu_0$, then since $L\big(\,(1+r(\lambda))\,x,v\big) \to L(x,v)$ uniformly as $\lambda \to 0^+$, we deduce that $-c(0) = \langle \nu_0, L\rangle$. Let $f \in \mathcal{G}_{0,\Omega}$, there exists $u \in \mathrm{C}(\overline{\Omega})$ such that $H_f(x, Du(x)) \le 0$ in Ω. Let us define $\tilde{u}$ as in (5.2.10), then

$$H_{f_\lambda}(x, D\tilde{u}(x)) \le 0 \quad \text{in } \Omega_\lambda$$

where $f_\lambda(x,v) = f\left(\frac{x}{1+r(\lambda)}, v\right)$. By definition of ν_λ we have

$$\langle \tilde{\nu}_\lambda, f(x,v)\rangle = \langle \nu_\lambda, f_\lambda(x,v)\rangle \ge 0.$$

Let $\lambda_j \to 0^+$ we deduce that $\langle \nu_0, f\rangle \ge 0$, thus $\nu_0 \in \mathcal{M}_0$. □

Proof of Theorem 5.1.4. Let us consider the case $r(\lambda) \ge 0$. Let w_λ be a solution to

$$\begin{cases} H(x, Dw_\lambda(x)) \le c(\lambda) & \text{in } \Omega_\lambda, \\ H(x, Dw_\lambda(x)) \ge c(\lambda) & \text{on } \overline{\Omega}_\lambda, \end{cases}$$

and $\tilde{w}_\lambda$ be its scaling as in (5.2.10), then

$$H_{L((1+r(\lambda))x,v)+c(\lambda)}\big(x, D\tilde{w}_\lambda(x)\big) \le 0 \qquad \text{in } \Omega.$$

Therefore $L((1+r(\lambda))x,v) + c(\lambda) \in \mathcal{F}_{0,\Omega}$, thus

$$\Big\langle \mu, L((1+r(\lambda))x,v) + c(\lambda)\Big\rangle \ge 0 \tag{5.4.3}$$

for any $\mu \in \mathcal{M}_0$. Using the fact that $-\langle \mu, L(x,v)\rangle = c_0$ we deduce that

$$\Big\langle \mu, L((1+r(\lambda))x,v) - L(x,v)\Big\rangle + (c(\lambda) - c(0)) \ge 0.$$

Thus if $r(\lambda) > 0$ then for all $\mu \in \mathcal{M}_0$ we have

$$\left\langle \mu, \frac{L((1+r(\lambda))x,v) - L(x,v)}{r(\lambda)}\right\rangle + \left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right) \ge 0.$$

Using (5.2.9) and the fact that $r(\lambda)$ is not identically zero near 0, as $\lambda \to 0^+$ we have

$$\langle \mu, x\cdot D_x L(x,v)\rangle + \liminf_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right) \ge 0 \qquad \text{for all } \mu \in \mathcal{M}_0. \tag{5.4.4}$$

Let $\lambda_j \to 0^+$ be the subsequence such that

$$\limsup_{\lambda\to 0^+}\left(\frac{c(\lambda)-c(0)}{r(\lambda)}\right) = \lim_{j\to\infty}\left(\frac{c(\lambda_j)-c(0)}{r(\lambda_j)}\right).$$

For simplicity we can assume that (up to subsequence) $\tilde{\nu}_{\lambda_j} \rightharpoonup \nu_0$ and $\nu_0 \in \mathcal{M}_0$. Let w be a solution to (S_0), then $\tilde{w}(x) = (1+r(\lambda))w\left((1+r(\lambda))^{-1}x\right)$ solves

$$H_{L\left(\frac{x}{1+r(\lambda)},v\right)+c(0)}(x, D\tilde{w}(x)) \le 0 \qquad \text{in } (1+r(\lambda))\Omega.$$

As $v_\lambda \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega_\lambda}$ and $\langle \nu_\lambda, L \rangle = -c(\lambda)$, we obtain that

$$\left\langle \nu_\lambda, L\left(\frac{x}{1+r(\lambda)}, v\right) - L(x,v) \right\rangle - c(\lambda) + c(0) \ge 0.$$

By definition of $\tilde{\nu}_\lambda$, it is equivalent to

$$\left\langle \tilde{\nu}_\lambda, L(x,v) - L((1+r(\lambda))x, v) \right\rangle \ge c(\lambda) - c(0). \tag{5.4.5}$$

As $r(\lambda_j) \ge 0$, let $\lambda_j \to 0^+$ we obtain

$$\langle \nu_0, (-x) \cdot D_x L(x,v) \rangle \ge \limsup_{\lambda \to 0^+} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right). \tag{5.4.6}$$

In (5.4.4), take $\mu = \nu_0 \in \mathcal{M}_0$ and together with (5.4.6) we conclude that

$$\lim_{\lambda \to 0^+} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) = \langle \nu_0, (-x) \cdot D_x L(x,v) \rangle = \sup_{\mu \in \mathcal{M}_0} \langle \mu, (-x) \cdot D_x L(x,v) \rangle.$$

Similarly, if $r(\lambda) \le 0$ as $\lambda \to 0^+$ then

$$\lim_{\lambda \to 0^+} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) = \min_{\mu \in \mathcal{M}_0} \langle \mu, (-x) \cdot D_x L(x,v) \rangle.$$

For an oscillating $r(\lambda)$ such that neither $r^-(\lambda) = \min\{0, r(\lambda)\}$ nor $r^+(\lambda) = \max\{0, r(\lambda)\}$ is identical to zero as $\lambda \to 0^+$, by applying the previous results we have (we consider the limit $(c(\lambda) - c(0))/r(\lambda)$ along subsequences where $r(\lambda) \neq 0$)

$$\lim_{\substack{\lambda \to 0^+ \\ r(\lambda) > 0}} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) = \max_{\mu \in \mathcal{M}_0} \langle \mu, (-x) \cdot D_x L(x,v) \rangle = c_+^{(1)},$$
$$\lim_{\substack{\lambda \to 0^+ \\ r(\lambda) < 0}} \left(\frac{c(\lambda) - c(0)}{r(\lambda)} \right) = \min_{\mu \in \mathcal{M}_0} \langle \mu, (-x) \cdot D_x L(x,v) \rangle = c_-^{(1)}.$$

For any given subsequence $\lambda_j \to 0^+$ along which $(c(\lambda_j) - c(0))/r(\lambda_j)$ converges, by decomposing λ_j into subsequences where $r(\lambda_j) > 0$ and $r(\lambda_j) < 0$ respectively, we see that $(c(\lambda_j) - c(0))/r(\lambda_j)$ can only converge either to $c_+^{(1)}$ or $c_-^{(1)}$, and therefore we obtain the conclusion of the theorem. $\square$

Proof of Corollary 5.1.5. If $\langle \mu, (-x) \cdot D_x L(x,v) \rangle = c^{(1)}$ for all $\mu \in \mathcal{M}_0$ then from Theorem 5.1.2 with $\gamma \in \mathbb{R}$ we have $\left\langle \mu, u^\gamma + \gamma c^{(1)} \right\rangle \le 0$ for all $\mu \in \mathcal{M}_0$, thus $u^\gamma + \gamma c^{(1)} \in \mathcal{E}$ and hence $u^\gamma + \gamma c_{(1)} \le u^0$. On the other hand, $\langle \mu, u^0 \rangle = \gamma c^{(1)} + \left\langle \mu, u^0 - \gamma c^{(1)} \right\rangle \le 0$ for all $\mu \in \mathcal{M}_0$, therefore

$$\gamma \langle \mu, (-x) \cdot D_x L(x,v) \rangle + \left\langle \mu, u^0 - \gamma c^{(1)} \right\rangle \le 0 \qquad \text{for all } \mu \in \mathcal{M}_0.$$

Thus $u^0 - \gamma c^{(1)} \in \mathcal{E}^\gamma$, hence $u^0 - \gamma c^{(1)} \le u^\gamma$. □

Remark 48. Here are some examples where $c_-^{(1)} = c_+^{(1)} = c^{(1)}$.

(i) If $H(x,p) = H(p) + V(x)$ with $x \cdot \nabla V(x) \le 0$ for all $x \in \Omega$. Indeed, Lemma 5.3.3 says that $\langle \mu, x \cdot \nabla V(x) \rangle \ge 0$ for all $\mu \in \mathcal{M}_0$, thus in this case we have $\langle \mu, (-x) \cdot D_x L(x,v) \rangle = 0$ for all $\mu \in \mathcal{M}_0$, hence $c_-^{(1)} = c_+^{(1)} = 0$.

(ii) If the Aubry set $\mathcal{A}$ of H is compactly supported in Ω, then by Theorem 5.2.12 we have $c(\lambda) = c(0)$ for all $\lambda > 0$ small enough, therefore $c^{(1)} = 0$.

(iii) Recall from Example 6 that if $H(x,p) = |p| - e^{-|x|}$ and $\Omega = (-1,1)$ then $c_+^{(1)} = c_-^{(1)} = c^{(1)} = e^{-1}$.

We state the following lemma concerning properties of limits of minimizing measures on Ω, which will be used to prove Corollary 5.1.6.

Lemma 5.4.2. *Let $v_\lambda \in \mathrm{C}(\overline{\Omega})$ be the solution to (5.3.3). For $z \in \Omega$, let $\sigma_\lambda \in \mathcal{P} \cap \mathcal{G}'_{z,\phi(\lambda),\Omega}$ be the minimizing measure such that $\phi(\lambda) v_\lambda(z) = \langle \sigma_\lambda, L \rangle$. Let us define*

$$\mathcal{U}_0(z) = \left\{ \mu \in \mathcal{P}(\overline{\Omega} \times \overline{B}_h) : \sigma_\lambda \rightharpoonup \mu \text{ in measures along some subsequences} \right\}$$

then

(i) $\langle \sigma_0, u^0 \rangle = 0$ for all $\sigma_0 \in \mathcal{U}_0(z)$.

(ii) $u^0(z) \ge u^\gamma(z) + \gamma \langle \sigma_0, (-x) \cdot D_x L(x,v) \rangle$ for all $\sigma_0 \in \mathcal{U}_0(z)$ and $\gamma \in \mathbb{R}$.

Proof of Lemma 5.4.2. It is clear that $\mathcal{U}_0(z) \subset \mathcal{M}_0$.

(i) For any w solves (S_0) we have

$$\left\langle \sigma_\lambda, L(x,v) + c(0) + \phi(\lambda) w(x) - \phi(\lambda) w(z) \right\rangle \ge 0. \tag{5.4.7}$$

Since $v_\lambda + \varphi(\lambda)^{-1} c(0) \to u^0$ by Theorem 5.2.8, from (5.4.7) we deduce that

$$u^0(z) + \langle \sigma_0, w \rangle \ge w(z) \tag{5.4.8}$$

for some $\sigma_0 \in \mathcal{U}_0(z)$. Let $w = u^0$ we obtain $\langle \sigma_0, u^0 \rangle \ge 0$, thus $\langle \sigma_0, u^0 \rangle = 0$ since $\langle \mu, u^0 \rangle \le 0$ for all $\mu \in \mathcal{M}_0$.

(ii) To connect u^0 with u^γ, we use the approximation u_λ on $(1+r(\lambda))\Omega$. Recall that after scaling $\tilde{u}_\lambda(x) = (1+r(\lambda))^{-1}u_\lambda\big((1+r(\lambda))x\big)$ for $x \in \overline{\Omega}$ we have

$$L((1+r(\lambda))x, v) - \phi(\lambda)r(\lambda)\tilde{u}_\lambda(x) \in \mathcal{F}_{z,\phi(\lambda),\Omega}.$$

Recall the definition of $\sigma_\lambda \in \mathcal{P} \cap \mathcal{G}'_{z,\phi(\lambda),\Omega}$ from Lemma 5.4.2, we have

$$\big\langle \sigma_\lambda, L((1+r(\lambda))x, v) - \phi(\lambda)r(\lambda)\tilde{u}_\lambda - \phi(\lambda)\tilde{u}_\lambda(z) \big\rangle \geq 0.$$

Using $-\langle \sigma_\lambda, L(x,v)\rangle + \phi(\lambda)v_\lambda(z) = 0$ where v_λ solves (5.3.3) we obtain

$$\frac{r(\lambda)}{\phi(\lambda)} \left\langle \sigma_\lambda, \frac{L((1+r(\lambda))x, v) - L(x,v)}{r(\lambda)} \right\rangle + v_\lambda(z) - r(\lambda)\langle \sigma_\lambda, \tilde{u}_\lambda\rangle \geq \tilde{u}_\lambda(z).$$

Taking into account the normalization, we deduce that

$$\begin{aligned}\frac{r(\lambda)}{\phi(\lambda)} \left\langle \sigma_\lambda, \frac{L((1+r(\lambda))x, v) - L(x,v)}{r(\lambda)} \right\rangle + \left(v_\lambda(z) + \frac{c(0)}{\phi(\lambda)} \right) \\ - r(\lambda) \left\langle \sigma_\lambda, \tilde{u}_\lambda + \frac{c(0)}{\phi(\lambda)} \right\rangle \geq \left(\tilde{u}_\lambda(z) + \frac{c(0)}{\phi(\lambda)} \right) - \frac{r(\lambda)}{\phi(\lambda)} c(0).\end{aligned}$$

Assume $\sigma_\lambda \rightharpoonup \sigma_0$ for some $\sigma_0 \in \mathcal{U}_0(z)$, then as $\lambda \to 0$ we have

$$\gamma\langle \sigma_0, (+x)\cdot D_x L(x,v)\rangle + u^0(z) \geq u^\gamma(z)$$

and thus the conclusion $u^0(z) \geq u^\gamma(z) + \gamma\,\langle \sigma_0, (-x)\cdot D_x L(x,v)\rangle$ follows.

□

Proof of Corollary 5.1.6. If $\gamma > 0$ then from Lemma 5.4.2 there exists $\sigma_0 \in \mathcal{U}_0(z)$ such that

$$0 = u^0(z) - u^\gamma(z) \geq \gamma\langle \sigma_0, (-x)\cdot D_x L(x,v)\rangle \geq \gamma c_-^{(1)} \geq 0$$

and thus $c_-^{(1)} = 0$. □

5.4.2 The additive eigenvalues as a function

A natural question that comes from Theorem 5.1.4 is when do we have the invariant

$$\langle \mu, (-x)\cdot D_x L(x,v)\rangle = c^{(1)}$$

for all $\mu \in \mathcal{M}_0$? In other words, when is the map $\lambda \mapsto c(\lambda)$ is differentiable at $\lambda = 0$? We can indeed study the map $\lambda \mapsto c(\lambda)$ on an open interval I including zero, and ask the question at what point λ where $c'(\lambda)$ exists. It is clear that $\lambda \mapsto c(\lambda)$ is Lipschitz, thus it is differentiable almost everywhere. We will show a stronger claim that indeed the set of points where $c'(\lambda)$ does not exists is almost countable. Without loss of generality (from Theorem 5.1.4) We can assume $r(\lambda) = \lambda$ for $\lambda \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ in this section.

Theorem 5.4.3. *Assume* $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$, $(\mathcal{H}_8)$ *and* $(\mathcal{A}_1)$ *and* $\lambda \in (-\varepsilon, \varepsilon)$.

(a) *The map $\lambda \mapsto c(\lambda)$ is left-differentiable and right-differentiable everywhere on its domain.*

(b) *The left derivative $\lambda \mapsto c'_-(\lambda)$ is left continuous and the right derivative $\lambda \mapsto c'_+(\lambda)$ is right continuous on their domains.*

(c) *The map $\lambda \mapsto c(\lambda)$ is differentiable except countably many points on its domain.*

Proof. For (a), by the same argument as in the proof of Theorem 5.1.4 we see that $\lambda \mapsto c(\lambda)$ is left and right differentiable with

$$c'_+(\lambda) = \max_{\mu \in \mathcal{M}_\lambda} \int_{\overline{\Omega}_\lambda \times \overline{B}_h} (-x) \cdot D_x L(x,v) d\mu(x,v)$$
$$c'_-(\lambda) = \min_{\mu \in \mathcal{M}_\lambda} \int_{\overline{\Omega}_\lambda \times \overline{B}_h} (-x) \cdot D_x L(x,v) d\mu(x,v)$$

where $\mathcal{M}_\lambda$ is the set of minimizing Mather measures on Ω_λ.

For $\lambda \in (-\varepsilon, \varepsilon)$ and ν_λ be any measure in $\mathcal{M}_\lambda$ then by the usual scaling as in Lemma 5.4.1 we have $-c(\lambda) = \langle \tilde{\nu}_\lambda, L((1+\lambda)x, v) \rangle$ and any subsequential weak limit $\tilde{\nu}_\lambda \rightharpoonup \nu_0$ in $\mathcal{P}(\overline{\Omega} \times \overline{B}_h)$ satisfies $\nu_0 \in \mathcal{M}_0$. We claim further that

$$\int_{\overline{\Omega} \times \overline{B}_h} (-x) \cdot D_x L(x,v) d\nu_0(x,v) = \begin{cases} c'_-(0) & \text{if } \lambda \to 0^-, \\ c'_+(0) & \text{if } \lambda \to 0^+. \end{cases}$$

It is rather clear from (5.4.3) and (5.4.5), since, for instance if $\lambda \to 0^-$ then

$$\left\langle \mu, \frac{L((1+\lambda)x,v) - L(x,v)}{\lambda} \right\rangle + \frac{c(\lambda) - c(0)}{\lambda} \leq 0 \qquad \text{for all } \mu \in \mathcal{M}_0$$
$$\left\langle \tilde{\nu}_\lambda, \frac{L(x,v) - L((1+\lambda)x,v)}{\lambda} \right\rangle \leq \frac{c(\lambda) - c(0)}{\lambda}.$$

Therefore, together with Theorem 5.1.4 we deduce that

$$c'_-(0) \leq \big\langle \mu, (-x) \cdot D_x L(x,v) \big\rangle \qquad \text{for all } \mu \in \mathcal{M}_0$$
$$\big\langle \nu_0, (-x) \cdot D_x L(x,v) \big\rangle \leq c'_-(0).$$

We conclude that

$$\big\langle \nu_0, (-x) \cdot D_x L(x,v) \big\rangle = c'_-(0).$$

Now (b) follows easily. To see that $\lambda \mapsto c'_-(\lambda)$ is left continuous, it suffices to show it is left continuous at 0. If $\lambda \to 0^-$, let $\nu_\lambda \in \mathcal{M}_\lambda$ that realizes $c'_-(\lambda)$, i.e.,

$$c'_-(\lambda) = \int_{\overline{\Omega}_\lambda \times \overline{B}_h} (-x) \cdot D_x L(x,v) d\nu_\lambda(x,v) = (1+\lambda) \int_{\overline{\Omega} \times \overline{B}_h} (-x) \cdot D_x L\big((1+\lambda)x, v\big) d\tilde{\nu}_\lambda(x,v).$$

From $(\mathcal{H}_8)$ we have that $(-x) \cdot D_x L((1+\lambda)x, v) \to (-x) \cdot D_x L(x,v)$ uniformly on $\overline{\Omega} \times \overline{B}_h$, and since the limit of the right hand side is $c'_-(0)$ independent of subsequence, we deduce that

$$\lim_{\lambda \to 0^-} c'_-(\lambda) = c'_-(0).$$

The case $\lambda \to 0^+$ can be done in the same manner. Finally the fact that $\lambda \mapsto c(\lambda)$ is differentiable except countably many points is standard, since $\lambda \mapsto c'(\lambda)$ is defined almost everywhere and is non-decreasing, or one can argue as in [62, Theorem 17.9] or [18, Theorem 4.2]. $\square$

With some additional information about the Hamiltonian, we can say something more about the map $\lambda \mapsto c(\lambda)$.

Lemma 5.4.4. *Assume* $(\mathcal{H}_0)$ – $(\mathcal{H}_5)$, $(\mathcal{H}_8)$ *and* $(\mathcal{A}_1)$ *and further that* $(x,p) \mapsto H(x,p)$ *is jointly convex, then* $\lambda \mapsto c(\lambda)$ *is convex.*

We omit the proof of this lemma as it is a simple modification of Corollary 5.1.3.

5.5 The second normalization: convergence and a counter example

From Theorems 5.1.2 and 5.1.4 we obtain the convergence of the second normalization (5.1.7) when γ is finite as in Corollary 5.1.7. In this section we provide an example where given any $r(\lambda)$, we can construct $\phi(\lambda)$ such that $\gamma = \infty$ and $\left\{u_\lambda + \phi(\lambda)^{-1}c(\lambda)\right\}_{\lambda>0}$ is divergent along some subsequence. To simplify notations, we will consider $r(\lambda) \geq 0$ and denote $c(\lambda)$ to be the eigenvalue of H in $\Omega_\lambda = (1 - r(\lambda))\Omega$. Let us consider the following Hamiltonian

$$H(x,p) = |p| - V(x), \qquad (x,p) \in \overline{\Omega} \times \mathbb{R}^n, \tag{5.5.1}$$

where $V : \overline{\Omega} \to \mathbb{R}$ is uniformly bounded continuous and is nonnegative. For a given $r(\lambda)$, we will construct $\phi(\lambda)$ so that $\left\{u_\lambda + \phi(\lambda)^{-1}c(\lambda)\right\}_{\lambda>0}$ is divergent as $\lambda \to 0^+$. The example is constructed based on an instability of the Aubry set $\mathcal{A}_{\Omega_\lambda}$ of H on Ω_λ, when $\lambda \to 0^+$. We recall from Theorem 5.2.10 that

$$-c(0) = \min_{\overline{\Omega}} V \qquad \text{and} \qquad \mathcal{A}_\Omega = \left\{ x \in \overline{\Omega} : V(x) = \min_{\overline{\Omega}} V \right\}.$$

Also, the Lagrangian is nonnegative in this case, since

$$L(x,v) = \begin{cases} V(x) & \text{if } |v| \leq 1, \\ +\infty & \text{if } |v| > 1. \end{cases} \tag{5.5.2}$$

Lemma 5.5.1. *Assume* $(\mathcal{H}_3)$, $(\mathcal{H}_4)$, $(\mathcal{H}_5)$ *and* $(\mathcal{A}_2)$. *Let*

$$S_\Omega(x,y) = \sup \left\{ u(x) - u(y) : u \text{ is a subsolution } H(x, Du(x)) \leq c(0) \text{ in } \Omega \right\}.$$

We can extend S_Ω *uniquely to* $\overline{\Omega} \times \overline{\Omega}$. *If* $\mathcal{A}_\Omega = \{z_0\}$ *is a singleton then* $u^0(x) \equiv S_\Omega(x, z_0)$ *where* u^0 *is the maximal solution on* Ω *defined in Theorem 5.2.8.*

Proof of Lemma 5.5.1. One can show that $\mathcal{A}_\Omega$ is a uniqueness set for (S_0) (see [55, 71, 72, 93]). From Lemma 5.4.2 there exists $\sigma_0 \in \mathcal{M}_0$ such that $\langle \sigma_0, u^0 \rangle = 0$. If $\mathcal{A}_\Omega = \{z_0\}$ then we can show that supp $(\sigma_0) \subset \{z_0\}$ and thus $\sigma_0 \equiv \delta_{z_0}$, hence

$$u^0(z_0) = \langle \sigma_0, u^0 \rangle = 0 = S_\Omega(z_0, z_0).$$

Therefore $u^0(x) \equiv S_\Omega(x, z_0)$. $\square$

Definition 22 (Definition of the potential $V(x)$)**.** *We will construct a potential V to use for the proof of Theorem 5.1.8 on $\Omega = (-1, 1)$. We start with the first step, the building block will be as follows.*

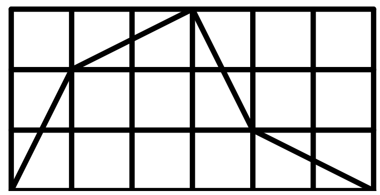

Figure 5.1: The first step.

Next, we apply the same construction but with a smaller scale, which gives us Figure 5.2.

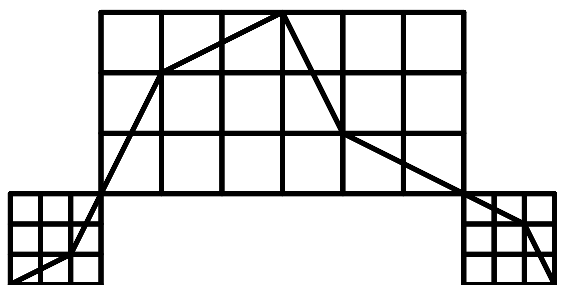

Figure 5.2: The second step.

Keep switching the small box with this construction and with an appropriate initial length to start with, the graph of V is given as in Figure 5.3.

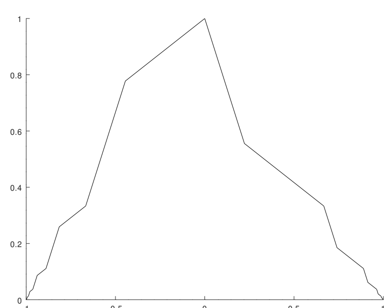

Figure 5.3: Graph of the function V.

Lemma 5.5.2. *Let $V(x)$ defined as in Definition 22 and $\Omega_\lambda = (-1 + r(\lambda), 1 - r(\lambda))$. Then the maximal solution on Ω_λ (as in Theorem 5.2.8), denoted by $u^0_\lambda(x)$, does not converge as $\lambda \to 0^+$.*

Proof of Lemma 5.5.2. By Theorem 5.2.10 the additive eigenvalue of H on Ω_λ, denoted by $c(\lambda)$, is given by $-c(\lambda) = \min_{x \in \overline{\Omega}_\lambda} V(x)$. By the construction of V, there are exactly two points, denoted by z_λ^+ and z_λ^- such that

$$\left\{z \in \overline{\Omega}_\lambda : V(z) = \min_{x \in \overline{\Omega}_\lambda} V(x) = -c(\lambda)\right\} = \{z_\lambda^+, z_\lambda^-\}. \tag{5.5.3}$$

We can find two subsequence of $\lambda_j \to 0^+$ and $\delta_j \to 0^+$ such that $\lim_{\lambda_j \to 0^+} z_{\lambda_j} = -1$ and $\lim_{\delta_j \to 0^+} z_{\delta_j} = 1$. We claim that

$$\lim_{\lambda_j \to 0^+} u^0_{\lambda_j}(x) = S_\Omega(x, -1) \qquad \text{and} \qquad \lim_{\delta_j \to 0^+} u^0_{\delta_j}(x) = S_\Omega(x, 1). \tag{5.5.4}$$

For those z_λ satisfying (5.5.3) we have $u^0_\lambda(x) \equiv S_{\Omega_\lambda}(x, z_\lambda)$ for $x \in \overline{\Omega}_\lambda$. We show that

$$\lim_{\lambda_j \to 0} S_{\Omega_{\lambda_j}}\left(x, z_{\lambda_j}\right) = S_\Omega(x, z_0) \qquad \text{for } x \in \Omega \tag{5.5.5}$$

where $z_0 = -1$. The other case is similar. If $x \in \Omega$ then for all λ small enough we have $x \in \Omega_\lambda$, by Theorem 5.2.13 we have

$$S_{\Omega_\lambda}(x, z_\lambda) = \inf\left\{\int_0^T \Big(c(\lambda) + L(\xi(s), \dot{\xi}(s))\Big) ds : \xi \in \mathrm{AC}\left([0,T]; \overline{\Omega}_\lambda\right), \xi(0) = z_\lambda, \xi(T) = x\right\},$$
$$S_\Omega(x, z_\lambda) = \inf\left\{\int_0^T \Big(c(0) + L(\xi(s), \dot{\xi}(s))\Big) ds : \xi \in \mathrm{AC}\left([0,T]; \overline{\Omega}\right), \xi(0) = z_\lambda, \xi(T) = x\right\}.$$

We show that $S_{\Omega_\lambda}(x, z_\lambda) \le S_\Omega(x, z_0)$. Take any $\xi \in \mathcal{F}_\Omega(x, z_0; 0, T)$ (defined in Theorem 5.2.13) and define $t_\lambda = \inf\left\{s > 0 : \xi(s) = z_\lambda\right\} \in (0, T)$, then

$$\eta(s) = \begin{cases} z_\lambda & s \in [0, t_\lambda], \\ \xi(s) & s \in [t_\lambda, T], \end{cases}$$

belongs to $\mathcal{F}_{\Omega_\lambda}(x, z_\lambda; 0, T)$, therefore together with (5.5.2) we have

$$\begin{aligned}
\int_0^T \Big(c(0) + L(\xi(s), \dot{\xi}(s))\Big) ds &= \int_0^{t_\lambda} \Big(c(0) + L(\xi(s), \dot{\xi}(s))\Big) ds + \int_0^T \Big(c(0) + L(\eta(s), \dot{\eta}(s))\Big) ds \\
&\ge \int_0^{t_\lambda} \Big(c(0) + L(\xi(s), \dot{\xi}(s))\Big) ds + \int_0^T \Big(c(\lambda) + L(\eta(s), \dot{\eta}(s))\Big) ds \\
&\ge \max\left\{S_\Omega(z_\lambda, z_0), 0\right\} + S_{\Omega_\lambda}(x, z_\lambda).
\end{aligned}$$

Therefore taking the infimum over all possible ξ we deduce that

$$S_\Omega(x, z_0) \ge \max\left\{S_\Omega(z_\lambda, z_0), 0\right\} + S_{\Omega_\lambda}(x, z_\lambda)$$

and thus

$$\limsup_{\lambda \to 0^+} S_{\Omega_\lambda}(x, z_\lambda) \le S_\Omega(x, z_0). \tag{5.5.6}$$

Now let us start with $\xi_n \in \mathcal{F}_{\Omega_\lambda}(x, z_\lambda; 0, T_n)$ such that

$$\int_0^{T_n} \Big(L(\xi(s), \dot{\xi}(s)) + c(\lambda)\Big) ds < S_{\Omega_\lambda}(x, z_\lambda) + \frac{1}{n}. \tag{5.5.7}$$

Let us connect z_0 and z_λ by the straight line $\zeta(s) = (1-s)z_0 + sz_\lambda$ for $s \in [0,1]$. Since $|\dot{\zeta}(s)| = |z_0 - z_\lambda| \ll 1$, therefore from (5.5.2) we have

$$\int_0^1 L\left(\zeta(s), \dot{\zeta}(s)\right)\, ds = \int_0^1 V(\zeta(s))\, ds \leq \max_{x\in[z_0,z_\lambda]} V(x) = -c(\lambda).$$

Therefore

$$\int_0^1 \Big(L(\zeta(s),\dot{\zeta}(s)) + c(\lambda)\Big) ds \leq 0. \tag{5.5.8}$$

Let us define

$$\eta_n(s) = \begin{cases} \zeta(s) & \text{for } s \in [0,1], \\ \xi(s-1) & \text{for } s \in [1, T_n+1] \end{cases}$$

then $\eta_n \in \mathcal{F}_\Omega(x, z_0; 0, T_{n+1})$. From (5.5.7) and (5.5.8) we have

$$\begin{aligned} S_{\Omega_\lambda}(x, z_\lambda) + \frac{1}{n} &> \int_0^1 \Big(L(\zeta(s),\dot{\zeta}(s)) + c(\lambda)\Big) ds + \int_0^{T_n} \Big(L(\xi(s),\dot{\xi}(s)) + c(\lambda)\Big) ds \\ &= \int_0^{T_n+1} \Big(L(\eta_n(s),\dot{\eta}_n(s)) + c(\lambda)\Big) ds \geq S_\Omega(x, z_0) \end{aligned}$$

since $\eta_n \in \mathcal{F}_\Omega(x, z_0; 0, T_{n+1})$. Let $\lambda \to 0^+$ and then $n \to \infty$ we have

$$\liminf_{\lambda\to 0^+} S_{\Omega_\lambda}(x, z_\lambda) \geq S_\Omega(x, z_0). \tag{5.5.9}$$

From (5.5.6) and (5.5.9) we obtain (5.5.5) and (5.5.4) follows. We finally observe that $S_\Omega(x,-1) \neq S_\Omega(x,1)$, since otherwise $S_\Omega(-1,1) = S_\Omega(1,-1) = 0$, which is impossible. Indeed, if $\xi \in \mathcal{F}_\Omega(1,-1;0,T)$ then, as (5.5.2) implies that $|\dot{\xi}(s)| \leq 1$ a.e., we deduce from (5.5.2) that

$$\int_0^T L(\xi(s),\dot{\xi}(s)) ds = \int_0^T V(\xi(s)) ds \geq \int_0^T V(\xi(s))\dot{\xi}(s) ds = \int_{-1}^1 V(x)\, dx = \|V\|_{L^1(\Omega)} > 0.$$

Therefore $S_\Omega(-1,1) > 0$. □

Proof of Theorem 5.1.8. Let H be defined as in (5.5.1), we consider the following discounted problems:

$$\begin{cases} \delta u_\delta(x) + H(x, Du_\delta(x)) \leq 0 & \text{in } \Omega_\lambda, \\ \delta u_\delta(x) + H(x, Du_\delta(x)) \geq 0 & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{5.5.10}$$

Let $c(\lambda)$ be the eigenvalue of H over Ω_λ. By Theorem 5.2.8 we know that

$$\lim_{\delta\to 0^+} \left(u_\delta(x) + \frac{c(\lambda)}{\delta}\right) \to u^0_\lambda(x)$$

uniformly on $\overline{\Omega}$, where $u^0_\lambda(x)$ is a maximal solution on Ω_λ. For each $\lambda > 0$, we can find $\tau(\lambda) > 0$ such that

$$\sup_{x\in\overline{\Omega}_\lambda} \left| \left(u_\delta(x) + \frac{c(\lambda)}{\delta}\right) - u^0_\lambda(x) \right| \leq r(\lambda) \qquad \text{for all } \delta \leq \tau(\lambda). \tag{5.5.11}$$

Set $\phi(\lambda) = \tau(\lambda)r(\lambda)^2$, then $\phi(\lambda) \to 0$ as $\lambda \to 0^+$ and $\gamma = \infty$. The function $\phi(\lambda)$ can be modified to be decreasing. Now by (5.5.11) and Lemma 5.5.2, along two subsequences λ_j and δ_j we have

$$\lim_{\lambda_j \to 0^+} \left(u_{\lambda_j}(x) + \frac{c(\lambda_j)}{\phi(\lambda_j)} \right) = S_\Omega(x, -1) \neq S_\Omega(x, 1) = \lim_{\delta_j \to 0^+} \left(u_{\delta_j}(x) + \frac{c(\delta_j)}{\phi(\delta_j)} \right).$$

Thus we have the divergence of $\left\{ u_\lambda + \phi(\lambda)^{-1} c(\lambda) \right\}_{\lambda > 0}$ in this case. □

Remark 49. Note that the parametrization here u_λ means $u_{\phi(\lambda)}$, which is the same as in the original definition (5.2.7). In (5.2.7) we should have used $u_{\phi(\lambda)}$ instead of u_λ but we simplify the notation for clarity.

5.6 Proof of vanishing discount on fixed bounded domains

In this section we provide proofs to all results on the vanishing discount on fixed bounded domains.

Proof of Theorem 5.2.7. By the priori estimate $\delta|u_\delta(x)| + |Du_\delta(x)| \leq C_H$ for $x \in \overline{\Omega}$. Fix $x_0 \in \overline{\Omega}$, then by Aezelà-Ascoli theorem there exists a subequence δ_j and $u \in \mathrm{C}(\overline{\Omega})$ such that $u_{\delta_j}(\cdot) - u_{\delta_j}(x_0) \to u(\cdot)$ uniformly on $\overline{\Omega}$ for some $u \in \mathrm{C}(\overline{\Omega})$. By Bolzano-Weiertrass theorem there exists $c \in \mathbb{R}$ such that (up to subsequence) $\delta_j u_{\delta_j}(x_0) \to -c$.

By stability of viscosity solution we have $H(x, Du(x)) = c$ in Ω. We will show $H(x, Du(x)) \geq c$ on $\overline{\Omega}$. Let $\tilde{x} \in \partial\Omega$ and $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $u - \varphi$ has a strict minimum over $\overline{\Omega}$ at $\tilde{x}$, we show that $H(\tilde{x}, D\varphi(\tilde{x})) \geq c$. Without loss of generality we can assume that $(u - \varphi)(x) \geq (u - \varphi)(\tilde{x}) = 0$ for $x \in \overline{\Omega}$.

Define $\varphi_\delta(x) = (1+\delta)\varphi\left(\frac{x}{1+\delta}\right)$ for $x \in (1+\delta)\overline{\Omega}$. Let us define

$$\Phi(x, y) = \varphi_\delta(x) - u_\delta(y) - \frac{|x-y|^2}{2\delta^2}, \qquad (x, y) \in (1+\delta)\overline{\Omega} \times \overline{\Omega}.$$

Assume Φ has maximum over $(1+\delta)\overline{\Omega} \times \overline{\Omega}$ at (x_δ, y_δ). As $\Phi(x_\delta, y_\delta) \geq \Phi(y_\delta, y_\delta)$, we obtain $|x_\delta - y_\delta| \leq C\delta$. By compactness we deduce that $(x_\delta, y_\delta) \to (\overline{x}, \overline{x})$ for $\overline{x} \in \overline{\Omega}$ as $\delta \to 0^+$. We deduce further that

$$\limsup_{\delta \to 0} \frac{|x_\delta - y_\delta|^2}{2\delta^2} \leq \limsup_{\delta \to 0} \Big(\varphi(x_\delta) - \varphi(y_\delta) \Big) = 0 \quad \Longrightarrow \quad |x_\delta - y_\delta| = o(\delta).$$

Also $\Phi(x_\delta, y_\delta) \geq \Phi(\tilde{x}, \tilde{x})$, let $\delta \to 0$ we have $\Phi(\overline{x}, \overline{x}) \geq \Phi(\tilde{x}, \tilde{x})$ which implies that $\overline{x} = \tilde{x}$. By (A2) we deduce that $x_\delta \in (1+\delta)\Omega$. Now by supersolution test as $y \mapsto \Phi^\delta(x_\delta, y)$ has a max at y_δ, we obtain

$$\delta u_\delta(y_\delta) + H\Big(y_\delta, \delta^{-2}(x_\delta - y_\delta)\Big) \geq 0.$$

As $x \mapsto \Phi(x, y_\delta)$ has a max at $x_\delta \in (1+\delta)\Omega$ as an interior point of $(1+\delta)\Omega$, we deduce that $D\varphi_\delta(x_\delta) = \delta^{-2}(x_\delta - y_\delta)$. Therefore

$$\delta u_\delta(y_\delta) + H\Big(y_\delta, D\varphi_\delta(x_\delta)\Big) \geq 0.$$

As $u_\delta(\cdot)$ is Lipschitz with constant C_H, we have $u_\delta(y_\delta) \to u(\tilde{x})$ along the subsequence δ_j. Therefore as $\delta_j \to 0$ we have $H(\tilde{x}, D\varphi(\tilde{x})) \geq c$.

Now with the help of comparison principle, we obtain the uniqueness of $c = c(0)$ and thus the convergence of the full sequence $\delta u_\delta(x_0) \to -c(0)$ follows. If we use the following normalization

$$\lim_{j\to\infty} \left(u_{\delta_j}(x) + \frac{c(0)}{\delta_j} \right) = w(x)$$

then by a similar argument we can show w solves (S_0) as well, and

$$\begin{aligned} u(x) &= \lim_{j\to\infty} \left(u_{\delta_j}(x) - u_{\delta_j}(x_0) \right) \\ &= \lim_{j\to\infty} \left(u_{\delta_j}(x) + \frac{c(0)}{\delta_j} \right) - \lim_{j\to\infty} \left(u_{\delta_j}(x_0) + \frac{c(0)}{\delta_j} \right) = w(x) - w(x_0). \end{aligned}$$

We have left to show (5.2.4). Let u be defined as the limit of $u_{\delta_j}(\cdot) - u_{\delta_j}(x_0)$, we have $|u(x)| + |Du(x)| \leq C$ for $x \in \Omega$ where C depends on C_H and $\mathrm{diam}(\Omega)$. It is clear that $u(x) - \delta^{-1}c(0) \pm C$ are, respectively, subsolution and supersolution to (HJ_δ), therefore by comparison principle we obtain (5.2.4). $\square$

Proof of Theorem 5.2.10. If $v \in \mathrm{C}(\overline{\Omega})$ is a solution to (E) then for a.e. $x \in \overline{\Omega}$ we have $-V(x) \leq |Dv(x)| - V(x) = c_\Omega$, therefore $c_\Omega \geq \max_{\overline{\Omega}}(-V) = -\min_{\overline{\Omega}} V$. Assume V attains its minimum over $\overline{\Omega}$ at x_0 then by supersolution test at that point we have $0 \geq -V(x_0) \geq c_\Omega$, therefore $c_\Omega = -\min_{\overline{\Omega}} V$.

Let $z \in \overline{\Omega}$ such that $V(z) = -c_\Omega$, we check that $x \mapsto S_\Omega(x,z)$ is a supersolution at $x = z$. Let $\omega(\cdot)$ be the modulus of continuity of V on $\overline{\Omega}$, we have $|V(x) + c_\Omega| \leq \omega(r)$ for all $x \in B(z,r) \cap \overline{\Omega}$. From (E) as $x \mapsto u(x) = S_\Omega(x,z)$ is a subsolution in Ω, we have

$$|Du(x)| - V(x) \leq c_\Omega \qquad \Longrightarrow \qquad |Du(x)| \leq V(x) + c_\Omega \leq \omega(r)$$

for a.e. $x \in B(z,r) \cap \overline{\Omega}$ and for all $r > 0$, thus

$$|u(x)| = |u(x) - u(z)| \leq \int_0^1 |Du(sx + (1-s)z) \cdot (x-z)|\, ds \leq \omega(r)r$$

for $x \in B(z,r) \cap \overline{U}$. That means $x \mapsto u(x)$ is differentiable at $x = z$ and $Du(z) = 0$, thus $x \mapsto u(x) = S_\Omega(x,z)$ is a solution to (E).

Conversely, if $V(z) = -c_\Omega + \varepsilon$ for some $\varepsilon > 0$, then at $x = z$ we have $0 \in D^- u(z)$ where $u(x) = S_\Omega(x,z)$, therefore if the supersolution test holds then we must have $-V(z) \geq c_\Omega$, hence $\varepsilon < 0$ which is a contradiction, thus $x \mapsto S_\Omega(x,z)$ fails to be a supersolution at $x = z$. $\square$

Proof of Theorem 5.2.12. Without loss of generality we assume $c_U = 0$. Let $z \in \mathcal{A}_U \subset \Omega$ and $w(x) = S_U(x,z)$ solves (E), we have $H(x, Dw(x)) = 0$ in Ω. We have

$$c_\Omega = \inf \left\{ c \in \mathbb{R} : H(x, Du(x)) = c \text{ admits a viscosity subsolution in } \Omega \right\} \leq 0.$$

Assume the contrary that $c_\Omega < 0$ then there exists $u \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{W}^{1,\infty}(\Omega)$ solves

$$H(x, Du(x)) \leq c(0) < 0 \qquad \text{in } \Omega.$$

Let us consider $g(x) = w(x)$ defined for $x \in \partial\Omega$ and the boundary value problem

$$\begin{cases} H(x, Dv(x)) = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases} \tag{5.6.1}$$

As there exists a solution u such that $H(x, Du(x)) < 0$ in Ω, by Theorem 5.6.1 the problem (5.6.1) cannot have more than one solution. On the other hand, the following function is a solution to (5.6.1)

$$\mathcal{V}(x) = \min_{y \in \partial\Omega} \Big\{ g(y) + S_U(x, y) \Big\}.$$

Indeed, for each $y \in \partial\Omega$, $x \mapsto g(y) + S_U(x, y)$ is a Lipschitz viscosity solution to $H(x, Dv(x)) = 0$ in Ω, therefore by the convexity of H is obtain $\mathcal{V}$ is a viscosity solution to $H(x, D\mathcal{V}(x)) = 0$ in Ω as well. On the boundary we see that $\mathcal{V}(x) \leq g(x)$, and also for any $y \in \partial\Omega$ then

$$g(y) + S_U(x, y) = S_U(y, z) + S_U(x, y) \geq S_U(x, z) = g(x),$$

which implies that $\mathcal{V}(x) = g(x)$ on $\partial\Omega$. Therefore we must have $\mathcal{V}(x) = S_U(x, z)$ for all $x \in \overline{\Omega}$, hence $\mathcal{V}(z) = S_U(z, z) = 0$ and as a consequence there exists $y \in \partial\Omega$ such that

$$S_U(y, z) + S_U(z, y) = 0.$$

This implies that $y \in \mathcal{A}_U$ (see [55, 71]), which is a contradiction since $\mathcal{A}_U$ is supported inside Ω, therefore we must have $c_\Omega = 0$. □

Proof of Lemma 5.3.1. From $(\mathcal{H}_8)$ for each $R > 0$ we can find a nondecreasing function $\omega_R : [0, \infty) \to [0, \infty)$ with $\omega_R(0) = 0$ such that $|D_x L(x, v) - D_x L(y, v)| \leq \omega_R(|x - y|)$ if $|x|, |v| \leq R$. Fix $(x, v) \in \overline{\Omega} \times \overline{B}_h$, we can assume h is large so that $\overline{\Omega} \subset \overline{B}_h$. Let $f(\delta) = L\left((1 - \delta)x, v\right)$ then $\delta \mapsto f(\delta)$ is continuously differentiable and

$$\begin{aligned} \left| \frac{f(\delta) - f(0)}{\delta} - f'(0) \right| &\leq \sup_{s \in [0, \delta]} |f'(s) - f'(0)| \\ &= \sup_{s \in [0, \delta]} |x| . \left| D_x L\left((1 - s)x, v\right) - D_x L(x, v) \right| \leq \big(\mathrm{diam}\ \Omega\big) \omega_h\left(\delta |x|\right). \end{aligned}$$

Therefore

$$\lim_{\delta \to 0^+} \left(\sup_{(x,v) \in \overline{\Omega} \times \overline{B}_h} \left| \frac{L\left((1 - \delta)x, v\right) - L(x, v)}{\delta} - (-x) \cdot D_x L(x, v) \right| \right) = 0.$$

The conclusions follow from here. □

Theorem 5.6.1 (Comparison principle for Dirichlet problem, [8])**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume $u_1, u_2 \in \mathrm{C}(\overline{\Omega})$ are, respectively viscosity subsolution and supersolution of $H(x, Du(x)) = 0$ in Ω with $u_1 \leq u_2$ on $\partial\Omega$. Assume further that*

- $|H(x,p) - H(y,p)| \leq \omega((1+|p|)|x-y|)$ *for all* $x, y \in \Omega$ *and* $p \in \mathbb{R}^n$.
- $p \mapsto H(x,p)$ *is convex for each* $x \in \Omega$.
- *There exists* $\varphi \in \mathrm{C}(\overline{\Omega})$ *such that* $\varphi \leq u_2$ *in* $\overline{\Omega}$ *and* $H(x, D\varphi(x)) < 0$ *in* Ω.

Then $u_1 \leq u_2$ *in* Ω.

Chapter 6

Second-order equation with state-constraint: rate of vanishing viscosity

This chapter is devoted to the study of the convergence rate in the vanishing viscosity process of the solutions to the subquadratic state-constraint Hamilton-Jacobi equations. Let Ω be an open, bounded and connected domain in $\mathbb{R}^n$ with C^2 boundary, $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. For $\varepsilon > 0$, let $u^\varepsilon \in C^2(\Omega)$ (see [78]) be the solution to

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

We are interested in studying *quantitatively* the asymptotic behavior of $\{u^\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0^+$. The blow-up behavior on the boundary makes it a nontrivial problem. Heuristically, (PDE$_\varepsilon$) can be written in the viscosity solution framework as a state-constraint boundary condition (see [78])

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \leq 0 & \text{in } \Omega, \\ u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{6.0.1}$$

It is natural to expect that, the solution of the second-order state-constraint problem converges to that of a first-order state-constraint problem associated with the deterministic optimal control, namely,

$$\begin{cases} u(x) + |Du(x)|^p - f(x) \leq 0 & \text{in } \Omega, \\ u(x) + |Du(x)|^p - f(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{PDE$_0$}$$

We show the following quantitative estimates.

- For nonnegative Lipschitz data that vanish on the boundary, the rate of convergence is $\mathcal{O}(\sqrt{\varepsilon})$ in the interior.

- The one-sided rate can be improved to $\mathcal{O}(\varepsilon)$ for nonnegative compactly supported data and $\mathcal{O}(\varepsilon^{1/p})$ (where $1 < p < 2$ is the exponent of the gradient term) for nonnegative data $f \in C^2(\overline{\Omega})$ such that $f = 0$ and $Df = 0$ on the boundary.

The materials of this chapter is taken mainly from [61] with some new remarks added.

Remark 50. One of the key ingredient in boosting the one-sided rate of convergence from $\mathcal{O}(\varepsilon^{1/2})$ to $\mathcal{O}(\varepsilon^{1/p})$ in the interior is to utilize refined estimates on the second derivative of solution u to (PDE$_0$), i.e., the semiconcavity of u. Heuristically, assume that $u^\varepsilon(x) - u(x)$ has a maximum over $\overline{\Omega}$ at some interior point $x_0 \in \Omega$. Then by the equation (PDE$_\varepsilon$) at x_0 and the supersolution test for (PDE$_0$) at x_0, we obtain

$$\max_{x\in\overline{\Omega}} \Big(u^\varepsilon(x) - u(x)\Big) \leq u^\varepsilon(x_0) - u(x_0) \leq \varepsilon \Delta u^\varepsilon(x_0).$$

If u is uniformly semiconcave in $\overline{\Omega}$, then $\Delta u^\varepsilon(x_0) \leq \Delta u(x_0) \leq C$. However we can only show this global semiconcavity of u if f is compactly supported in Ω and can be extended to a global semiconcave function in $\mathbb{R}^n$. Later in this chapter we show that

$$u(x+h) - 2u(x) + u(x-h) \leq \frac{C}{\mathrm{dist}(x,\partial\Omega)}|h|^2 \tag{6.0.2}$$

for all $h \in \mathbb{R}^n$ small enough.

If one can improve the semiconcavity modulus of u then one can also improve the one-sided rate of convergence. Recently, in [60], the author establishes a global semiconcavity modulus under some conditions on f for solution to (PDE$_0$). As a consequence, the upper bound of $u^\varepsilon - u$ can be improved to $\mathcal{O}(\varepsilon)$ in the interior as long as u is global semiconcave.

6.1 Introduction

Let Ω be an open, bounded and connected domain in $\mathbb{R}^n$ with C^2 boundary, $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. For $\varepsilon > 0$, let $u^\varepsilon \in C^2(\Omega)$ (see [78] for the existence and the uniqueness) be the solution to

$$\begin{cases} u^\varepsilon(x) + H(Du^\varepsilon(x)) - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty, \end{cases} \tag{6.1.1}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ is a given continuous Hamiltonian. The solution that blows up uniformly on the boundary is also called a *large solution*. A typical Hamiltonian that has been considered in the literature is $H(\xi) = |\xi|^p$ for $\xi \in \mathbb{R}^n$ where $1 < p \leq 2$, and equation (6.1.1) becomes

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

It turns out that for this specific subquadratic Hamiltonian, u^ϵ is also the unique solution to the second-order state-constrait problem (see [78])

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \leq 0 & \text{in } \Omega, \\ u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{6.1.2}$$

We focus on this Hamiltonian in our paper, which follows the setting of [78], where the specific structure of the Hamiltonian enables more explicit estimates for the solution of (PDE_ε). In fact, for $1 < p \leq 2$, the solution to equation (PDE_ε) is the value function associated with a minimization problem in stochastic optimal control theory with state constraints ([52, 78]). We briefly recall the setting and all the domains and target spaces are omitted for simplicity. For a given stochastic control $\alpha(\cdot)$, we can solve for a solution (a state process) of the feedback control system

$$\begin{cases} dX_t = \alpha\,(X_t)\,dt + \sqrt{2\varepsilon}\,d\mathbb{B}_t & \text{for } t > 0, \\ X_0 = x. \end{cases} \tag{6.1.3}$$

Here, $\mathbb{B}_t \sim \mathcal{N}(0,t)$ is the Brownian motion with mean zero and variance t. To constrain the state X_t inside $\overline{\Omega}$, we define

$$\widehat{\mathcal{A}}_x = \left\{\alpha(\cdot) \in \mathrm{C}(\Omega) : \mathbb{P}(X_t \in \Omega) = 1 \text{ for all } t \geq 0\right\}$$

and hope to minimize a cost function in expectation to get the value function

$$u^\varepsilon(x) = \inf_{\alpha \in \widehat{\mathcal{A}}_x} \mathbb{E}\left[\int_0^\infty e^{-t} L\big(X_t, -\alpha(X_t)\big)\,dt\right], \tag{6.1.4}$$

where $L(x,v) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is the running cost. More specifically, $L(x,v) = c\,|v|^q + f(x)$ is the Legendre transform of $H(x,\xi) := |\xi|^p - f(x)$ with $q > 1$, $f \in \mathrm{C}(\overline{\Omega})$ nonnegative, and some constant c. Using the Dynamic Programming Principle (see [78]), we expect the value function (6.1.4) to solve (6.1.2), which means that u^ε is a subsolution in Ω and a supersolution on $\overline{\Omega}$.

We are interested in studying the asymptotic behavior of $\{u^\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0^+$. Heuristically, the solution of the second-order state-constraint equation converges to that of a first-order state-constraint equation associated with the deterministic optimal control problem, namely,

$$\begin{cases} u(x) + |Du(x)|^p - f(x) \leq 0 & \text{in } \Omega, \\ u(x) + |Du(x)|^p - f(x) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{PDE_0}$$

and indeed equation (PDE_0) admits a unique viscosity solution $u \in \mathrm{C}(\overline{\Omega})$(see [30, 107]). From the viewpoint of optimal control theory, as $\varepsilon \to 0^+$, the stochastic control system (6.1.3) becomes a deterministic control system. In particular, let $\mathcal{A}_x = \{\zeta \in \mathrm{AC}([0,\infty);\overline{\Omega}) : \zeta(0) = x\}$ and we have

$$u(x) = \inf_{\zeta \in \mathcal{A}_x} \int_0^\infty e^{-t} L\big(\zeta(t), -\dot{\zeta}(t)\big)\,dt$$

where $L(x, v)$ is again the Legendre transform of $H(x, \xi) := |\xi|^p - f(x)$.

The problem is interesting since in the limit there is no blowing up behavior near the boundary, as $u \in C(\overline{\Omega})$. In this paper, we investigate the rate of convergence of $u^\varepsilon \to u$ as $\varepsilon \to 0^+$. What is intriguing and delicate here is the blow-up behavior of u^ε in a narrow strip near $\partial\Omega$ as $\varepsilon \to 0^+$. This is often called the boundary layer theory in the literature.

Note that a comparison principle holds for (PDE_0) since we always assume Ω is an open, bounded and connected domain in $\mathbb{R}^n$ with C^2 boundary ([30, 107]).

6.1.1 Relevant literature

There is a vast amount of work in the literature on viscosity solutions with state constraints and large solutions. We would like to first mention that the problem (PDE_0) with general Hamiltonian is a huge subject of research interest, started with the pioneer work [107] (see also [67, 68]). Some of the recent work related to the asymptotic behavior of solutions of (PDE_0) can be found in [70, 75, 95, 114]. The problem (PDE_ε) was first studied in [78] and subsequently many works have been done in understanding deeper the properties of solutions (see [7, 89, 101, 102] and the references therein). The time-dependent version of (6.1.1) was also studied by many works, for instance, [11, 12, 81, 96] and the references therein.

In terms of rate of convergence, that is, the convergence rate of $u^\varepsilon \to u$ as $\varepsilon \to 0^+$, to the best of our knowledge, such a question has not been studied in the literature. For the case where (PDE_ε) is equipped with the Dirichlet boundary condition, a rate $\mathcal{O}(\sqrt{\varepsilon})$ is well known with multiple proofs (see [8, 43, 111]).

6.1.2 Main results

For $1 < p \leq 2$, define

$$\alpha = \frac{2-p}{p-1} \in [0, \infty).$$

Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. For small $\delta > 0$, denote $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$ and $\Omega^\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \overline{\Omega}) < \delta\}$.

Definition 23. *Define*

$$\delta_{0,\Omega} = \frac{1}{2} \sup \left\{ \delta > 0 : x \mapsto \mathrm{dist}(x, \partial\Omega) \text{ is } C^2 \text{ in } \Omega^\delta \backslash \overline{\Omega}_\delta \right\}. \tag{6.1.5}$$

We will write δ_0 instead of $\delta_{0,\Omega}$ when the underlying domain is understood.

The reader is referred to [59] for the regularity of the distance function defined in $\Omega^{\delta_0} \backslash \Omega_{\delta_0}$. We then extend $\mathrm{dist}(x, \partial\Omega)$ to a function $d(x) \in C^2(\mathbb{R}^n)$ such that

$$\begin{cases} d(x) \geq 0 \text{ for } x \in \Omega \text{ with } d(x) = +\mathrm{dist}(x, \partial\Omega) \text{ for } x \in \Omega \backslash \Omega_{\delta_0}, \\ d(x) \leq 0 \text{ for } x \notin \Omega \text{ with } d(x) = -\mathrm{dist}(x, \partial\Omega) \text{ for } x \in \Omega^{\delta_0} \backslash \Omega. \end{cases} \tag{6.1.6}$$

Assumption on f. We assume that $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with $f = \min_{\overline{\Omega}}$ on $\partial\Omega$.

By replacing f by $f - \min_{\overline{\Omega}}$, without loss of generality, we can assume $\min_{\overline{\Omega}} f = 0$ and $f = 0$ on $\partial\Omega$. The reason why this assumption is needed is elaborated in Remark 1. The main results of the paper are the following theorems.

Theorem 6.1.1. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with C^2 boundary. Assume that $1 < p \leq 2$ and f is nonnegative and Lipschitz with $f = 0$ on $\partial\Omega$. Let u^ε be the unique solution to* (PDE$_\varepsilon$) *and u be the unique solution to* (PDE$_0$). *Then there exists a constant C independent of $\varepsilon \in (0,1)$ such that for $x \in \Omega$,*

$$
\begin{aligned}
-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) &\leq C\left(\sqrt{\varepsilon} + \frac{\varepsilon^{\alpha+1}}{d(x)^\alpha}\right), \qquad 1 < p < 2, \\
-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) &\leq C\left(\sqrt{\varepsilon} + \varepsilon\left|\log(d(x))\right|\right), \qquad p = 2.
\end{aligned}
$$

Remark 51. To the best of our knowledge, this theorem is new in the literature. The precise boundary behavior is very delicate and deserves further investigation. The condition $f = 0$ on $\partial\Omega$ is a little bit restrictive but is naturally needed in the proof. As is illustrated in the proof of Theorem 6.1.1, we will first show the result for f that is compactly supported and nonnegative. Then, to further generalize the main result, if we make the assumption that $f = 0$ on $\partial\Omega$ and f is nonnegative, we can approximate f *uniformly* in $L^\infty(\Omega)$ by a sequence of compactly supported Lipschitz functions with uniformly bounded Lipschitz constants. Using the previous result obtained for the case where f is compactly supported and nonnegative, we can pass to the limit and prove Theorem 6.1.1 for nonnegative f with $f = 0$ on $\partial\Omega$, which is more general than the compactly supported case. At the current moment, we do not yet know how to extend the result to general f where f does not vanish or is not equal to its minimum on the boundary.

To prove the result for the case where f is compactly supported and nonnegative, it is natural to consider the doubling variable method. Indeed, for instance, if $1 < p < 2$, one would consider constructing an auxiliary function with

$$
\psi^\varepsilon(x) := u^\varepsilon(x) - \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} \tag{6.1.7}
$$

and $u(x)$, where $C_\alpha \varepsilon^{\alpha+1} d(x)^{-\alpha}$ is the leading order term in the asymptotic expansion of $u^\varepsilon(x)$ as $d(x) \to 0^+$ with $C_\alpha := \alpha^{-1}(\alpha+1)^{\alpha+1}$. If we take the derivative of (6.1.7) formally, it becomes

$$
D\psi^\varepsilon(x) = Du^\varepsilon(x) + C_\alpha \alpha \left(\frac{\varepsilon}{d(x)}\right)^{\alpha+1} Dd(x). \tag{6.1.8}
$$

We will see that $D\psi^\varepsilon(x)$ is uniformly bounded if $d(x) \geq \varepsilon$ (Lemma 6.5.2). Indeed,

$$
-C_\alpha \alpha \left(\frac{\varepsilon}{d(x)}\right)^{\alpha+1} Dd(x)
$$

is more or less the leading order term in the asymptotic expansion of Du^ε near $\partial\Omega$. Heuristically, this means that the boundary layer is $\mathcal{O}(\varepsilon)$ from the boundary.

However, to get a useful estimate by the doubling variable method, at the maximum point x_0 of $\psi^\varepsilon(x) - u(x)$, we need to have $d(x_0) \geq \varepsilon^\gamma$ for $\gamma < 1$ so that the latter term in (6.1.8) vanishes as $\varepsilon \to 0^+$. Otherwise, we cannot obtain a convergence rate via the doubling variable method as there are still nonvanishing constant terms. In the other case where $d(x_0) < \varepsilon^\gamma$, we introduce a new localization idea, that is, we construct a blow-up solution in the ball of radius ε^γ from the boundary. Finally, a technical (and common for the doubling variable method) computation leads to $\gamma = 1/2$.

As a different approach, the convexity of $|\xi|^p$ and the semiconcavity of the solution to (PDE_0) give us a better one-sided $\mathcal{O}(\varepsilon)$ estimate for nonnegative compactly supported f which is semiconcave in its support (see Theorem 6.1.2). Such an one-sided $\mathcal{O}(\varepsilon)$ rate is well known for the Dirichlet boundary problem (see [8, 111]). Moreover, the result in Theorem 6.1.2 further provides us with a better one-sided estimate $\mathcal{O}(\varepsilon^{1/p})$ than that in Theorem 6.1.1, as in Corollary 6.1.3. We recall that f is (uniformly) semiconcave in $\overline{\Omega}$ with linear modulus (or semiconcavity constant) $c > 0$ if

$$f(x+h) - 2f(x) + f(x-h) \leq c\,|h|^2, \quad \forall x, h \in \mathbb{R}^n \text{ such that } x+h, x, \text{and } x-h \in \overline{\Omega}.$$

Note that any $f \in \mathrm{C}^2_c(\Omega)$ is semiconcave on its support with the constant

$$c = \max\left\{D^2 f(x)\xi \cdot \xi : |\xi| = 1, x \in \Omega\right\}$$

in the above definition. It is well known that the solution u to (PDE_0) is *locally* semiconcave given f is uniformly semiconcave in $\overline{\Omega}$. Using tools from the optimal control theory, we provide the explicit blow-up rate of the semiconcavity modulus of $u(x)$ when x approaches $\partial\Omega$. As an application, we can improve the rate of convergence as follows.

Theorem 6.1.2 (One-sided $\mathcal{O}(\varepsilon)$ rate for nonnegative compactly supported data)**.** *Under the conditions of Theorem 6.1.1, suppose f also satisfies the following conditions:*

- *f is semiconcave in its support;*
- *f has a compact support in $\Omega_\kappa := \{x \in \Omega : \mathrm{dist}(x, \Omega) > \kappa\}$ for some $\kappa \in (0, \delta_0)$, $0 < \delta_0 < 1$ defined in (6.1.5).*

Then there exist two constants $\nu > 1$ and C independent of ε and κ such that $\forall x \in \Omega$,

$$u^\varepsilon(x) - u(x) \leq \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\left(\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right) + \frac{Cn\varepsilon}{\kappa}, \qquad \text{if } p < 2,$$
$$u^\varepsilon(x) - u(x) \leq \nu\varepsilon \log\left(\frac{1}{d(x)}\right) + C\left(\left(\frac{\varepsilon}{\kappa}\right) + \left(\frac{\varepsilon}{\kappa}\right)^2\right) + \frac{Cn\varepsilon}{\kappa}, \qquad \text{if } p = 2.$$

Remark 52. If $f \in \mathrm{C}^2_c(\Omega)$, then the last term $(Cn\varepsilon)\,\kappa^{-1}$ in the equations above can be improved to $nc\varepsilon$, where c is the semiconcavity constant of f. This improvement is due to the fact that we can prove u is uniformly semiconcave with a semiconcavity constant that only depends on the semiconcavity constant c of f (see Theorem 6.7.1 in the Section 6.7). Hence, in the proof of Theorem 6.1.2, in equation (6.5.7), instead of $C\kappa^{-1}$, we can bound $c(x_0)$ by the semiconcavity constant c of f, independent of κ. Similarly, see Remark 59 for this improvement on the last term. It turns out that in general, if f can be extended to a semiconcave function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $f = 0$ on Ω^c, then u is uniformly semiconcave, and hence this improvement happens. See Fig. 6.1 for two examples where f can and cannot be extended to a semiconcave function in the whole space by setting $f = 0$ outside Ω.

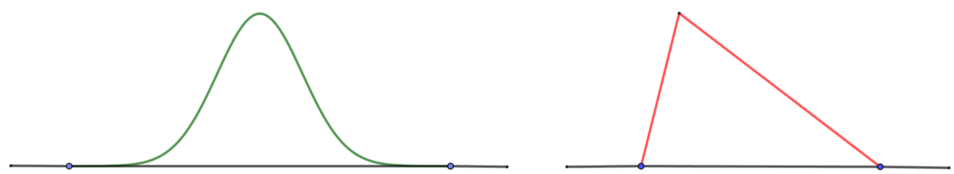

Figure 6.1: The one on the right corresponds to a general f in Theorem 6.1.2, while the one on the left corresponds to the situation in Remark 52 where the improvement happens.

Corollary 6.1.3 (One-sided $\mathcal{O}(\varepsilon^{1/p})$ rate)**.** *Let $1 < p < 2$. If $f \in C^2(\overline{\Omega})$ is nonnegative, $f = 0$ and $Df = 0$ on $\partial\Omega$, then there exists a constant C independent of $\varepsilon \in (0,1)$ such that*

$$-C\varepsilon^{1/2} \leq u^\varepsilon(x) - u(x) \leq C\left(\varepsilon^{1/p} + \frac{\varepsilon^{\alpha+1}}{d(x)^\alpha}\right)$$

for all $x \in \Omega$.

Remark 53. While the second approach looks more powerful, we need the gradient bound of u^ε (Lemma 6.5.2), the blow-up rate of the semiconcavity constant of u (Theorem 6.5.1), and higher regularity on f. On the other hand, the first approach by doubling variable is relatively simple and does not require any explicit asymptotic behavior of Du^ε, except the fact that it is locally bounded.

6.2 Preliminaries on the well-posedness of blow-up solutions

Let $K_0 := \max_{x\in\overline{\Omega}} |d(x)|$, $K_1 := \max_{x\in\overline{\Omega}} |Dd(x)|$, and $K_2 := \max_{x\in\overline{\Omega}} |\Delta d(x)|$. Note that $d(x) = \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$ and $|Dd(x)| = 1$ in the classical sense in $\Omega^{\delta_0}\backslash\Omega_{\delta_0}$. Denote by $\mathcal{L}^\varepsilon : C^2(\Omega) \to C(\Omega)$ the operator

$$\mathcal{L}^\varepsilon[u](x) := u(x) + |Du(x)|^p - f(x) - \varepsilon\Delta u(x), \qquad x \in \Omega.$$

6.2.1 Local gradient estimate

For $\varepsilon \in (0,1)$ and $p > 1$, we state an a priori estimate for C^2 solutions to (PDE$_\varepsilon$) ([78, Appendix]). Since we are working with smooth solutions, the proof is relatively simple by the classical Bernstein method ([15, 78, 84]). Another proof using Berstein's method inside a doubling variable argument is given in [6].

Theorem 6.2.1. *Let $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and $u^\varepsilon \in C^2(\Omega)$ be a solution to $\mathcal{L}^\varepsilon[u^\varepsilon] = 0$ in Ω with $1 < p \leq 2$. Let $m := \max_{\overline{\Omega}} f(x)$. Then for $\delta > 0$, there exists $C_\delta = C(m, p, \delta, \|Df\|_{L^\infty(\Omega)})$ such that*

$$\sup_{x \in \overline{\Omega}_\delta} \Big(|u^\varepsilon(x)| + |Du^\varepsilon(x)| \Big) \leq C_\delta$$

for ε small enough.

Proof of Theorem 6.2.1. Let $\theta \in (0,1)$ be chosen later, $\varphi \in C_c^\infty(\Omega)$, $0 \leq \varphi \leq 1$, supp $\varphi \subset \Omega$ and $\varphi = 1$ on Ω_δ such that

$$|\Delta\varphi(x)| \leq C\varphi^\theta \qquad \text{and} \qquad |D\varphi(x)|^2 \leq C\varphi^{1+\theta}, \quad \forall x \in \Omega, \tag{6.2.1}$$

where $C = C(\delta, \theta)$ is a constant depending on δ, θ.

Define $w(x) := |Du^\varepsilon(x)|^2$ for $x \in \Omega$. The equation for w is given by

$$-\varepsilon\Delta w + 2p\,|Du^\varepsilon|^{p-2}\, Du^\varepsilon \cdot Dw + 2w - 2Df \cdot Du^\varepsilon + 2\varepsilon\left|D^2u^\varepsilon\right|^2 = 0 \qquad \text{in } \Omega.$$

Then an equation for (φw) can be derived as follows.

$$\begin{aligned}
&-\varepsilon\Delta(\varphi w) + 2p\,|Du^\varepsilon|^{p-2}\, Du^\varepsilon \cdot D(\varphi w) + 2(\varphi w) + 2\varepsilon\varphi\left|D^2u^\varepsilon\right|^2 + 2\varepsilon\frac{D\varphi}{\varphi} \cdot D(\varphi w) \\
&= \varphi(Df \cdot Du^\varepsilon) + 2p\,|Du^\varepsilon|^{p-2}\,(Du^\varepsilon \cdot D\varphi)w - \varepsilon w\Delta\varphi + 2\varepsilon\frac{|D\varphi|^2}{\varphi}w \qquad \text{in supp } \varphi.
\end{aligned}$$

Assume that φw achieves its maximum over $\overline{\Omega}$ at $x_0 \in \Omega$. And we can further assume that $x_0 \in$ supp φ, since otherwise the maximum of φw over $\overline{\Omega}$ is zero. By the classical maximum principle,

$$-\varepsilon\Delta(\varphi w)(x_0) \geq 0 \qquad \text{and} \qquad |D(\varphi w)(x_0)| = 0.$$

Use this in the equation of φw above to obtain

$$\varepsilon\varphi\left|D^2u^\varepsilon\right|^2 \leq \varphi(Df \cdot Du^\varepsilon) + 2p\,|Du^\varepsilon|^{p-1}\,|D\varphi|\,w + \varepsilon w\,|\Delta\varphi| + 2\varepsilon w\frac{|D\varphi|^2}{\varphi},$$

where all terms are evaluated at x_0. From (6.2.1), we have

$$\varepsilon\varphi\left|D^2u^\varepsilon\right|^2 \leq \varphi\,|Df|\,w^{\frac{1}{2}} + 2Cpw^{\frac{p-1}{2}+1}\varphi^{\frac{1+\theta}{2}} + C\varepsilon w\varphi^\theta + 2C\varepsilon w\varphi^\theta. \tag{6.2.2}$$

By Cauchy-Schwartz inequality, $n\left|D^2u^\varepsilon\right|^2 \geq (\Delta u^\varepsilon)^2$. Thus, if $n\varepsilon < 1$, then

$$\begin{aligned}
\varepsilon\left|D^2u^\varepsilon\right|^2 &\geq \frac{(\varepsilon\Delta u^\varepsilon)^2}{n\varepsilon} \geq (\varepsilon\Delta u^\varepsilon)^2 = \left(u^\varepsilon + |Du^\varepsilon|^p - f\right)^2 \\
&\geq |Du^\varepsilon|^{2p} - 2C\,|Du^\varepsilon|^p \geq \frac{|Du^\varepsilon|^{2p}}{2} - 2C,
\end{aligned} \tag{6.2.3}$$

where C depends on $\max_{\overline{\Omega}} f$ only. Using (6.2.3) in (6.2.2), we obtain that

$$\varphi\left(\frac{1}{2}w^p - 2C\right) \leq \varphi\,|Df|\,w^{\frac{1}{2}} + 2Cpw^{\frac{p-1}{2}+1}\varphi^{\frac{1+\theta}{2}} + 3C\varepsilon w\varphi^\theta.$$

Multiply both sides by φ^{p-1} to deduce that

$$(\varphi w)^p \leq 4C\varphi^{p-1} + 2\|Df\|_{L^\infty}\varphi^p w^{\frac{1}{2}} + 4Cp\varphi^{\frac{2p+\theta-1}{2}} w^{\frac{p+1}{2}} + 6C\varepsilon\varphi^{p+\theta-1}w.$$

Choose $2p+\theta-1 \geq p+1$, i.e., $p+\theta \geq 2$. This is always possible with the requirement $\theta \in (0,1)$, as $1 < p < \infty$. Then we get

$$(\varphi w)^p \leq C\left(1 + (\varphi w)^{\frac{1}{2}} + (\varphi w)^{\frac{p+1}{2}} + (\varphi w)\right). \tag{6.2.4}$$

As a polynomial in $z = (\varphi w)(x_0)$, this implies that $(\varphi w)(x_0) \leq C$ where C depends on coefficients of the right hand side of (6.2.4), which gives our desired gradient bound since $w(x) = (\varphi w)(x) \leq (\varphi w)(x_0)$ for $x \in \overline{\Omega}_\delta \subset \operatorname{supp} \varphi$. □

6.2.2 Well-posedness

In this section, we recall the existence and the uniqueness of solutions to (PDE$_\varepsilon$) for $1 < p \leq 2$ and $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. In fact, the assumption of f can be relaxed to $f \in L^\infty(\Omega)$ (see [78]).

Theorem 6.2.2. *Let $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. There exists a unique solution $u^\varepsilon \in C^2(\Omega)$ of (PDE$_\varepsilon$) such that:*

(i) If $1 < p < 2$, then

$$\lim_{d(x)\to 0} \left(u^\varepsilon(x)\, d(x)^\alpha\right) = C_\alpha \varepsilon^{\alpha+1}, \tag{6.2.5}$$

where $\alpha = (p-1)^{-1}(2-p)$ and $C_\alpha = \alpha^{-1}(\alpha+1)^{\alpha+1}$.

(ii) If $p = 2$, then

$$\lim_{d(x)\to 0} \left(-\frac{u^\varepsilon(x)}{\log(d(x))}\right) = \varepsilon. \tag{6.2.6}$$

Furthermore, u^ε is the maximal subsolution among all the subsolutions $v \in W^{2,r}_{\mathrm{loc}}(\Omega)$ for all $r \in [1,\infty)$ of (PDE$_\varepsilon$).

This is Theorem I.1 in [78] with an explicit dependence on ε. The proof of this theorem is carried out explicitly in Section 6.7 for later use. Also, it is useful to note that $\alpha + 1 = (p-1)^{-1}$. More results on the behavior of the gradient of u^ε can be found in [102] and Lemma 6.5.2, where we show

$$|Du^\varepsilon| \leq C + C\left(\frac{\varepsilon}{d(x)}\right)^{\alpha+1}.$$

We believe Lemma 6.5.2 is new in the literature.

6.2.3 Convergence results

We first state the following lemma ([30]), which characterizes the solution to the first-order state-constraint equation (PDE_0).

Lemma 6.2.3. *Let $u \in \mathrm{C}(\overline{\Omega})$ be a viscosity subsolution of (PDE_0) such that, for any viscosity subsolution $v \in \mathrm{C}(\overline{\Omega})$ of (PDE_0), one has $v \le u$ on $\overline{\Omega}$. Then u is a viscosity supersolution of (PDE_0) on $\overline{\Omega}$.*

Again, the proof of Lemma 6.2.3 is given in Section 6.7 for the reader's convenience.

Lemma 6.2.4. *Assume $1 < p \le 2$. Let $u^\varepsilon \in \mathrm{C}^2(\Omega)$ be the solution to (PDE_ε) and $u \in \mathrm{C}(\overline{\Omega})$ be the solution to (PDE_0). We have $\{u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded from below by a constant independent of ε. More precisely, $u^\varepsilon \ge \min_\Omega f$ and $u \ge \min_\Omega f$.*

Proof. For $m \in \mathbb{N}$, let $u^\varepsilon_m \in \mathrm{C}^2(\Omega) \cap \mathrm{C}(\overline{\Omega})$ solve the Dirichlet problem

$$\begin{cases} u^\varepsilon_m(x) + |Du^\varepsilon_m(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon_m(x) = 0 & \text{in } \Omega, \\ u^\varepsilon_m(x) = m & \text{on } \partial\Omega. \end{cases} \tag{$\mathrm{PDE}_{\varepsilon,m}$}$$

We have $u^\varepsilon_m(x) \to u^\varepsilon(x)$ in Ω as $m \to \infty$. Let $\varphi(x) \equiv \inf_\Omega f$ for $x \in \overline{\Omega}$. Then $\varphi(x)$ is a classical subsolution of ($\mathrm{PDE}_{\varepsilon,m}$) in Ω with

$$\varphi(x) = \inf_\Omega f \le m = u^\varepsilon_m(x) \qquad \text{on } \partial\Omega$$

for m large enough. By the comparison principle of the uniformly elliptic equation ($\mathrm{PDE}_{\varepsilon,m}$),

$$\inf_\Omega f \le u^\varepsilon_m(x) \qquad \text{for all } x \in \Omega.$$

As $m \to \infty$, we obtain $u^\varepsilon \ge \min_\Omega f$. The inequality $u \ge \min_\Omega f$ follows from the comparison principle of (PDE_0) applied to the supersolution u on $\overline{\Omega}$ and the subsolution φ in Ω. □

We present here a simple proof of the convergence $u^\varepsilon \to u$ using Lemma 6.2.3. See also [30, Theorem VII.3].

Theorem 6.2.5 (Vanishing viscosity)**.** *Let u^ε be the solution to (PDE_ε). Then there exists $u \in \mathrm{C}(\overline{\Omega})$ such that $u^\varepsilon \to u$ locally uniformly in Ω as $\varepsilon \to 0$ and u solves (PDE_0).*

Proof. By the a priori estimate (Theorem 6.2.1),

$$|u^\varepsilon(x)| + |Du^\varepsilon(x)| \le C_\delta \qquad \text{for } x \in \overline{\Omega}_\delta. \tag{6.2.7}$$

By the Arzelà–Ascoli theorem, there exists a subsequence $\varepsilon_j \to 0$ and a function $u \in \mathrm{C}(\Omega)$ such that $u^{\varepsilon_j} \to u$ locally uniformly in Ω. From the stability of viscosity solutions, we easily deduce that

$$u(x) + |Du(x)|^p - f(x) = 0 \qquad \text{in } \Omega. \tag{6.2.8}$$

From Lemma 6.2.4, $u^\varepsilon(x) \ge \min_\Omega f$ and $u(x) \ge \min_\Omega f$ for all $x \in \Omega$. Together with (6.2.8), we obtain $|\xi|^p \le \max_\Omega f - \min_\Omega f$ for all $\xi \in D^+u(x)$ and $x \in \Omega$. This implies there exists a constant C_0 such that

$$|u(x) - u(y)| \le C_0\,|x - y| \qquad \text{for all } x, y \in \Omega. \tag{6.2.9}$$

Thus, we can extend u uniquely to $u \in C(\overline{\Omega})$. We use Lemma 6.2.3 to show that u is a supersolution of (PDE$_0$) on $\overline{\Omega}$.

It suffices to show that $u \geq w$ on $\overline{\Omega}$, where $w \in C(\overline{\Omega})$ is the unique solution to (PDE$_0$). For $\delta > 0$, let $u_\delta \in C(\overline{\Omega}_\delta)$ be the unique viscosity solution to

$$\begin{cases} u_\delta(x) + |Du_\delta(x)|^p - f(x) \leq 0 & \text{in } \Omega_\delta, \\ u_\delta(x) + |Du_\delta(x)|^p - f(x) \geq 0 & \text{on } \overline{\Omega}_\delta. \end{cases} \tag{6.2.10}$$

Since $u_\delta \to w$ locally uniformly as $\delta \to 0^+$ (see [75]) and w is bounded, $\{u_\delta\}_{\delta>0}$ is uniformly bounded. Let $v_\delta^\varepsilon \in C^2(\Omega_\delta) \cap C(\overline{\Omega}_\delta)$ be the unique solution to the Dirichlet problem

$$\begin{cases} v_\delta^\varepsilon(x) + \left|Dv_\delta^\varepsilon(x)\right|^p - f(x) = \varepsilon\Delta v_\delta^\varepsilon(x) & \text{in } \Omega_\delta, \\ v_\delta^\varepsilon = u_\delta & \text{on } \partial\Omega_\delta. \end{cases} \tag{6.2.11}$$

It is well known that $v_\delta^\varepsilon \to u_\delta$ uniformly on $\overline{\Omega}_\delta$ as $\varepsilon \to 0$ ([43, 56, 110]).

For δ small enough, $u_\delta \leq u^\varepsilon$ on $\partial\Omega_\delta$. Hence, by the maximum principle, $v_\delta^\varepsilon \leq u^\varepsilon$ on $\overline{\Omega}_\delta$. Now we first let $\varepsilon \to 0$ to obtain $u_\delta \leq u$ on $\overline{\Omega}_\delta$. Then let $\delta \to 0$ to get $w \leq u$ in Ω, which implies $w \leq u$ on $\overline{\Omega}$ since both w, u belong to $C(\overline{\Omega})$. □

6.3 Rate of convergence

In this section, we focus on the rate of convergence for the case where

$$\boxed{f \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \text{ is nonnegative.}}$$

As a consequence, $u^\varepsilon(x), u(x) \geq 0$ for $x \in \Omega$ by Lemma 6.2.4. In our main results, we have an additional assumption that $f = 0$ on $\partial\Omega$.

Before we show any result about the rate of convergence, we would like to mention a lower bound of $u^\varepsilon - u$ and some properties of u from its optimal control formulation.

Theorem 6.3.1. *Let u^ε be the unique solution to (PDE$_\varepsilon$) and u be the unique solution to (PDE$_0$). Then there exists a constant C independent of ε such that*

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \qquad \text{for all } x \in \Omega. \tag{6.3.1}$$

Proof. The proof relies on a well-known rate of convergence for vanishing viscosity of the viscous Hamilton–Jacobi equation with the Dirichlet boundary condition (see [43, 49, 56, 110]). Let $g(x) = u(x)$ for $x \in \partial\Omega$. Let $v^\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique viscosity solution to

$$\begin{cases} v^\varepsilon(x) + |Dv^\varepsilon(x)|^p - f(x) - \varepsilon\Delta v^\varepsilon(x) = 0 & \text{in } \Omega, \\ v^\varepsilon(x) = g(x) & \text{on } \partial\Omega. \end{cases}$$

It is well known that $v^\varepsilon \to u$ ([43, 56, 110]). Furthermore, there exists a positive constant C independent of $\varepsilon \in (0,1)$ such that

$$|v^\varepsilon(x) - u(x)| \leq C\sqrt{\varepsilon} \qquad \text{for } x \in \overline{\Omega}. \tag{6.3.2}$$

By the comparison principle for (PDE$_\varepsilon$), we have

$$v^\varepsilon(x) \le u^\varepsilon(x) \qquad \text{for } x \in \Omega. \tag{6.3.3}$$

From (6.3.2) and (6.3.3), we obtain the lower bound (6.3.1). $\square$

Lemma 6.3.2. *Assume $f \ge 0$ in Ω. Then $u(x) = 0$ if and only if $f(x) = 0$. In particular, $f \equiv 0$ implies $u \equiv 0$.*

Proof. It is clear to see that $f \equiv 0$ implies $u \equiv 0$ by the uniqueness of (PDE$_0$).

It is not hard to prove the converse by contradiction. Suppose $u \equiv 0$ and $f(x_0) > 0$. Then there exists ε, $\delta > 0$ such that $f(x) > \varepsilon$ for all $x \in B_\delta(x_0)$. Let $\eta \in AC([0,\infty); \overline{\Omega})$ such that $\eta(0) = x_0$ and t be the time that η first hits $\partial B_\delta(x_0)$. Note that t could be $+\infty$. Then

$$\begin{aligned}\int_0^\infty e^{-s}\left(|\dot\eta(s)|^q + f(\eta(s))\right) ds &\ge \int_0^t e^{-s}\left(|\dot\eta(s)|^q + f(\eta(s))\right) ds \\ &\ge \frac{1}{e^t t^{q-1}}\left|\int_0^t \dot\eta(s) ds\right|^q + \varepsilon\left(1 - e^{-t}\right) \\ &\ge \frac{\delta^q}{e^t t^{q-1}} + \varepsilon\left(1 - e^{-t}\right),\end{aligned}$$

where we used Jensen's inequality in the second line. This implies $u(x_0) > 0$ since $q \ge 2$, which is a contradiction. $\square$

The following lemma is about a crucial estimate that will be used. It is a refined construction of a supersolution for (PDE$_\varepsilon$).

Lemma 6.3.3. *Let δ_0 be defined as in (6.1.5). There exist positive constants $\nu = \nu(\delta_0) > 1$ and $C_{\delta_0} = \mathcal{O}\left(\delta_0^{-(\alpha+2)}\right)$ such that*

$$w(x) = \begin{cases} \dfrac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + \max f + C_{\delta_0}\varepsilon^{\alpha+2}, & p < 2, \\[2ex] \nu\varepsilon \log\left(\dfrac{1}{d(x)}\right) + \max f + C_{\delta_0}\varepsilon^2, & p = 2, \end{cases} \tag{6.3.4}$$

is a supersolution of (PDE$_\varepsilon$) *in* Ω.

Proof. Let us first consider $1 < p < 2$. Recall from Theorem 6.2.2 that $C_\alpha^p \alpha^p = C_\alpha \alpha(\alpha+1)$ and $p(\alpha+1) = \alpha + 2$. Compute

$$|Dw(x)|^p = \nu^p \frac{(C_\alpha \alpha)^p \varepsilon^{p(\alpha+1)}}{d(x)^{p(\alpha+1)}} |Dd(x)|^p = \nu^p \frac{C_\alpha \alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x)^{\alpha+2}} |Dd(x)|^p$$

and

$$\varepsilon \Delta w(x) = \nu \frac{C_\alpha \alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x)^{\alpha+2}} |Dd(x)|^2 - \nu \frac{C_\alpha \alpha \varepsilon^{\alpha+2} \Delta d(x)}{d(x)^{\alpha+1}}.$$

We have

$$\begin{aligned}\mathcal{L}^\varepsilon[w] =& \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + \max f - f(x) + C_{\delta_0}\varepsilon^{\alpha+2} \\ &+ \frac{C_\alpha \alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x)^{\alpha+2}}\left[\nu^p |Dd(x)|^p - \nu |Dd(x)|^2 + \nu \frac{d(x)\Delta d(x)}{\alpha+1}\right].\end{aligned}$$

Case 1. If $0 < d(x) \le \delta_0$, we have $|Dd(x)| = 1$. Recall that $K_2 = \|\Delta d\|_{L^\infty}$ and observe

$$\left|\frac{d(x)\Delta d(x)}{\alpha+1}\right| \le \frac{\delta_0 \|\Delta d\|_{L^\infty}}{\alpha+1} \le \frac{K_2\delta_0}{\alpha+1} \le K_2\delta_0.$$

Therefore,

$$\nu^p - \nu + \nu\frac{d(x)\Delta d(x)}{(\alpha+1)} \ge \nu^p - \nu - \nu K_2\delta_0 = \nu\Big(\nu^{p-1} - (1+K_2\delta_0)\Big). \tag{6.3.5}$$

We will choose ν as follows. For $\gamma > 1$, we have the inequality

$$\big||x+y|^\gamma - |x|^\gamma\big| \le \gamma\left(|x|+|y|\right)^{\gamma-1}|y| \tag{6.3.6}$$

for $x, y \in \mathbb{R}$, which implies that

$$0 \le (1+K_2\delta_0)^{\alpha+1} - 1 \le \underbrace{(\alpha+1)\left(1+K_2\delta_0\right)^\alpha K_2}_{C_2}\delta_0.$$

Hence, $(1+K_2\delta_0)^{\alpha+1} \le 1 + C_2\delta_0$. Since $\alpha + 1 = \frac{1}{p-1}$,

$$(1+K_2\delta_0) \le (1+C_2\delta_0)^{\frac{1}{\alpha+1}} = (1+C_2\delta_0)^{p-1}. \tag{6.3.7}$$

Choose $\nu = 1 + C_2\delta_0$ in (6.3.5) and we obtain $\mathcal{L}[w] \ge 0$ in $\{x \in \Omega_\delta : \delta < d(x) \le \delta_0\}$.

Case 2. If $d(x) \ge \delta_0$, recall that $K_0 = \|d\|_{L^\infty}$ and $K_1 = \|Dd\|_{L^\infty}$. And we have

$$\begin{aligned}
\mathcal{L}[w] =& \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + \max_\Omega f - f(x) \\
&+ \nu^p \frac{C_\alpha \alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x)^{\alpha+2}}|Dd(x)|^p - \nu\frac{C_\alpha\alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x)^{\alpha+2}}|Dd(x)|^2 \\
&+ \nu\frac{C_\alpha \alpha \varepsilon^{\alpha+2}\Delta d(x)}{d(x)^{\alpha+1}} + C_{\delta_0}\varepsilon^{\alpha+2} \\
\ge& \frac{C_\alpha\alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x)^{\alpha+2}}\left(\nu^p|Dd(x)|^p - \nu|Dd(x)|^2 + \nu\frac{d(x)\Delta d(x)}{\alpha+1}\right) + C_{\delta_0}\varepsilon^{\alpha+2} \\
\ge& \left[C_{\delta_0} - C_3\left(\frac{1}{\delta_0}\right)^{\alpha+2}\right]\varepsilon^{\alpha+2},
\end{aligned}$$

where

$$C_3 = C_\alpha\alpha(\alpha+1)\left(\nu^p K_1^p + \nu K_1^2 + \nu\frac{K_0K_2}{\alpha+1}\right).$$

We can choose $C_{\delta_0} = C_3\delta_0^{-(\alpha+2)}$ to obtain $\mathcal{L}[w] \ge 0$ in $\{x \in \Omega_\delta : d(x) \ge \delta_0\}$.

If $p = 2$, then $\alpha = 0$. We can easily see that the similar calculation holds true with $\nu := 1 + K_2\delta_0$ and $C_{\delta_0} := \delta_0^{-2}\nu(\nu K_1^2 + K_1^2 + K_0K_2)$. □

Now we begin to present the rate of convergence for the special case where $f = C_f$ in Ω for some constant C_f.

Theorem 6.3.4 (Constant data)**.** *Assume $f \equiv C_f$ in Ω. Let u^ε be the unique solution to (PDE$_\varepsilon$) and $u \equiv C_f$ be the unique solution to (PDE$_0$). Then there exists a constant C independent of $\varepsilon \in (0,1)$ such that*

$$
\begin{aligned}
&0 \le u^\varepsilon(x) - u(x) \le C\left(\frac{\varepsilon^{\alpha+1}}{d(x)^\alpha} + \frac{\varepsilon^{\alpha+2}}{\delta_{0,\Omega}^{\alpha+2}}\right), && \text{if } 1 < p < 2,\\
&0 \le u^\varepsilon(x) - u(x) \le C\left(\varepsilon \log\left(\frac{1}{d(x)}\right) + \frac{\varepsilon^2}{\delta_{0,\Omega}^2}\right), && \text{if } p = 2,
\end{aligned}
$$

for $x \in \Omega$, where $\delta_{0,\Omega}$ is defined as in (6.1.5). In particular,

(i) if $1 < p < 2$, we have $C_f \le u^\varepsilon(x) \le C_f + C\varepsilon$ for $x \in \Omega_\varepsilon$, and

(ii) for any $K \subset\subset \Omega$, there holds $\|u^\varepsilon - u\|_{L^\infty(K)} \le C\varepsilon^{\alpha+1}$.

Proof. Lemma 6.3.2 implies $u \equiv C_f$ in Ω. And Lemma 6.2.4 tells us $u^\varepsilon - u = u^\varepsilon - C_f \ge 0$. By the comparison principle of (PDE$_\varepsilon$) and Lemma 6.3.3, the conclusion follows. □

Remark 54. The conclusion of Theorem 6.3.4 also holds if $f = C_f + \mathcal{O}(\varepsilon^\beta)$ for $\beta \ge \alpha + 1$.

Even this special case (Theorem 6.3.4) is new in the literature. As an immediate consequence, we obtain the rate of convergence on any compact subset that is disjoint from the support of f.

Corollary 6.3.5. *Assume f is Lipschitz with compact support and K is a connected compact subset of Ω that is disjoint from* $\mathrm{supp}(f)$. *Then there exists a constant $C = C(K)$ independent of $\varepsilon \in (0,1)$ such that*

$$\|u^\varepsilon - u\|_{L^\infty(K)} \le C\varepsilon^{\alpha+1}.$$

Proof. We choose an open, bounded and connected set U such that ∂U is C^2 and $K \subset\subset U \subset\subset \Omega$. Let w^ε be the solution to (PDE$_\varepsilon$) with Ω replaced by U. Then by Theorem 6.3.4, we have

$$0 \le w^\varepsilon(x) \le C\left(\varepsilon^{\alpha+1} + \varepsilon^{\alpha+2}\right), \qquad x \in K,$$

where C depends on $\mathrm{dist}(K, \partial U)$ and U. Recall that $u = 0$ outside the support of f. By the comparison principle in U, we see that $u^\varepsilon \le w^\varepsilon$ and thus the conclusion follows. □

For the general result of nonnegative compactly supported data, we have the following theorem.

Theorem 6.3.6 (Nonnegative compactly supported data)**.** *Assume that f is nonnegative and Lipschitz with compact support in Ω_κ for some $\kappa > 0$. Let u^ε be the unique solution to (PDE$_\varepsilon$) and u be the unique solution to (PDE$_0$). Then there exists a constant C independent of $\varepsilon \in (0,1)$ and κ such that*

$$- C\sqrt{\varepsilon} \le u^\varepsilon(x) - u(x) \le C\left(\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right) + \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad p < 2, \tag{6.3.8}$$

$$- C\sqrt{\varepsilon} \le u^\varepsilon(x) - u(x) \le C\left(\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{2}\right) + \nu\varepsilon \log\left(\frac{1}{d(x)}\right), \qquad p = 2, \tag{6.3.9}$$

for any $x \in \Omega$. *As a consequence,* $|u^{\varepsilon}(x) - u(x)| \leq C\sqrt{\varepsilon}$ *for all* $x \in \Omega_{\varepsilon}$.

We state the following lemma as a preparation.

Lemma 6.3.7. *Let* $0 < \kappa < \delta_0$ *and* $U_\kappa = \{x \in \Omega : 0 < \operatorname{dist}(x, \partial\Omega) < \kappa\} = \Omega \backslash \overline{\Omega}_\kappa$. *There holds*

$$\operatorname{dist}(x, \partial\Omega_\kappa) = \kappa - \operatorname{dist}(x, \partial\Omega) \qquad \text{for all } x \in U_\kappa.$$

As a consequence, $x \mapsto \operatorname{dist}(x, \partial U_\kappa) = \min\{\operatorname{dist}(x, \partial\Omega_k), \operatorname{dist}(x, \partial\Omega)\}$ *is twice continuously differentiable for* $x \in \Omega \backslash \overline{\Omega}_{\kappa/2}$. *Hence, we can choose*

$$\delta_{0,U_\kappa} \geq \frac{\kappa}{4} \tag{6.3.10}$$

where $\delta_{0,\Omega}$ *is defined as in* (6.1.5).

Proof. By the definition of $\delta_0 = \delta_{0,\Omega}$, we have $d(x) = \operatorname{dist}(x, \partial\Omega)$ is twice continuously differentiable in the region $U_{\delta_0} = \Omega \backslash \overline{\Omega}_{\delta_0}$. The proof follows from [59, p. 355]. □

Proof of Theorem 6.3.6. Without loss of generality, assume that f is supported in Ω_κ where $0 < \kappa < \delta_0$. Let $g_\kappa = u^\varepsilon$ on $\partial\Omega_\kappa$. Then the solution u^ε of (PDE$_\varepsilon$) also solves

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } U_\kappa, \\ \qquad\qquad\qquad\qquad u^\varepsilon(x) = +\infty & \text{on } \partial\Omega, \\ \qquad\qquad\qquad\qquad u^\varepsilon(x) = g_\kappa & \text{on } \partial\Omega_\kappa, \end{cases}$$

in $U_\kappa = \Omega \setminus \overline{\Omega}_\kappa = \{x \in \Omega : 0 < d(x) < \kappa\}$. Let $\tilde{u}^\varepsilon \in C^2(U_\kappa)$ be the solution to the following problem

$$\begin{cases} \tilde{u}^\varepsilon(x) + |D\tilde{u}^\varepsilon(x)|^p - \varepsilon\Delta \tilde{u}^\varepsilon(x) = 0 & \text{in } U_\kappa, \\ \qquad\qquad\qquad\qquad \tilde{u}^\varepsilon(x) = +\infty & \text{on } \partial U_\kappa = \partial\Omega \cup \partial\Omega_\kappa, \end{cases}$$

whose existence is guaranteed by Theorem 6.2.2. Here the boundary condition is understood in the sense that $\tilde{u}^\varepsilon(x) \to \infty$ as $d_\kappa(x) \to 0$, where $d_\kappa(\cdot)$ is the distance function from the boundary of U_κ, i.e.,

$$d_\kappa(x) = \min\{\operatorname{dist}(x, \partial\Omega_\kappa), \operatorname{dist}(x, \partial\Omega)\} \leq d(x) \qquad \text{for } x \in U_\kappa.$$

Since $f = 0$ in $\overline{U}_\kappa$, by Lemma 6.3.2, $u = 0$ in $\overline{U}_\kappa$. Hence, u is also the unique state-constraint solution to

$$\begin{cases} u(x) + |Du(x)|^p = 0 & \text{in } U_\kappa, \\ u(x) + |Du(x)|^p \geq 0 & \text{on } \partial U_\kappa = \partial\Omega \cup \partial\Omega_\kappa. \end{cases}$$

The vanishing viscosity of $\tilde{u}^\varepsilon \to 0$ in U_κ can be quantified by Theorem 6.3.4, which gives us

$$0 \leq \tilde{u}^\varepsilon(x) \leq \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d_\kappa(x)^\alpha} + C_3\left(\frac{\varepsilon}{\delta_{0,U_\kappa}}\right)^{\alpha+2} \qquad \text{for } p < 2,$$

$$0 \leq \tilde{u}^\varepsilon(x) \leq \nu\varepsilon \log\left(\frac{1}{d_\kappa(x)}\right) + C\left(\frac{\varepsilon}{\delta_{0,U_\kappa}}\right)^2 \qquad \text{for } p = 2,$$

for $x \in U_\kappa$. From (6.3.10) and the comparison principle in U_κ, we have

$$0 \le u^\varepsilon(x) \le \tilde{u}^\varepsilon(x) \le \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d_\kappa(x)^\alpha} + C_3 \left(\frac{4\varepsilon}{\kappa}\right)^{\alpha+2} \qquad \text{for } p < 2, \tag{6.3.11}$$

$$0 \le u^\varepsilon(x) \le \tilde{u}^\varepsilon(x) \le \nu\varepsilon \log\left(\frac{1}{d_\kappa(x)}\right) + C\left(\frac{4\varepsilon}{\kappa}\right)^2 \qquad \text{for } p = 2, \tag{6.3.12}$$

for $x \in U_\kappa$. We proceed with the doubling variable method. For $p < 2$, consider the auxiliary functional

$$\Phi(x,y) = u^\varepsilon(x) - u(y) - \frac{C_0 |x-y|^2}{\sigma} - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega},$$

where C_0 is the Lipschitz constant of u from (6.2.9), $\sigma \in (0,1)$. The fact that $d(x)^\alpha u^\varepsilon(x) \to C_\alpha \varepsilon^{\alpha+1}$ as $d(x) \to 0^+$ implies

$$\max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi(x,y) = \Phi(x_\sigma, y_\sigma) \qquad \text{for some } (x_\sigma, y_\sigma) \in \Omega \times \overline{\Omega}.$$

From $\Phi(x_\sigma, y_\sigma) \ge \Phi(x_\sigma, x_\sigma)$, we can deduce that

$$|x_\sigma - y_\sigma| \le \sigma. \tag{6.3.13}$$

If $d(x_\sigma) \ge \frac{1}{2}\kappa$, since $x \mapsto \Phi(x, y_\sigma)$ has a maximum over Ω at $x = x_\sigma$, the subsolution test for $u^\varepsilon(x)$ gives us

$$\begin{aligned} u^\varepsilon(x_\sigma) &+ \left| \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} - \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1} Dd(x_\sigma)}{d(x_\sigma)^{\alpha+1}} \right|^p - f(x_\sigma) \\ &- \varepsilon \left(\frac{2nC_0}{\sigma} + \frac{\nu C_\alpha \alpha(\alpha+1)\varepsilon^{\alpha+1} |Dd(x_\sigma)|^2}{d(x_\sigma)^{\alpha+2}} - \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1} \Delta d(x_\sigma)}{d(x_\sigma)^{\alpha+1}} \right) \le 0. \end{aligned} \tag{6.3.14}$$

Since $y \mapsto \Phi(x_\sigma, y)$ has a maximum over $\overline{\Omega}$ at $y = y_\sigma$, the supersolution test for $u(y)$ gives us

$$u(y_\sigma) + \left| \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} \right|^p - f(y_\sigma) \ge 0. \tag{6.3.15}$$

For simplicity, define

$$\xi_\sigma := \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} \qquad \text{and} \qquad \zeta_\sigma := -\frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1} Dd(x_\sigma)}{d(x_\sigma)^{\alpha+1}}.$$

From (6.3.13) and $d(x_\sigma) \ge \frac{1}{2}\kappa$,

$$|\xi_\sigma| \le 2C_0, \qquad \text{and} \qquad |\zeta_\sigma| \le \nu K_1 C_\alpha \alpha \left(\frac{\varepsilon}{d(x_\sigma)}\right)^{\alpha+1} \le \nu K_1 C_\alpha \alpha \left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+1}.$$

Using the inequality (6.3.6) with $\gamma = p > 1$, we deduce that

$$\begin{aligned}\big||\xi_\sigma + \zeta_\sigma|^p - |\xi_\sigma|^p\big| &\le p\Big(|\xi_\sigma| + |\zeta_\sigma|\Big)^{p-1}|\zeta_\sigma| \\ &\le p\left[2C_0 + \nu K_1 C_\alpha \alpha\left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+1}\right]^{p-1}\nu K_1 C_\alpha \alpha\left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+1}. \qquad (6.3.16)\end{aligned}$$

Combine (6.3.16) together with (6.3.14), (6.3.15) and $|f(x_\sigma) - f(y_\sigma)| \le C|x_\sigma - y_\sigma| \le C\sigma$ to obtain

$$\begin{aligned}u^\varepsilon(x_\sigma) - u(y_\sigma) \le & p\left(2C_0 + \nu K_1 C_\alpha \alpha\left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+1}\right)^{p-1}\nu K_1 C_\alpha \alpha\left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+1} + C\sigma \\ & + 2nC_0\left(\frac{\varepsilon}{\sigma}\right) + \nu K_1^2 C_\alpha \alpha(\alpha+1)\left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+2} + \nu K_2 C_\alpha \alpha\left(\frac{2\varepsilon}{\kappa}\right)^{\alpha+1}\varepsilon \\ \le & C\left[\sigma + \frac{\varepsilon}{\sigma} + \left(1 + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1}\right)^{p-1}\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right].\end{aligned}$$

By the fact that $(1+x)^\gamma \le 1 + x^\gamma$ for $x \in [0,1]$ and $\gamma \in [0,1]$, we know

$$\left(1 + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1}\right)^{p-1} \le 1 + \left(\frac{\varepsilon}{\kappa}\right),$$

as $0 < p - 1 \le 1$. Therefore,

$$u^\varepsilon(x_\sigma) - u(y_\sigma) \le C\left[\sigma + \frac{\varepsilon}{\sigma} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right],$$

where C is independent of κ and ε. Now choose $\sigma = \sqrt{\varepsilon}$ to get (with κ fixed)

$$\Phi(x_\sigma, y_\sigma) \le u^\varepsilon(x_\sigma) - u(y_\sigma) \le C\sqrt{\varepsilon}. \qquad (6.3.17)$$

If $d(x_\sigma) < \frac{1}{2}\kappa$, then $x_\sigma \in U_\kappa$ and furthermore $\mathrm{dist}(x_\sigma, \partial\Omega_\kappa) > \frac{1}{2}\kappa$. Indeed, for any $y \in \partial\Omega$ and $z \in \partial\Omega_k$, we have $|x_\sigma - z| + |x_\sigma - y| \ge |y - z|$. Taking the infimum over all $y \in \partial\Omega$, we deduce that

$$|x_\sigma - z| + d(x_\sigma) \ge \inf_{y \in \partial\Omega}|y - z| = d(z) = \kappa$$

since $z \in \partial\Omega_k = \{x \in \Omega : d(x) = \kappa\}$. Thus, $|x_\sigma - z| \ge \kappa - d(x_\sigma) > \frac{1}{2}\kappa$ for all $z \in \partial\Omega_k$, which implies that $\mathrm{dist}(x_\sigma, \partial\Omega_k) > \frac{1}{2}\kappa$ and hence $d_\kappa(x_\sigma) = d(x_\sigma)$. By (6.3.11) and the fact that $u \ge 0$, we have

$$\Phi(x_\sigma, y_\sigma) \le u^\varepsilon(x_\sigma) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x_\sigma)^\alpha} \le C_3\left(\frac{4\varepsilon}{\kappa}\right)^{\alpha+2}. \qquad (6.3.18)$$

Since $\Phi(x, x) \le \Phi(x_\sigma, y_\sigma)$ for all $x \in \Omega$, we obtain from (6.3.17) and (6.3.18) that

$$u^\varepsilon(x) - u(x) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} \le C\sqrt{\varepsilon} + C_3\left(\frac{4\varepsilon}{\kappa}\right)^{\alpha+2}$$

and thus (6.3.8) follows.

For $p = 2$, we consider instead the functional

$$\Phi(x,y) = u^\varepsilon(x) - u(y) - \frac{C_0 |x-y|^2}{\sigma} - \nu\varepsilon\log\left(\frac{1}{d(x)}\right), \qquad (x,y) \in \overline{\Omega}\times\overline{\Omega}.$$

Similar to the previous case where $1 < p < 2$, the maximum of Φ occurs at some point $(x_\sigma, y_\sigma) \in \Omega \times \overline{\Omega}$ and $|x_\sigma - y_\sigma| \le \sigma$. If $d(x_\sigma) \ge \frac{1}{2}\kappa$, by the subsolution test for $u^\varepsilon(x)$, we have

$$\begin{aligned} u^\varepsilon(x_\sigma) + \left|\frac{2C_0(x_\sigma - y_\sigma)}{\sigma} - \nu\varepsilon\frac{Dd(x_\sigma)}{d(x_\sigma)}\right|^2 - f(x_\sigma) \\ - 2nC_0\left(\frac{\varepsilon}{\sigma}\right) - \nu\,|Dd(x_\sigma)|^2\left(\frac{\varepsilon}{d(x_\sigma)}\right)^2 + \nu\Delta d(x_\sigma)\left(\frac{\varepsilon^2}{d(x_\sigma)}\right) \le 0. \end{aligned} \tag{6.3.19}$$

By the supersolution test for $u(y)$, we have

$$u(y_\sigma) + \left|\frac{2C_0(x_\sigma - y_\sigma)}{\sigma}\right|^2 - f(y_\sigma) \ge 0. \tag{6.3.20}$$

Subtract (6.3.20) from (6.3.19) to get

$$\begin{aligned} u^\varepsilon(x_\sigma) - u(y_\sigma) \le \left(4C_0 + \nu\varepsilon\frac{Dd(x_\sigma)}{d(x_\sigma)}\right)\left(\nu\varepsilon\frac{Dd(x_\sigma)}{d(x_\sigma)}\right) \\ + C\sigma + 2nC_0\left(\frac{\varepsilon}{\sigma}\right) + \nu\,|Dd(x_\sigma)|^2\left(\frac{\varepsilon}{d(x_\sigma)}\right)^2 + \nu\,|\Delta d(x_\sigma)|\,\frac{\varepsilon^2}{d(x_\sigma)}. \end{aligned}$$

Using $d(x_\sigma) \ge \frac{1}{2}\kappa$ and bounds on $d(x)$, we see that

$$\begin{aligned} \Phi(x_\sigma, y_\sigma) &\le u^\varepsilon(x_\sigma) - u(y_\sigma) \\ &\le 4K_1^2\nu(1+\nu)\left(\frac{\varepsilon}{\kappa}\right)^2 + C\sigma + 2nC_0\left(\frac{\varepsilon}{\sigma}\right) + 2\nu(K_2\varepsilon + 4C_0K_1)\left(\frac{\varepsilon}{\kappa}\right) \\ &\le C\left(\sigma + \frac{\varepsilon}{\sigma} + \frac{\varepsilon}{\kappa} + \left(\frac{\varepsilon}{\kappa}\right)^2\right) \le C\sqrt{\varepsilon} \end{aligned} \tag{6.3.21}$$

if we choose $\sigma = \sqrt{\varepsilon}$.

If $d(x_\sigma) < \frac{1}{2}\kappa$, then $x_\sigma \in U_\kappa$. Again, we have $d_\kappa(x_\sigma) = d(x_\sigma)$ and from (6.3.12)

$$\Phi(x_\sigma, y_\sigma) \le u^\varepsilon(x_\sigma) - \nu\varepsilon\log\left(\frac{1}{d(x_\sigma)}\right) \le C\left(\frac{4\varepsilon}{\kappa}\right)^2. \tag{6.3.22}$$

Since $\Phi(x,x) \le \Phi(x_\sigma, y_\sigma)$ for $x \in \Omega$, we obtain from (6.3.21) and (6.3.22) that

$$u^\varepsilon(x) - u(x) - \nu\varepsilon\log\left(\frac{1}{d(x)}\right) \le C\sqrt{\varepsilon} + C\left(\frac{4\varepsilon}{\kappa}\right)^2$$

and thus (6.3.9) follows. $\square$

Remark 55. For general nonnegative Lipschitz data $f \in C(\overline{\Omega})$, it is natural to try a cutoff function argument. Let $\chi_\kappa \in C_c^\infty(\Omega)$ such that $0 \leq \chi_\kappa \leq 1$, $\chi_\kappa = 1$ in $\Omega_{2\kappa}$ and supp $\chi_\kappa \subset \Omega_\kappa$. Let $u_\kappa^\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ solve (PDE$_\varepsilon$) with data $f\chi_\kappa$. Then $u_\kappa^\varepsilon \to u^\varepsilon$ as $\kappa \to 0$ (since $f\chi_\kappa \to f$ in the weak* topology of $L^\infty(\Omega)$ and we have the continuity of the solution to (PDE$_\varepsilon$) with respect to data in this topology [78, Remark II.1]). However, it is not clear at the moment how to quantify this rate of convergence, since $f\chi_\kappa$ does not converge to f in the uniform norm, unless $f = 0$ on $\partial\Omega$.

6.4 A rate for nonnegative zero boundary data

We prove the rate of convergence for the case where f is nonnegative with $f = 0$ on $\partial\Omega$.

Proof of Theorem 6.1.1. Let $L = \|Df\|_{L^\infty(\Omega)}$ be the Lipschitz constant of f. For $\kappa > 0$ small such that $0 < \kappa < \delta_0$ and $x \in \Omega \backslash \Omega_\kappa$, let x_0 be the projection of x onto $\partial\Omega$. We observe that

$$f(x) = f(x) - f(x_0) \leq L\,|x - x_0| = L\kappa. \tag{6.4.1}$$

Define

$$g_\kappa(x) = \begin{cases} 0 & \text{if } 0 \leq d(x) \leq \kappa/2, \\ 2L\,(d(x) - \kappa/2) & \text{if } \kappa/2 \leq d(x) \leq \kappa. \end{cases}$$

It is clear that for $x \in \partial\Omega_\kappa$, $g_\kappa(x) = L\kappa \geq f(x)$ since (6.4.1). Therefore, we can define the following continuous function

$$f_\kappa(x) = \begin{cases} 0 & \text{if } 0 \leq d(x) \leq \kappa/2, \\ \min\{g_\kappa(x), f(x)\} & \text{if } \kappa/2 \leq d(x) \leq \kappa, \\ f(x) & \text{if } \kappa \leq d(x). \end{cases} \tag{6.4.2}$$

A graph of f_κ is given in Figure 6.2. The continuity at $x \in \partial\Omega_\kappa$ comes from the fact that

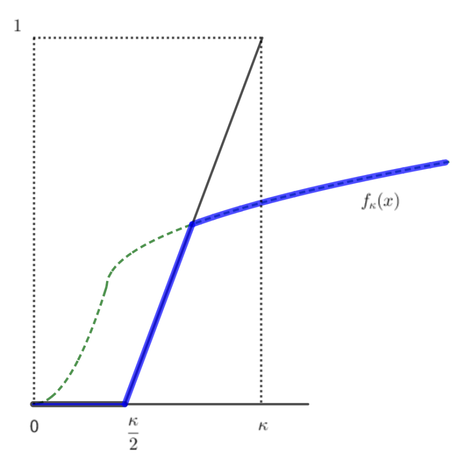

Figure 6.2: Graph of the function f_κ.

when $d(x) = \kappa$, we have $g_k(x) = L\kappa \geq f(x)$ by (6.4.1). It is clear that f_κ is Lipschitz with $\|f_\kappa\|_{L^\infty(\Omega)} \leq L$ as well and $f_\kappa \to f$ uniformly as $\kappa \to 0$. Indeed, we have $0 \leq f_\kappa \leq f$ and

$$0 \leq \max_{x\in\overline{\Omega}}(f(x) - f_\kappa(x)) \leq \max_{x\in\overline{\Omega}\setminus\overline{\Omega}_\kappa}(f(x) - f_\kappa(x)) = \max_{x\in\overline{\Omega}\setminus\overline{\Omega}_\kappa} f(x) \leq L\kappa.$$

Let $u^\varepsilon_\kappa \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solution to (PDE$_\varepsilon$) with data $f\chi_\kappa$ and $u_k \in C(\overline{\Omega})$ be the corresponding solution to (PDE$_0$) with data $f\chi_\kappa$. By the comparison principle ([78, Corollary II.1]), we have

$$0 \leq u^\varepsilon(x) - u^\varepsilon_\kappa(x) \leq L\kappa \qquad \text{for } x \in \Omega. \tag{6.4.3}$$

By the comparison principle for (PDE$_0$), we also have

$$0 \leq u(x) - u_\kappa(x) \leq L\kappa \qquad \text{for } x \in \Omega. \tag{6.4.4}$$

If $1 < p < 2$, by Theorem 6.3.6, there exists a constant C independent of κ such that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon_\kappa(x) - u_\kappa(x) \leq C\left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2} + \frac{\varepsilon^{\alpha+1}}{d(x)^\alpha}\right], \qquad x \in \Omega. \tag{6.4.5}$$

Combining (6.4.3), (6.4.4) and (6.4.5), we obtain

$$\begin{aligned}-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) &= \Big(u^\varepsilon(x) - u^\varepsilon_\kappa(x)\Big) + \Big(u^\varepsilon_\kappa(x) - u_\kappa(x)\Big) + \Big(u_\kappa(x) - u(x)\Big)\\ &\leq L\kappa + C\left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2} + \frac{\varepsilon^{\alpha+1}}{d(x)^\alpha}\right], \qquad x \in \Omega.\end{aligned}$$

Choose $\kappa = \sqrt{\varepsilon}$ and we deduce that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C\sqrt{\varepsilon} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}$$

for $x \in \Omega$. Thus, the conclusion follows.

If $p = 2$, by Theorem 6.3.6, there exists a constant C independent of κ such that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon_\kappa(x) - u_\kappa(x) \leq C\left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^2 + \varepsilon\log\left(\frac{1}{d(x)}\right)\right], \qquad x \in \Omega. \tag{6.4.6}$$

Combining (6.4.3), (6.4.4) and (6.4.6), we obtain

$$\begin{aligned}-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) &= \Big(u^\varepsilon(x) - u^\varepsilon_\kappa(x)\Big) + \Big(u^\varepsilon_\kappa(x) - u_\kappa(x)\Big) + \Big(u_\kappa(x) - u(x)\Big)\\ &\leq L\kappa + C\left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^2 + \varepsilon\log\left(\frac{1}{d(x)}\right)\right], \qquad x \in \Omega.\end{aligned}$$

Choose $\kappa = \varepsilon$ and we deduce that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C\sqrt{\varepsilon} + \varepsilon\log\left(\frac{1}{d(x)}\right)$$

for $x \in \Omega$. Thus, the conclusion follows. □

6.5 Improved one-sided rate of convergence

In this section, we assume $f \in C^2(\overline{\Omega})$ (or uniformly semiconcave in $\overline{\Omega}$) such that $f = 0$ on $\partial\Omega$ and $f \geq 0$. It is known that for the problem on $\mathbb{R}^n$, namely,

$$u(x) + |Du|^p - f(x) = 0 \qquad \text{in } \mathbb{R}^n,$$

if f is semiconcave in the whole space $\mathbb{R}^n$, then the solution u is also semiconcave (Theorem 6.7.1, see also [20]).

Remark 56. The heuristic idea that we will use in this section is the following. Assume that $u^\varepsilon(x) - u(x)$ has a maximum over $\overline{\Omega}$ at some interior point $x_0 \in \Omega$. Then by the equation (PDE_ε) at x_0 and the supersolution test for (PDE_0) at x_0, we obtain

$$\max_{x\in\overline{\Omega}} \Big(u^\varepsilon(x) - u(x)\Big) \leq u^\varepsilon(x_0) - u(x_0) \leq \varepsilon \Delta u^\varepsilon(x_0).$$

If u is uniformly semiconcave in $\overline{\Omega}$, then $\Delta u^\varepsilon(x_0) \leq \Delta u(x_0) \leq C$. Thus, we obtain a better one-sided rate $\mathcal{O}(\varepsilon)$ for $u^\varepsilon - u$. However, there are a couple of problems with this argument. Firstly, as $u^\varepsilon = +\infty$ on $\partial\Omega$, we need to subtract an appropriate term from u^ε to make a maximum over $\overline{\Omega}$ happen in the interior. Secondly, unless $f \in C^2_c(\Omega)$, in general, u is not uniformly semiconcave but only *locally semiconcave*. In this section, we provide estimates on the local semiconcavity constant of u and rigorously show how the upper bound of $u^\varepsilon - u$ can be obtained.

From Lemma 6.3.2, we have $u = 0$ on $\partial\Omega$. It is clear that the solution u to (PDE_0) is also the unique solution to the following Dirichlet boundary problem

$$\begin{cases} u(x) + |Du(x)|^p = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5.1}$$

Since $H(x,\xi) = |\xi|^p - f(x)$, the corresponding Legendre transform is

$$L(x,v) = C_p\,|v|^q + f(x)$$

where $p^{-1} + q^{-1} = 1$ and C_p is defined in Lemma 6.3.2. Let us extend f to a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $\tilde{f}(x) = 0$ for $x \notin \Omega$.

Definition 24. *Define*

$$C^k_0(\overline{\Omega}) = \left\{\varphi \in C^k(\overline{\Omega}) : D^\beta\varphi(x) = 0 \text{ on } \partial\Omega \text{ with } |\beta| \in [0,k]\right\},$$

where β is a multiindex and $|\beta|$ is its order.

We summarize the results about the semiconcavity of u as follows.

Theorem 6.5.1 (Semiconcavity)**.** *Assume $f \geq 0$, $f = 0$ on $\partial\Omega$ and f is uniformly semiconcave in $\overline{\Omega}$ with semiconcavity constant c. Let u be the solution to* (PDE_0).

(i) If $\tilde{f}$ is uniformly semiconcave in $\mathbb{R}^n$, then u is uniformly semiconcave in $\overline{\Omega}$.

(ii) In general, u is locally semiconcave. More specifically, there exists a constant $C > 0$ independent of $x \in \Omega$ such that $\forall x \in \Omega$,

$$u(x+h) - 2u(x) + u(x-h) \leq \frac{C}{d(x)} |h|^2, \tag{6.5.2}$$

$\forall h \in \mathbb{R}^n$ with $|h| \leq M_x$ for some constant M_x that depends on x.

***Remark* 57.** Equation (6.5.2) can be refined as follows (see [60]). For $x \in \Omega$ and a minimizer path $\xi \in \mathrm{AC}([0,\infty); \overline{\Omega})$ such that $\xi(0) = x$, we define

$$T_{x,\xi} = \inf\{t \geq 0 : \xi(t) \in \partial\Omega\}$$

with the standard convention $\inf \emptyset = +\infty$. Then for any $T \in [0, T_{x,\xi}]$ there holds

$$u(x+h) - 2u(x) + u(x-h) \leq \left(1 + \frac{C}{T}\right) |h|^2,$$

where C depends on $\|\dot{\xi}\|$ and $q \geq 2$ (provided that $|h|$ is small enough). The proof remains the same by replacing $T_{x,\xi}$ by T.

The proof of Theorem 6.5.1 is given at the end of this section.

***Remark* 58.** If $f \in \mathrm{C}^2_c(\mathbb{R}^n)$ (or $\mathrm{C}^2_0(\overline{\Omega})$), then f is uniformly semiconcave with semiconcavity constant

$$c = \max\left\{D^2 f(x)\xi \cdot \xi : |\xi| = 1, x \in \mathbb{R}^n\right\} \geq 0. \tag{6.5.3}$$

Also, the condition that $\tilde{f}$ is semiconcave in $\mathbb{R}^n$ holds for $\mathrm{C}^2_c(\Omega)$ and $\mathrm{C}^2_0(\overline{\Omega})$.

The following lemma is a refined version of the local gradient bound in Theorem 6.2.1. We follow [6, Theorem 3.1] where the authors use Bernstein's method inside a doubling variable argument and explicitly keep track of all the dependencies. We refer the reader to [9, 29] and the references therein for related versions of the gradient bound. We believe this result is new in the literature since it is uniform in ε, namely, we give the explicit dependence of the gradient bound on $d(x)$. It also indicates that the boundary layer is a strip of size $\mathcal{O}(\varepsilon)$ from the boundary.

Lemma 6.5.2. *For all ε small enough, there exists a constant C independent of ε such that*

$$|Du^\varepsilon(x)| \leq C\left(1 + \left(\frac{\varepsilon}{d(x)}\right)^{\alpha+1}\right) \qquad \text{for } x \in \Omega. \tag{6.5.4}$$

Proof of Lemma 6.5.2. Fix $x_0 \in \Omega \backslash \Omega_{\delta_0}$. Let $\delta := \frac{1}{4} d(x_0)$ and

$$v(x) := \frac{1}{\delta} u^\varepsilon(x_0 + \delta x), \qquad x \in B(0,2).$$

Then v solves

$$\delta v(x) + |Dv(x)|^p - \tilde{f}(x) - \frac{\varepsilon}{\delta} \Delta v(x) = 0 \qquad \text{in } B(0,2), \tag{6.5.5}$$

where $\tilde{f}(x) := f(x_0 + \delta x)$ on $\overline{B(0,2)}$. Note that $\|\tilde{f}\|_{L^\infty} \le \|f\|_{L^\infty}$ and

$$B(x_0, 2\delta) \subset \Omega_{2\delta} \subset\subset \Omega.$$

By Lemma 6.3.3, there is a constant C independent of δ, ε such that

$$\delta\|v\|_{L^\infty\left(B\left(0,\frac{3}{2}\right)\right)} \le \|u^\varepsilon\|_{L^\infty(\Omega_{2\delta}))} \le C\left(1 + \frac{\varepsilon^{\alpha+1}}{\delta^\alpha}\right).$$

Apply Theorem 3.1 in [6] to obtain

$$\begin{aligned}\sup_{x\in B(0,1)} |Dv(x)| &\le C\left[\left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{p-1}} + \left(\|f\|_{L^\infty} + \delta\|v\|_{L^\infty\left(B\left(0,\frac{3}{2}\right)\right)}\right)^{\frac{1}{p}}\right] \\ &\le C\left[\left(\frac{\varepsilon}{\delta}\right)^{\alpha+1} + \left(1 + \frac{\varepsilon^{\alpha+1}}{\delta^\alpha}\right)^{\frac{\alpha+1}{\alpha+2}}\right] \le C\left(1 + \left(\frac{\varepsilon}{\delta}\right)^{\alpha+1}\right),\end{aligned}$$

where $p = \frac{\alpha+2}{\alpha+1}$ and $\alpha + 1 = \frac{1}{p-1}$. Plugging in $\delta = \frac{1}{4}d(x_0)$, we obtain

$$|Du^\varepsilon(x_0)| = |Dv(0)| \le C\left(1 + \left(\frac{\varepsilon}{d(x_0)}\right)^{\alpha+1}\right).$$

In other words, we have (6.5.4) for all $x \in \Omega\backslash\Omega_{\delta_0}$. On the other hand, from Theorem 6.2.1, there exists a constant C independent of ε such that $|Du^\varepsilon(x)| \le C$ for all $x \in \Omega_{\delta_0}$. Thus, the proof is complete. $\square$

Proof of Theorem 6.1.2. For $1 < p < 2$, we proceed as in the proof of Theorem 6.3.6 to obtain

$$0 \le u^\varepsilon(x) \le \tilde{u}^\varepsilon(x) \le \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d_\kappa(x)^\alpha} + C_3\left(\frac{4\varepsilon}{\kappa}\right)^{\alpha+2} \tag{6.5.6}$$

for $x \in U_\kappa$. Let

$$\psi^\varepsilon(x) := u^\varepsilon(x) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad x \in \Omega,$$

where $\nu > 1$ is chosen as in Lemma 6.3.3. It is clear that $u - \psi^\varepsilon$ has a local minimum at some point $x_0 \in \Omega$ since $\psi^\varepsilon(x) \to -\infty$ as $x \to \partial\Omega$. The normal derivative test gives us

$$D\psi^\varepsilon(x_0) = Du^\varepsilon(x_0) + \nu C_\alpha \alpha\left(\frac{\varepsilon}{d(x_0)}\right)^{\alpha+1} Dd(x_0) \in D^- u(x_0).$$

There are two cases to consider:

- If $d(x_0) < \dfrac{1}{2}\kappa$, then as in the proof of Theorem 6.3.6, $x_0 \in U_\kappa$ and $d_\kappa(x_0) = d(x_0)$. By the definition of x_0, for any $x \in \Omega$, there holds

$$u(x) - \left(u^\varepsilon(x) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}\right) \ge u(x_0) - \left(u^\varepsilon(x_0) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x_0)^\alpha}\right).$$

Therefore,

$$u^\varepsilon(x) - u(x) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} \le \left(u^\varepsilon(x_0) - \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x_0)^\alpha}\right) - u(x_0) \le C_3 \left(\frac{4\varepsilon}{\kappa}\right)^{\alpha+2}$$

thanks to (6.5.6). Thus, in this case

$$u^\varepsilon(x) - u(x) \le \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C_3 \left(\frac{4\varepsilon}{\kappa}\right)^{\alpha+2}, \qquad x \in \Omega.$$

- If $d(x_0) \ge \dfrac{1}{2}\kappa$, from the fact that u is semiconcave in Ω with a linear modulus $c(x)$ as in Theorem 6.5.1, we have

$$D^2\psi^\varepsilon(x_0) \le c(x_0)\, \mathbb{I}_n,$$

where $\mathbb{I}_n$ denotes the identity matrix of size n. This implies that

$$\Delta\psi^\varepsilon(x_0) \le nc(x_0) \le \frac{Cn}{d(x_0)} \le \frac{Cn}{\kappa}. \tag{6.5.7}$$

In other words, we have

$$\varepsilon\Delta u^\varepsilon(x_0) - \frac{\nu C_\alpha \alpha(\alpha+1)\varepsilon^{\alpha+2}}{d(x_0)^{\alpha+2}} |Dd(x_0)|^2 + \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+2}}{d(x_0)^{\alpha+1}} \Delta d(x_0) \le \frac{Cn\varepsilon}{\kappa}.$$

Since $d(x_0) \ge \frac{1}{2}\kappa$, we can further deduce that

$$\varepsilon\Delta u^\varepsilon(x_0) \le \frac{Cn\varepsilon}{\kappa} + \frac{C\varepsilon^{\alpha+2}}{d(x_0)^{\alpha+2}} \le \frac{Cn\varepsilon}{\kappa} + C\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}, \tag{6.5.8}$$

where C is independent of ε. Since $\psi^\varepsilon \in \mathrm{C}^2(\Omega)$, the viscosity supersolution test for u gives us

$$u(x_0) + \left| Du^\varepsilon(x_0) + \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1}}{d(x_0)^{\alpha+1}} Dd(x_0) \right|^p - f(x_0) \ge 0. \tag{6.5.9}$$

On the other hand, since u^ε solves (PDE_ε), we have

$$u^\varepsilon(x_0) + |Du^\varepsilon(x_0)|^p - f(x_0) - \varepsilon\Delta u^\varepsilon(x_0) = 0. \tag{6.5.10}$$

Combine (6.5.9) and (6.5.10) to obtain that

$$u^\varepsilon(x_0) - u(x_0) \le \left| Du^\varepsilon(x_0) + \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1}}{d(x_0)^{\alpha+1}} Dd(x_0) \right|^p - |Du^\varepsilon(x_0)|^p + \varepsilon\Delta u^\varepsilon(x_0). \tag{6.5.11}$$

By Lemma 6.5.2, we can bound $Du^\varepsilon(x_0)$ as

$$|Du^\varepsilon(x_0)| \le C + C\left(\frac{\varepsilon}{d(x_0)}\right)^{\alpha+1} \le C + C\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} \tag{6.5.12}$$

since $d(x_0) \geq \frac{1}{2}\kappa$. We estimate the gradient terms on the right hand side of (6.5.11) using (6.5.12) as follows.

$$\begin{aligned}
&\left|Du^\varepsilon(x_0) + \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1}}{d(x_0)^{\alpha+1}} Dd(x_0)\right|^p - |Du^\varepsilon(x_0)|^p \\
&\qquad \leq p\left(|Du^\varepsilon(x_0)| + \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1}}{d(x_0)^{\alpha+1}}|Dd(x_0)|\right)^{p-1} \frac{\nu C_\alpha \alpha \varepsilon^{\alpha+1}}{d(x_0)^{\alpha+1}}|Dd(x_0)| \\
&\qquad \leq p\left(C + C\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1}\right)^{p-1} C\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} \leq C\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1}\left(1 + \left(\frac{\varepsilon}{\kappa}\right)\right),
\end{aligned}$$

where C is a constant depending only on ν, α, and d. Plugging (6.5.8) and (6.5.13) in the right hand side of (6.5.11), we get

$$u^\varepsilon(x_0) - u(x_0) \leq \frac{Cn\varepsilon}{\kappa} + C\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1}\left(1 + \left(\frac{\varepsilon}{\kappa}\right)\right).$$

Therefore,

$$u^\varepsilon(x) - u(x) \leq \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\left(\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right) + \frac{Cn\varepsilon}{\kappa}, \qquad x \in \Omega.$$

For $p = 2$, the argument is similar. We take $\psi^\varepsilon(x) := u^\varepsilon(x) - \nu\varepsilon \log\left(\frac{1}{d(x)}\right)$ instead and still $u - \psi^\varepsilon$ attains a local minimum at some point $x_0 \in \Omega$. Carrying out the similar computations as in the case of $1 < p < 2$, we have:

- If $d(x_0) < \dfrac{1}{2}\kappa$, then

$$u^\varepsilon(x) - u(x) \leq \nu\varepsilon \log\left(\frac{1}{d(x)}\right) + C\left(\frac{4\varepsilon}{\kappa}\right)^2, \quad x \in \Omega.$$

- If $d(x_0) \geq \dfrac{1}{2}\kappa$, then

$$u^\varepsilon(x) - u(x) \leq \nu\varepsilon \log\left(\frac{1}{d(x)}\right) + C\left(\left(\frac{\varepsilon}{\kappa}\right) + \left(\frac{\varepsilon}{\kappa}\right)^2\right) + \frac{Cn\varepsilon}{\kappa}, \quad x \in \Omega.$$

From these two cases, the conclusion for $p = 2$ follows. □

Remark *59.* If $f \in C^2(\overline{\Omega})$ with $f = 0, Df = 0$ and $D^2 f = 0$ on $\partial\Omega$, then (6.5.7) can be improved to $\Delta\psi^\varepsilon(x_0) \leq nc$ where c is the semiconcavity constant of f, and thus the final estimate becomes

$$u^\varepsilon(x) - u(x) \leq \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\left(\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right) + nc\varepsilon, \qquad x \in \Omega.$$

Remark *60.*

- We only need the local gradient bound in Theorem 6.2.1 to obtain the local rate of convergence $\mathcal{O}(\varepsilon)$ in (6.5.13). However, to make the dependence on κ explicit, we need to bound $Du^{\varepsilon}(x_0)$ as in (6.5.13).
- Another way to get (6.5.12) without using Lemma 6.5.2 (which is true for all $x \in \Omega$) is using the fact that $D\psi^{\varepsilon}(x_0) \in D^{-}u(x_0)$, which implies

$$|D\psi^{\varepsilon}(x_0)| = \left| Du^{\varepsilon}(x_0) + \nu C_{\alpha}\alpha \left(\frac{\varepsilon}{d(x_0)}\right)^{\alpha+1} Dd(x_0) \right| \leq C_0$$

since u is Lipschitz with constant C_0.

Before giving the proof of Corollary 6.1.3, we need to modify the construction of the cutoff function in the proof of Theorem 6.1.1.

Lemma 6.5.3. *Assume $f \in C^2(\overline{\Omega})$ such that $f = 0$ and $Df = 0$ on $\partial\Omega$. For all $\kappa > 0$ small enough, there exists $f_\kappa \in C^2_c(\Omega)$ such that*

$$\|f_\kappa - f\|_{L^\infty(\Omega)} \leq C\kappa \qquad \text{and} \qquad \|D^2 f_\kappa\|_{L^\infty(\Omega)} \leq C$$

where C is independent of κ.

Proof. Choose a smooth function $\chi \in C^\infty(\mathbb{R})$ such that $\chi \geq 0$, $\chi = 0$ if $x \leq 1$, $\chi = 1$ if $x \geq 2$ and $0 \leq \chi' \leq 2$ in $\mathbb{R}$.

For $\kappa > 0$ such that $0 < 2\kappa < \delta_0$ and $x \in \Omega \backslash \Omega_{2\kappa}$, let x_0 be the projection of x onto $\partial\Omega$ and denote by $\nu(x_0)$ the outward unit normal vector at x_0. Write $x = x_0 - d(x)\nu(x_0)$ where $d(x) \leq 2\kappa$. We have

$$f(x) = f(x_0) - Df(x_0)\cdot\nu(x_0)d(x) + \int_0^{d(x)} (d(x)-s)\nu(x_0)\cdot D^2 f(x_0 - s\nu(x_0))\cdot\nu(x_0)ds.$$

Since $f = 0$ and $Df = 0$ on $\partial\Omega$, we deduce that

$$|f(x)| \leq \left(\left\| \frac{1}{2} D^2 f \right\|_{L^\infty(\overline{\Omega})} \right) d(x)^2 \leq C\kappa^2 \qquad \text{and} \qquad |Df(x)| \leq C\kappa \tag{6.5.13}$$

for all $d(x) \leq 2\kappa$. Define

$$f_\kappa(x) = f(x)\chi\left(\frac{d(x)}{\kappa}\right) \qquad \text{for } x \in \overline{\Omega}.$$

It is clear that $0 \leq f_\kappa(x) \leq f(x)$ for all $x \in \overline{\Omega}$ and $f_\kappa(x) = f(x)$ if $d(x) \geq 2\kappa$. Furthermore, we observe that

$$0 \leq \max_{x\in\overline{\Omega}} \left(f(x) - f_\kappa(x)\right) \leq \max_{0\leq d(x)\leq 2\kappa} \left(f(x) - f_\kappa(x)\right) \leq \max_{0\leq d(x)\leq 2\kappa} f(x) \leq C\kappa^2.$$

We have

$$Df_\kappa(x) = Df(x)\chi\left(\frac{d(x)}{\kappa}\right) + f(x)\chi'\left(\frac{d(x)}{\kappa}\right)\frac{Dd(x)}{\kappa}$$

and

$$\begin{aligned} D^2 f_\kappa(x) =& D^2 f(x)\chi\left(\frac{d(x)}{\kappa}\right) + 2\chi'\left(\frac{d(x)}{\kappa}\right)\frac{Df(x)\otimes Dd(x)}{\kappa} \\ &+ f(x)\left(\chi''\left(\frac{d(x)}{\kappa}\right)\frac{Dd(x)\otimes Dd(x)}{\kappa^2} + \chi'\left(\frac{d(x)}{\kappa}\right)\frac{D^2 d(x)}{\kappa}\right) \end{aligned}$$

is uniformly bounded thanks to (6.5.13). □

Proof of Corollary 6.1.3. Let $u_\kappa^\varepsilon \in C^2(\Omega)\cap C(\overline{\Omega})$ be the solution to (PDE$_\varepsilon$) and u_k be the solution to (PDE$_0$) with f replaced by f_k, respectively. It is clear that

$$0 \le u^\varepsilon(x) - u_\kappa^\varepsilon(x) \le C\kappa \qquad \text{for } x\in\Omega$$

and

$$0 \le u(x) - u_\kappa(x) \le C\kappa \qquad \text{for } x\in\Omega.$$

Therefore,

$$u^\varepsilon(x) - u(x) \le 2C\kappa + \left(u_\kappa^\varepsilon(x) - u_\kappa(x)\right). \tag{6.5.14}$$

By Theorem 6.1.2 and Remark 59, as $f_\kappa \in C_c^2(\Omega)$ with a uniform bound on $D^2 f_\kappa$, we have

$$\begin{aligned} u_\kappa^\varepsilon(x) - u_\kappa(x) &\le \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\left(\left(\frac{\varepsilon}{\kappa}\right)^{\alpha+1} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha+2}\right) + 4nC\varepsilon, \qquad p<2, \\ u_\kappa^\varepsilon(x) - u_\kappa(x) &\le \nu\varepsilon\log\left(\frac{1}{d(x)}\right) + C\left(\left(\frac{\varepsilon}{\kappa}\right) + \left(\frac{\varepsilon}{\kappa}\right)^2\right) + 4nC\varepsilon, \qquad p=2 \end{aligned}$$

for some constant C independent of κ. Choose $\kappa = \varepsilon^\gamma$ with $\gamma\in(0,1)$. Then (6.5.14) becomes

$$\begin{aligned} u^\varepsilon(x) - u(x) &\le C\varepsilon^\gamma + C\varepsilon + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha} + C\varepsilon^{(1-\gamma)(\alpha+1)}, \qquad && p<2, \\ u^\varepsilon(x) - u(x) &\le C\varepsilon^\gamma + C\varepsilon + C\varepsilon\left|\log d(x)\right| + C\varepsilon^{1-\gamma}, && p=2. \end{aligned}$$

If $p=2$, then $\gamma = 1/2$ is the best value to choose, which implies the $\mathcal{O}(\sqrt{\varepsilon})$ estimate in Theorem 6.1.1. If $p<2$, by setting $\gamma = (1-\gamma)(\alpha+1)$, we can get the best value of γ, that is,

$$\gamma = \frac{\alpha+1}{\alpha+2} = \frac{1}{p} > \frac{1}{2},$$

and we obtain a better estimate $\mathcal{O}(\varepsilon^{1/p})$. □

Remark 61. If we do not assume $Df = 0$ on $\partial\Omega$, then the best we can get from the above argument is

$$u^\varepsilon(x) - u(x) \le C\varepsilon^\gamma + C\varepsilon^{1-\gamma} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha} + C\varepsilon^{(1-\gamma)(\alpha+1)}, \qquad p<2$$

and we obtain the rate $\mathcal{O}(\varepsilon^{1/2})$ again.

Proof of Theorem 6.5.1.

(i) It is clear that

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega}, \\ 0 & \text{if } x \notin \overline{\Omega}, \end{cases}$$

solves the equation $\tilde{u}(x) + |D\tilde{u}(x)|^p - \tilde{f}(x) = 0$ in $\mathbb{R}^n$. Now we can use a classical doubling variable argument to show that $-D^2u \geq -c\,\mathbb{I}_n$ in $\mathbb{R}^n$ where

$$c = \max\left\{D^2 f(x)\xi \cdot \xi : |\xi| = 1, x \in \mathbb{R}^n\right\} \geq 0.$$

We give the proof of this fact in Section 6.7 for the reader's convenience (see also [20]).

(ii) Fix $x \in \Omega$ and let η be a minimizing curve for $u(x)$. Then

$$u(x) = \int_0^\infty e^{-s}\left(C_q\,|\dot{\eta}(s)|^q + f(\eta(s))\right) ds.$$

Since $\eta(0) = x \in \Omega$, then there exists $T > 0$ such that $\eta(s) \in \Omega, \forall 0 \leq s \leq T$. In fact, we can choose

$$T \geq \frac{d(x)}{C_0}$$

for some constant C_0 independent of x, since $\|\dot{\eta}\|_\infty \leq C$ where C is independent of x. By the dynamic programming principle we have

$$u(x) = \int_0^T e^{-s}\left(C_q\,|\dot{\eta}(s)|^q + f(\eta(s))\right) ds + e^{-T}u(\eta(T)). \tag{6.5.15}$$

Define $\tilde{\eta} : [0, +\infty) \to \mathbb{R}^n$ by

$$\tilde{\eta}(s) := \begin{cases} \eta(s) + \left(1 - \frac{s}{T}\right) h, & \text{if } 0 \leq s \leq T, \\ \eta(T), & \text{if } s \geq T. \end{cases}$$

Choose h small enough so that $\tilde{\eta}(s) \in \Omega, \forall s \geq 0$ (this can be done because there exists $r > 0$ such that $B(\eta(s), r) \subset \Omega$, for all $0 \leq s \leq T$). By the optimal control formula of $u(x+h)$ and $u(x-h)$, we have

$$u(x+h) \leq \int_0^T e^{-s}\left(C_q\left|\dot{\eta}(s) - \frac{h}{T}\right|^q + f\left(\eta(s) + \left(1 - \frac{s}{T}\right)h\right)\right) ds + e^{-T}u(\eta(T)), \tag{6.5.16}$$

and

$$u(x-h) \leq \int_0^T e^{-s}\left(C_q\left|\dot{\eta}(s) + \frac{h}{T}\right|^q + f\left(\eta(s) - \left(1 - \frac{s}{T}\right)h\right)\right) ds + e^{-T}u(\eta(T)). \tag{6.5.17}$$

Hence, from (6.5.15), (6.5.16), and (6.5.17), for h small enough,

$$
\begin{aligned}
&u(x+h)+u(x-h)-2u(x)\\
&\quad\leq \int_0^T e^{-s}C_q\left(\left|\dot{\eta}(s)-\frac{h}{T}\right|^q+\left|\dot{\eta}(s)+\frac{h}{T}\right|^q-2\left|\dot{\eta}(s)\right|^q\right)ds\\
&\quad+\int_0^T e^{-s}\left(f\left(\eta(s)+\left(1-\frac{s}{T}\right)h\right)+f\left(\eta(s)-\left(1-\frac{s}{T}\right)h\right)-2f\left(\eta(s)\right)\right)ds.
\end{aligned}
\tag{6.5.18}
$$

Using the semiconcavity of f, we deduce that

$$
\begin{aligned}
&u(x+h)+u(x-h)-2u(x)\\
&\quad\leq \int_0^T e^{-s}C_q\left(\left|\dot{\eta}(s)-\frac{h}{T}\right|^q+\left|\dot{\eta}(s)+\frac{h}{T}\right|^q-2\left|\dot{\eta}(s)\right|^q\right)ds\\
&\quad+C\left|h\right|^2\int_0^T e^{-s}\left(1-\frac{s}{T}\right)^2 ds.
\end{aligned}
\tag{6.5.19}
$$

By Taylor's theorem, for any $h\in\mathbb{R}^n$ we have

$$|x+h|^q-2|x|^q+|x-h|^q=q(q-1)|h|^2\int_0^1\left(|1+th|^{q-2}+|1-th|^{q-2}\right)(1-t)dt.$$

Thus for h small enough, as $|q|\geq 2$ we obtain

$$|x+h|^q-2|x|^q+|x-h|^q=q(q-1)|h|^2\leq C|h|^2.$$

Therefore

$$\left|\dot{\eta}(s)-\frac{h}{T}\right|^q+\left|\dot{\eta}(s)+\frac{h}{T}\right|^q-2\left|\dot{\eta}\right|^q\leq C\left(\left|\frac{h}{T}\right|^2+\left|\frac{h}{T}\right|^q\right)\leq C\left|\frac{h}{T}\right|^2 \tag{6.5.20}$$

where $q\geq 2$, $C=C(q,\|\dot{\eta}\|_\infty)$, and h is chosen to be small enough so that $\left|\dfrac{h}{T}\right|\leq 1$. Plugging (6.5.20) into (6.5.18), we get

$$
\begin{aligned}
u(x+h)+u(x-h)-2u(x)&\leq C\left|h\right|^2\int_0^T\frac{e^{-s}}{T^2}ds+C\left|h\right|^2\int_0^T e^{-s}\left(1-\frac{s}{T}\right)^2 ds\\
&\leq C\frac{|h|^2}{T}\int_0^1 e^{-sT}ds+C\left|h\right|^2\int_0^T e^{-s}ds\\
&\leq C\left(1+\frac{1}{T}\right)|h|^2\leq C\left(1+\frac{1}{d(x)}\right)|h|^2\leq\frac{C}{d(x)}\left|h\right|^2
\end{aligned}
\tag{6.5.21}
$$

since $T\geq\dfrac{d(x)}{C_0}$.

$\square$

6.6 Future work

In the end, we would like to mention some questions that are worth investigating in the future.

- GENERAL f. As is mentioned earlier, one interesting question is to figure out the rate of convergence for the case of general f where f is not equal to its minimum on the boundary.
- GENERAL H. In our proof, an explicit estimate of the asymptotic behavior of the solution u^ε near the boundary is obtained due to the specific form of Hamiltonian $H(\xi) = |\xi|^p$. We believe that a similar but more technical computation can be done to establish such an estimate of the asymptotic behavior of the solution for Hamiltonian that satisfies
$$\delta^{\frac{p}{p-1}} H\left(\delta^{\frac{-1}{p-1}}\xi\right) = |\xi|^p$$
locally uniformly in ξ as $\delta \to 0$. This condition is mentioned in [104]. For more general Hamiltonian, the question is still open.
- THE CASE $p > 2$. In this case, the solution to the second order state-constraint equation is no longer blowing up near the boundary and we do not know any explicit boundary information, which becomes a main difficulty. In fact, loss of boundary data can happen in this case, that is, the Dirichlet boundary problem may not be solvable for any boundary condition in the classical sense ([11]).

6.7 Property of blow-up solutions

6.7.1 Well-posedness proof

Proof of Theorem 6.2.2. If $p \in (1, 2)$, we use the ansatz $u(x) = C_\varepsilon d(x)^{-\alpha}$ to find a solution to (PDE$_\varepsilon$). Plug the ansatz into (PDE$_\varepsilon$) and compute

$$|Du(x)|^p = \frac{(\alpha C_\varepsilon)^p}{d(x)^{p(\alpha+1)}} |Dd(x)|^p,$$
$$\varepsilon\Delta u(x) = \frac{\varepsilon C_\varepsilon \alpha(\alpha+1)}{d(x)^{\alpha+2}} |Dd(x)|^2 - \frac{\varepsilon C_\varepsilon \alpha}{d(x)^{\alpha+1}} \Delta d(x).$$

Since $|Dd(x)| = 1$ for x near $\partial\Omega$, as $x \to \partial\Omega$, the explosive terms of the highest order are

$$C_\varepsilon^p \alpha^p d^{-(\alpha+1)p} - \varepsilon C_\varepsilon \alpha(\alpha+1) d^{-(\alpha+2)}.$$

Set the above to be zero to obtain that

$$\alpha = \frac{2-p}{p-1} \tag{6.7.1}$$

and

$$C_\varepsilon = \left(\frac{1}{\alpha}(\alpha+1)^{\frac{1}{p-1}}\right) \varepsilon^{\frac{1}{p-1}} = \frac{1}{\alpha}(\alpha+1)^{\alpha+1}\varepsilon^{\alpha+1}.$$

For $0 < \delta < \frac{1}{2}\delta_0$ and η small, define

$$\overline{w}_{\eta,\delta}(x) := \frac{(C_\alpha + \eta)\varepsilon^{\alpha+1}}{(d(x)-\delta)^\alpha} + M_\eta, \qquad x \in \Omega_\delta,$$
$$\underline{w}_{\eta,\delta}(x) := \frac{(C_\alpha - \eta)\varepsilon^{\alpha+1}}{(d(x)+\delta)^\alpha} - M_\eta, \qquad x \in \Omega^\delta,$$

where $C_\alpha := \frac{1}{\alpha}(\alpha+1)^{\alpha+1}$, M_η to be chosen. Next, we show that $\overline{w}_{\eta,\delta}$ is a supersolution of (PDE_ε) in Ω_δ, while $\underline{w}_{\eta,\delta}$ is a subsolution of (PDE_ε) in Ω^δ. Compute

$$\begin{aligned}
\mathcal{L}^\varepsilon\left[\overline{w}_{\eta,\delta}\right](x) =& \frac{(C_\alpha + \eta)\varepsilon^{\alpha+1}}{(d(x)-\delta)^\alpha} + M_\eta + \frac{(C_\alpha+\eta)^p\alpha^p\varepsilon^{\alpha+2}}{(d(x)-\delta)^{\alpha+2}}\left|Dd(x)\right|^p - f(x) \\
& - \frac{(C_\alpha+\eta)\alpha(\alpha+1)\varepsilon^{\alpha+2}}{(d(x)-\delta)^{\alpha+2}}\left|Dd(x)\right|^2 + \frac{(C_\alpha+\eta)\alpha\varepsilon^{\alpha+2}}{(d(x)-\delta)^{\alpha+1}}\Delta d(x) \\
\geq& M_\eta - f(x) \\
& + \underbrace{\frac{\nu C_\alpha\alpha(\alpha+1)\varepsilon^{\alpha+2}}{(d(x)-\delta)^{\alpha+2}}\left[\nu^{p-1}\left|Dd(x)\right|^p - \left|Dd(x)\right|^2 + \frac{(d(x)-\delta)\Delta d(x)}{\alpha+1}\right]}_{I},
\end{aligned}$$

where we use $(C_\alpha\alpha)^p = C_\alpha\alpha(\alpha+1)$ and $\nu = \frac{C_\alpha+\eta}{C_\alpha} \in (1,2)$ for small η. Let

$$\delta_\eta := \frac{\alpha+1}{K_2}\left[\nu^{p-1} - 1\right]$$

and $\delta_\eta \to 0$ as $\eta \to 0$. To get $\mathcal{L}^\varepsilon\left[\overline{w}_{\eta,\delta}\right] \geq 0$, there are two cases to consider, depending on how large $d(x) - \delta$ is.

- If $0 < d(x) - \delta < \delta_\eta < \delta_0$ for η small and fixed, then $|Dd(x)| = 1$, and thus $I \geq 0$. Hence, $\mathcal{L}^\varepsilon\left[\overline{w}_{\eta,\delta}\right] \geq 0$ if we choose $M_\eta \geq \max_{\overline{\Omega}} f$.
- If $d(x) - \delta \geq \delta_\eta$, then

$$I \leq \left(\frac{1}{\delta_\eta}\right)^{\alpha+2}\nu C_\alpha\alpha(\alpha+1)\left[\nu^{p-1}K_1^p + K_1^2 + K_2K_0\right]\varepsilon^{\alpha+2}.$$

  Thus, we can choose $M_\eta = \max_{\overline{\Omega}} f + C\varepsilon^{\alpha+2}$ for C large enough (depending on η) so that $\mathcal{L}^\varepsilon\left[\overline{w}_{\eta,\delta}\right] \geq 0$.

Therefore, $\overline{w}_{\eta,\delta}$ is a supersolution in Ω_δ.

Similarly, we have

$$\begin{aligned}
&\mathcal{L}_\varepsilon\left[\underline{w}_{\eta,\delta}\right](x)\\
=&\frac{(C_\alpha-\eta)\varepsilon^{\alpha+1}}{(d(x)+\delta)^\alpha}-M_\eta+\frac{(C_\alpha-\eta)^p\alpha^p\varepsilon^{\alpha+2}}{(d(x)+\delta)^{\alpha+2}}|Dd(x)|^p-f(x)\\
&-\frac{(C_\alpha-\eta)\alpha(\alpha+1)\varepsilon^{\alpha+2}}{(d(x)+\delta)^{\alpha+2}}|Dd(x)|^2+\frac{(C_\alpha-\eta)\alpha\varepsilon^{\alpha+2}}{(d(x)+\delta)^{\alpha+1}}\Delta d(x)\\
=&-M_\eta-f(x)\\
&+\underbrace{\frac{\nu C_\alpha\alpha(\alpha+1)\varepsilon^{\alpha+2}}{(d(x)+\delta)^{\alpha+2}}\left[\nu^{p-1}|Dd(x)|^p-|Dd(x)|^2+\frac{(d(x)+\delta)\Delta d(x)}{\alpha+1}+\frac{(d(x)+\delta)^2}{\alpha(\alpha+1)\varepsilon}\right]}_{J},
\end{aligned}$$

where $\nu=\frac{C_\alpha-\eta}{C_\alpha}\in(0,1)$ for small η. Let

$$\delta_\eta:=\left(1-\nu^{p-1}\right)\left(\frac{\alpha(\alpha+1)\varepsilon}{1+K_2\alpha\varepsilon}\right)$$

and $\delta_\eta\to 0$ as $\eta\to 0$. To obtain $\mathcal{L}^\varepsilon\left[\underline{w}_{\eta,\delta}\right]\leq 0$, there are two cases to consider depending on how large $d(x)+\delta$ is.

- If $0<d(x)+\delta<\delta_\eta<\delta_0$ for η small and fixed, then $|Dd(x)|=1$, and thus $J\leq 0$. Hence, $\mathcal{L}^\varepsilon\left[\underline{w}_{\eta,\delta}\right]\leq 0$ if we choose $M_\eta\geq -\max_\Omega f$.
- If $d(x)+\delta\geq\delta_\eta$, then

$$|J|\leq\left(\frac{1}{\delta_\eta}\right)^{\alpha+2}\nu C_\alpha\alpha(\alpha+1)\left[\nu^{p-1}K_1^p+K_1^2+\frac{(K_0+1)K_2}{\alpha+1}+\frac{(K_0+1)^2}{\alpha(\alpha+1)\varepsilon}\right]\varepsilon^{\alpha+2}$$

  Thus, we can choose $M_\eta=-\max_{\overline{\Omega}}f-C\varepsilon^{\alpha+2}$ for C large enough (depending on η) so that $\mathcal{L}^\varepsilon\left[\underline{w}_{\eta,\delta}\right]\leq 0$.

Therefore, $\underline{w}_{\eta,\delta}$ is a subsolution in Ω^δ.

For $p=2$, we use the ansatz $u(x)=-C_\varepsilon\log(d(x))$ instead. Similar to the previous case, one can find $u(x)=-\varepsilon\log(d(x))$. For $0<\delta<\frac{1}{2}\delta_0$, define

$$\begin{aligned}
\overline{w}_{\eta,\delta}(x)&=-(1+\eta)\varepsilon\log(d(x)-\delta)+M_\eta, && x\in\Omega_\delta,\\
\underline{w}_{\eta,\delta}(x)&=-(1-\eta)\varepsilon\log(d(x)+\delta)-M_\eta, && x\in\Omega^\delta,
\end{aligned}$$

where M_η is to be chosen so that $\overline{w}_{\eta,\delta}(x)$ is a supersolution in Ω_δ and $\underline{w}_{\eta,\delta}$ is a subsolution in Ω^δ. The computations are omitted here, as they are similar to the previous case.

We divide the rest of the proof into 3 steps. We first construct a minimal solution, then a maximal solution to (PDE$_\varepsilon$), and finally show that they are equal to conclude the existence and the uniqueness of the solution to (PDE$_\varepsilon$).

Step 1. There exists a minimal solution $\underline{u} \in \mathrm{C}^2(\Omega)$ of (PDE$_\varepsilon$) such that $v \geq \underline{u}$ for any other solution $v \in \mathrm{C}^2(\Omega)$ solving (PDE$_\varepsilon$).

Proof. Let $w_{\eta,\delta} \in \mathrm{C}^2(\Omega)$ solve

$$\begin{cases} \mathcal{L}^\varepsilon \left[w_{\eta,\delta}\right] = 0 & \text{in } \Omega, \\ \quad w_{\eta,\delta} = \underline{w}_{\eta,\delta} & \text{on } \partial\Omega. \end{cases} \tag{6.7.2}$$

- Fix $\eta > 0$. As $\delta \to 0^+$, the value of $\underline{w}_{\eta,\delta}$ blows up on the boundary. Therefore, by the standard comparison principle for the second-order elliptic equation with the Dirichlet boundary, $\delta_1 \leq \delta_2$ implies $w_{\eta,\delta_1} \geq w_{\eta,\delta_2}$ on $\overline{\Omega}$.

- For $\delta' > 0$, since $\underline{w}_{\eta,\delta'}$ is a subsolution in $\overline{\Omega}$ with finite boundary,

$$0 < \delta \leq \delta' \qquad \Longrightarrow \qquad \underline{w}_{\eta,\delta'} \leq w_{\eta,\delta'} \leq w_{\eta,\delta} \qquad \text{on } \overline{\Omega}. \tag{6.7.3}$$

- Similarly, since $\overline{w}_{\eta,\delta'}$ is a supersolution on $\Omega_{\delta'}$ with infinity value on the boundary $\partial\Omega_{\delta'}$, by the comparison principle,

$$w_{\eta,\delta} \leq \overline{w}_{\eta,\delta'} \quad \text{in } \Omega_{\delta'} \qquad \Longrightarrow \qquad w_{\eta,\delta} \leq \overline{w}_{\eta,0} \quad \text{in } \Omega. \tag{6.7.4}$$

From (6.7.3) and (6.7.4), we have

$$0 < \delta \leq \delta' \qquad \Longrightarrow \qquad \underline{w}_{\eta,\delta'} \leq w_{\eta,\delta'} \leq w_{\eta,\delta} \leq \overline{w}_{\eta,0} \qquad \text{in } \Omega. \tag{6.7.5}$$

Thus, $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $L^\infty_{\mathrm{loc}}(\Omega)$ ($\{w_{\eta,\delta}\}_{\delta>0}$ is uniformly bounded from below). Using the local gradient estimate for $w_{\eta,\delta}$ solving (6.7.2), we deduce that $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Since $w_{\eta,\delta}$ solves (6.7.2), we further have that $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $W^{2,r}_{\mathrm{loc}}(\Omega)$ for all $r < \infty$ by Calderon-Zygmund estimates.

Local boundedness of $\{w_{\eta,\delta}\}_{\delta>0}$ in $W^{2,r}_{\mathrm{loc}}(\Omega)$ implies weak* compactness, that is, there exists a function $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ such that (via subsequence and monotonicity)

$$w_{\eta,\delta} \rightharpoonup u \qquad \text{weakly in } W^{2,r}_{\mathrm{loc}}(\Omega), \qquad \text{and} \qquad w_{\eta,\delta} \to u \qquad \text{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega).$$

In particular, $w_{\eta,\delta} \to u$ in $\mathrm{C}^1_{\mathrm{loc}}(\Omega)$ thanks to Sobolev compact embedding. Let us rewrite the equation $\mathcal{L}^\varepsilon \left[w_{\eta,\delta}\right] = 0$ as $\varepsilon\Delta w_{\eta,\delta}(x) = F[w_{\eta,\delta}](x)$ for $x \in U \subset\subset \Omega$, where

$$F[w_{\eta,\delta}](x) = w_{\eta,\delta}(x) + H(x, Dw_{\eta,\delta}(x)).$$

Since $w_{\eta,\delta} \to u$ in $\mathrm{C}^1(U)$ as $\delta \to 0$, we have $F[w_{\eta,\delta}](x) \to F(x)$ uniformly in U as $\delta \to 0$, where

$$F(x) = u(x) + H(x, Du(x)).$$

In the limit, we obtain that $u \in L^2(U)$ is a weak solution of $\varepsilon\Delta u = F$ in U where F is continuous. Thus, $u \in \mathrm{C}^2(\Omega)$ and by stability, u solves $\mathcal{L}^\varepsilon[u] = 0$ in Ω. From (6.7.5), we also have

$$\underline{w}_{\eta,0} \leq u \leq \overline{w}_{\eta,0} \qquad \text{in } \Omega.$$

Moreover, $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ with the precise rate like (6.2.5) or (6.2.6). Note that by construction, u may depend on η. But next, we will show that u is independent of η, by proving u is the unique minimal solution of $\mathcal{L}^\varepsilon[u] = 0$ in Ω with $u = +\infty$ on $\partial\Omega$.

Let $v \in C^2(\Omega)$ be a solution to (PDE$_\varepsilon$). Fix $\delta > 0$. Since $v(x) \to \infty$ as $x \to \partial\Omega$ while $w_{\eta,\delta}$ remains bounded on $\partial\Omega$, the comparison principle yields

$$v \geq w_{\eta,\delta} \qquad \text{in } \Omega.$$

Let $\delta \to 0$ and we deduce that $v \geq u$ in Ω. This concludes that u is the minimal solution in $C^2(\Omega)(\forall\, r < \infty)$ and thus u is independent of η. $\square$

Step 2. There exists a maximal solution $\overline{u} \in C^2(\Omega)$ of (PDE$_\varepsilon$) such that $v \leq \overline{u}$ for any other solution $v \in C^2(\Omega)$ solving (PDE$_\varepsilon$).

Proof. For each $\delta > 0$, let $u_\delta \in C^2(\Omega_\delta)$ be the minimal solution to $\mathcal{L}^\varepsilon[u_\delta] = 0$ in Ω_δ with $u_\delta = +\infty$ on $\partial\Omega_\delta$. By the comparison principle, for every $\eta > 0$, there holds

$$\underline{w}_{\eta,\delta} \leq u_\delta \leq \overline{w}_{\eta,\delta} \qquad \text{in } \Omega_\delta,$$

and

$$0 < \delta < \delta' \qquad \Longrightarrow \qquad u_\delta \leq u_\delta' \qquad \text{in } \Omega_{\delta'}.$$

The monotoniciy, together with the local boundedness of $\{u_\delta\}_{\delta>0}$ in $W^{2,r}_{\mathrm{loc}}(\Omega)$, implies that there exists $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ for all $r < \infty$ such that $u_\delta \to u$ strongly in $C^1_{\mathrm{loc}}(\Omega)$. Using the equation $\mathcal{L}^\varepsilon[u_\delta] = 0$ in Ω_δ and the regularity of Laplace's equation, we can further deduce that $u \in C^2(\Omega)$ solves (PDE$_\varepsilon$) and

$$\underline{w}_{\eta,0} \leq u \leq \overline{w}_{\eta,0} \qquad \text{in } \Omega$$

for all $\eta > 0$. As u_δ is independent of η by the previous argument in Step 1, it is clear that u is also independent of η. Now we show that u is the maximal solution of (PDE$_\varepsilon$). Let $v \in C^2(\Omega)$ solve (PDE$_\varepsilon$). Clearly $v \leq u_\delta$ on Ω_δ. Therefore, as $\delta \to 0$, we have $v \leq u$. $\square$

In conclusion, we have found a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ in $C^2(\Omega)$ such that

$$\underline{w}_{\eta,0} \leq \underline{u} \leq \overline{u} \leq \overline{w}_{\eta,0} \qquad \text{in } \Omega \tag{6.7.6}$$

for any $\eta > 0$. This extra parameter η now enables us to show that $\overline{u} = \underline{u}$ in Ω. The key ingredient here is the convexity in the gradient slot of the operator.

Step 3. We have $\overline{u} \equiv \underline{u}$ in Ω. Therefore, the solution to (PDE$_\varepsilon$) in $C^2(\Omega)$ is unique.

Proof. Let $\theta \in (0, 1)$. Define $w_\theta = \theta\overline{u} + (1-\theta)\inf_\Omega f$. It can be verified that w_θ is a subsolution to (PDE$_\varepsilon$). Then one may argue that by the comparison principle,

$$w_\theta = \theta\overline{u} + (1-\theta)\inf_\Omega f \leq \underline{u} \qquad \text{in } \Omega,$$

and conclude that $\overline{u} \leq \underline{u}$ by letting $\theta \to 1$. But we have to be careful here. As they are both explosive solutions, to use the comparison principle, we need to show that $w_\theta \leq \underline{u}$ in a neighborhood of $\partial\Omega$. From (6.7.6), we see that

$$1 \leq \frac{\overline{u}(x)}{\underline{u}(x)} \leq \frac{\overline{w}_{\eta,0}(x)}{\underline{w}_{\eta,0}(x)} = \frac{(C_\alpha + \eta) + M_\eta d(x)^\alpha}{(C_\alpha - \eta) - M_\eta d(x)^\alpha}, \qquad 1 < p < 2,$$
$$1 \leq \frac{\overline{u}(x)}{\underline{u}(x)} \leq \frac{\overline{w}_{\eta,0}(x)}{\underline{w}_{\eta,0}(x)} = \frac{-(1+\eta)\log(d(x)) + M_\eta}{-(1-\eta)\log(d(x)) - M_\eta}, \qquad p = 2,$$

for $x \in \Omega$. Hence,

$$1 \leq \lim_{d(x)\to 0} \left(\frac{\overline{u}(x)}{\underline{u}(x)}\right) \leq \frac{C_\alpha + \eta}{C_\alpha - \eta}, \qquad 1 < p < 2,$$
$$1 \leq \lim_{d(x)\to 0} \left(\frac{\overline{u}(x)}{\underline{u}(x)}\right) \leq \frac{-(1+\eta)}{-(1-\eta)}, \qquad p = 2.$$

Since $\eta > 0$ is chosen arbitrary, we obtain

$$\lim_{d(x)\to 0} \left(\frac{\overline{u}(x)}{\underline{u}(x)}\right) = 1.$$

This means for any $\varsigma \in (0,1)$, there exists $\delta_1(\varsigma) > 0$ small such that

$$\frac{\overline{u}(x)}{\underline{u}(x)} \leq (1+\varsigma) \Longrightarrow \left(\frac{1}{1+\varsigma}\right)\overline{u}(x) \leq \underline{u}(x) \qquad \text{in } \Omega\backslash\Omega_{\delta_1}.$$

For a fixed $\theta \in (0,1)$, one can always choose ς small enough so that $(1+\varsigma)^{-1} \geq \dfrac{1+\theta}{2}$. Since $\overline{u}(x) \to +\infty$ as $d(x) \to 0$, there exists $\delta_2 > 0$ such that $\overline{u}(x) \geq 2\inf_\Omega f$ for all $x \in \Omega \setminus \Omega_{\delta_2}$. Now we have

$$\underline{u}(x) \geq \left(\frac{1}{1+\varsigma}\right)\overline{u}(x) \geq \theta\overline{u}(x) + \left(\frac{1-\theta}{2}\right)\overline{u}(x) \geq \theta\overline{u}(x) + (1-\theta)\left(\inf_\Omega f\right)$$

for all $x \in \Omega \setminus \Omega_\delta$ where $\delta := \min\{\delta_1, \delta_2\}$. This implies for any fixed $\theta \in (0,1)$, $w_\theta \leq \underline{u}$ in a neighborhood of $\partial\Omega$. Hence, by the comparison principle,

$$w_\theta = \theta\overline{u} + (1-\theta)\inf_\Omega f \leq \underline{u} \qquad \text{in } \Omega,$$

for any $\theta \in (0,1)$. Then let $\theta \to 1$ to get the conclusion. □

This finishes the proof of the well-posedness of (PDE_ε) for $1 < p \leq 2$. □

Proof of Lemma 6.2.3. The proof is a variation of Perron's method (see [30]) and we proceed by contradiction. Let $\varphi \in C(\overline{\Omega})$ and $x_0 \in \overline{\Omega}$ such that $u(x_0) = \varphi(x_0)$ and $u - \varphi$ has a global strict minimum over $\overline{\Omega}$ at x_0 with

$$\varphi(x_0) + H(x_0, D\varphi(x_0)) < 0. \tag{6.7.7}$$

Let $\varphi^\varepsilon(x) = \varphi(x) - |x - x_0|^2 + \varepsilon$ for $x \in \overline{\Omega}$. Let $\delta > 0$. We see that for $x \in \partial B(x_0, \delta) \cap \overline{\Omega}$,

$$\varphi^\varepsilon(x) = \varphi(x) - \delta^2 + \varepsilon \leq \varphi(x) - \varepsilon$$

if $2\varepsilon \leq \delta^2$. We observe that

$$\begin{aligned} \varphi^\varepsilon(x) - \varphi(x_0) &= \varphi(x) - \varphi(x_0) + \varepsilon - |x - x_0|^2 \\ D\varphi^\varepsilon(x) - D\varphi(x_0) &= D\varphi(x) - D\varphi(x_0) - 2(x - x_0) \end{aligned}$$

for $x \in B(x_0, \delta) \cap \overline{\Omega}$. By the continuity of $H(x,p)$ near $(x_0, D\varphi(x_0))$ and the fact that $\varphi \in \mathrm{C}^1(\overline{\Omega})$, we can deduce from (6.7.7) that if δ is small enough and $0 < 2\varepsilon < \delta^2$, then

$$\varphi^\varepsilon(x) + H(x, D\varphi^\varepsilon(x)) < 0 \qquad \text{for } x \in B(x_0, \delta) \cap \overline{\Omega}. \tag{6.7.8}$$

We have found $\varphi^\varepsilon \in \mathrm{C}^1(\overline{\Omega})$ such that $\varphi^\varepsilon(x_0) > u(x_0)$, $\varphi^\varepsilon < u$ on $\partial B(x_0, \delta) \cap \overline{\Omega}$ and (6.7.8). Let

$$\tilde{u}(x) = \begin{cases} \max\{u(x), \varphi^\varepsilon(x)\} & x \in B(x_0, \delta) \cap \overline{\Omega}, \\ u(x) & x \notin B(x_0, \delta) \cap \overline{\Omega}. \end{cases}$$

We see that $\tilde{u} \in \mathrm{C}(\overline{\Omega})$ is a subsolution of (PDE_0) in Ω with $\tilde{u}(x_0) > u(x_0)$, which is a contradiction. Thus, u is a supersolution of (PDE_0) on $\overline{\Omega}$. $\square$

6.7.2 Semiconcavity

We present a proof for the semiconcavity of solution to first-order Hamilton–Jacobi equation using the doubling variable method (see also [20]).

Theorem 6.7.1. *Let $H(x,p) = G(p) - f(x)$ where $G \geq 0$ with $G(0) = 0$ is a convex function from $\mathbb{R}^n \to \mathbb{R}^n$ and $f \in \mathrm{C}_c^2(\mathbb{R}^n)$. Let $u \in \mathrm{C}_c(\mathbb{R}^n)$ be a viscosity solution to $u + H(x, Du) = 0$ in $\mathbb{R}^n$. Then u is semiconcave, i.e., u is a viscosity solution of $-D^2 u \geq -c\, \mathbb{I}_n$ in $\mathbb{R}^n$ where*

$$c = \max\{D_{\xi\xi} f(x) : |\xi| = 1, x \in \mathbb{R}^n\} \geq 0.$$

Proof. Consider the auxiliary functional

$$\Phi(x,y,z) = u(x) - 2u(y) + u(z) - \frac{\alpha}{2}|x - 2y + z|^2 - \frac{c}{2}|y - x|^2 - \frac{c}{2}|y - z|^2$$

for $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. By the a priori estimate, u is bounded and Lipschitz. Thus, we can assume Φ achieves its maximum over $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ at $(x_\alpha, y_\alpha, z_\alpha)$. The viscosity solution tests give us

$$\begin{aligned} &u(x_\alpha) + G\big(p_\alpha + c(x_\alpha - y_\alpha)\big) \leq f(x_\alpha) \\ &u(z_\alpha) + G\big(p_\alpha + c(z_\alpha - y_\alpha)\big) \leq f(z_\alpha) \\ &u(y_\alpha) + G\left(p_\alpha + \frac{c}{2}(x_\alpha - y_\alpha) + \frac{c}{2}(z_\alpha - y_\alpha)\right) \geq f(y_\alpha), \end{aligned}$$

where $p_\alpha = \alpha(x_\alpha - 2y_\alpha + z_\alpha)$. By the convexity of G, we have

$$2G\left(p_\alpha + \frac{c}{2}(x_\alpha - y_\alpha) + \frac{c}{2}(z_\alpha - y_\alpha)\right) \leq G\big(p_\alpha + c(x_\alpha - y_\alpha)\big) + G\big(p_\alpha + c(z_\alpha - y_\alpha)\big)$$

Therefore,

$$u(x_\alpha) - 2u(y_\alpha) + u(z_\alpha) \leq f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha).$$

- $\Phi(x_\alpha, y_\alpha, z_\alpha) \geq \Phi(0,0,0)$ gives us

$$\frac{\alpha}{2}|x_\alpha - 2y_\alpha + z_\alpha|^2 + \frac{c}{2}|y_\alpha - x_\alpha|^2 + \frac{c}{2}|y_\alpha - z_\alpha|^2 \leq C.$$

  Thus, $(x_\alpha - y_\alpha) \to h_0$ and $(y_\alpha - z_\alpha) \to h_0$ as $\alpha \to \infty$ for some $h_0 \in \mathbb{R}^n$.

- $\Phi(x_\alpha, y_\alpha, z_\alpha) \geq \Phi(y_\alpha + h_0, y_\alpha, y_\alpha - h_0)$ gives us

$$\begin{aligned} u(x_\alpha) - 2u(y_\alpha) + u(z_\alpha) - \frac{\alpha}{2}|x_\alpha - 2y_\alpha + z_\alpha|^2 - \frac{c}{2}|x_\alpha - y_\alpha|^2 - \frac{c}{2}|y_\alpha - z_\alpha|^2 \\ \geq u(y_\alpha + h_0) - 2u(y_\alpha) + u(y_\alpha - h_0) - c|h_0|^2. \end{aligned}$$

  Therefore, by the fact that u is Lipschitz, we have

$$\begin{aligned} \frac{\alpha}{2}|x_\alpha - 2y_\alpha + z_\alpha|^2 \leq & c\left(\frac{2|h_0|^2 - |x_\alpha - y_\alpha|^2 - |y_\alpha - z_\alpha|^2}{2}\right) \\ & + C\Big(|(x_\alpha - y_\alpha) - h_0| + |(z_\alpha - y_\alpha) + h_0|\Big) \to 0 \end{aligned}$$

  as $\alpha \to \infty$.

For any $x \in \mathbb{R}^n$, we have $\Phi(x_\alpha, y_\alpha, z_\alpha) \geq \Phi(x+h, x, x-h)$, i.e.,

$$\begin{aligned} & u(x+h) - 2u(x) + u(x-h) - c|h|^2 \\ \leq & f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha) \\ & - \frac{\alpha}{2}|x_\alpha - 2y_\alpha + z_\alpha|^2 - \frac{c}{2}|y_\alpha - x_\alpha|^2 - \frac{c}{2}|y_\alpha - z_\alpha|^2. \end{aligned}$$

If $\{y_\alpha\}$ is unbounded, then since $f \in C_c^2(\mathbb{R}^n)$, we have $f(y_\alpha) \to 0$ as $\alpha \to \infty$. As a consequence, $x_\alpha, z_\alpha \to \infty$ as well and thus $f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha) \to 0$ as $\alpha \to \infty$. Therefore,

$$u(x+h) - 2u(x) + u(x-h) - c|h|^2 \leq 0.$$

If $\{y_\alpha\}$ is bounded, then $y_\alpha \to y_0$ for some $y_0 \in \mathbb{R}^n$ as $\alpha \to \infty$. Thus,

$$u(x+h) - 2u(x) + u(x-h) - c|h|^2 \leq f(y_0 + h_0) - 2f(y_0) + f(y_0 - h_0) - c|h_0|^2.$$

Let $\xi = h_0$ and we have

$$\begin{cases} f(y_0 + h_0) - f(y_0) = \displaystyle\int_0^1 D_x f(y_0 + t\xi) \cdot \xi dt, \\ f(y_0) - f(y_0 - h_0) = \displaystyle\int_0^1 D_x f(y_0 - \xi + t\xi) \cdot \xi dt. \end{cases}$$

Therefore,

$$\begin{aligned} f(y_0 + h_0) - 2f(y_0) + f(y_0 - h_0) &= \int_0^1 \Big(D_x f(y_0 + t\xi) - D_x f(y_0 - \xi + t\xi)\Big) \cdot \xi dt \\ &= \int_0^1 \int_0^1 \xi^{\mathsf{T}} D^2 f(y_0 - \xi + t\xi + s\xi)\xi \, ds dt. \end{aligned}$$

which implies

$$|f(y_0 + h_0) - 2f(y_0) + f(y_0 - h_0)| \leq \left(\max_{|\xi|=1} D_{\xi\xi} f \right) |\xi|^2 .$$

Hence,

$$u(x + h) - 2u(x) + u(x - h) - c\,|h|^2 \leq 0$$

and thus u is semiconcave. It is easy to see that if φ is smooth and $u - \varphi$ has a local min at x, then $D^2\varphi(x) \leq c\ \mathbb{I}$, i.e., $-D^2\varphi(x) \geq -c\ \mathbb{I}$. □

Appendix A

Notations

- $\mathbb{M}^n$: the set of all real $n \times n$ matrices.
- $\mathbb{S}^n$: the set of all real $n \times n$ matrices that are symmetric that have determinant equal to 1.
- $A \prec 0$ where $A \in \mathbb{M}^n$: the matrix A is negative definite, i.e., all of eigenvalues of A are negative. We also write $A \preceq 0$ to denote that the matrix A is nonpositive (or semi-negative) definite, i.e., all of eigenvalues of A are nonpositive. The same definition is given for $A \succ 0$ or $A \succeq 0$.
- $\mathrm{USC}(\mathcal{O})$: the set of all real-valued functions that are upper semicontinuous at all points in $\mathcal{O}$, i.e.,
$$\limsup_{\mathcal{O}\ni y\to x} u(y) \le u(x) \qquad \text{for all } x \in \mathcal{O}.$$
- $\mathrm{LSC}(\mathcal{O})$: the set of all real-valued functions that are lower semicontinuous at all points in $\mathcal{O}$, i.e.,
$$\liminf_{\mathcal{O}\ni y\to x} u(y) \ge u(x) \qquad \text{for all } x \in \mathcal{O}.$$
- $\mathrm{BUC}(\mathcal{O})$: the space of all bounded, uniformly continuous functions from $\mathbf{O}$ to $\mathbb{R}$.
- $x \cdot y$ or $\langle x, y\rangle$ for $x, y \in \mathbb{R}^n$: the dot product, i.e., if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then
$$x \cdot y = \langle x, y\rangle = x_1y_1 + \ldots + x_ny_n.$$

Appendix B

Assumptions

We list here some assumptions that are commonly used and referred to during the thesis.

Assumptions on the Hamiltonian

($\mathcal{H}_0$) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$.

($\mathcal{H}_1$) There exists $C_1 > 0$ such that $H(x,p) \geq -C_1$ for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$.

($\mathcal{H}_2$) There exists $C_2 > 0$ such that $|H(x,0)| \leq C_2$ for all $x \in \overline{\Omega}$.

($\mathcal{H}_3$) For each $R > 0$ there exists a constant C_R such that

$$\begin{cases} |H(x,p) - H(y,p)| \leq C_R|x-y|, \\ |H(x,p) - H(x,q)| \leq C_R|p-q|, \end{cases} \tag{B.0.1}$$

for $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$ with $|p|, |q| \leq R$.

($\mathcal{H}_4$) H satisfies the coercivity assumption

$$\lim_{|p|\to\infty} \left(\inf_{x\in\overline{\Omega}} H(x,p) \right) = +\infty. \tag{B.0.2}$$

($\mathcal{H}_5$) $p \mapsto H(x,p)$ is convex for each $x \in \overline{\Omega}$.

($\mathcal{H}_6$) $p \mapsto H(x,p)$ is superlinear uniformly for $x \in \overline{\Omega}$, that is,

$$\lim_{|p|\to\infty} \left(\inf_{x\in\Omega} \frac{H(x,p)}{|p|} \right) = +\infty. \tag{B.0.3}$$

($\mathcal{H}_7$) There exist some positive constants A, B such that

$$A^{-1}|v|^2 - B \leq H(x,p) \leq A|v|^2 + B \qquad \text{for } (x,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{H7}$$

($\mathcal{H}_8$) For $v \in \mathbb{R}^n$, $x \mapsto L(x,v)$ is continuously differentiable on $\overline{U}$, where the Lagrangian L of H is defined as

$$L(x,v) = \sup_{p\in\mathbb{R}^n} \Big(p\cdot v - H(x,p) \Big), \qquad (x,v) \in \overline{U} \times \mathbb{R}^n.$$

Assumptions on the Lagrangian

($\mathcal{L}_1$) $L(x,0) \le C_1$ for all $x \in \overline{\Omega}$;

($\mathcal{L}_2$) $L(x,v) \ge -C_1$ for all $(x,v) \in \overline{\Omega} \times \mathbb{R}^n$;

($\mathcal{L}_3$) For each $R > 0$ there exists a modulus $\tilde{\omega}_R(\cdot)$ such that

$$|L(x,v) - L(y,v)| \le \tilde{\omega}_R(|x-y|) \qquad \text{for all } x, y \in \overline{\Omega}, |v| \le R.$$

($\mathcal{L}_5$) $p \mapsto H(x,p)$ is convex for each $x \in \overline{\Omega}$.

($\mathcal{L}_6$) $p \mapsto H(x,p)$ is superlinear uniformly for $x \in \overline{\Omega}$, that is,

$$\lim_{|p|\to\infty} \left(\inf_{x\in\Omega} \frac{H(x,p)}{|p|} \right) = +\infty. \tag{B.0.4}$$

($\mathcal{L}_7$) There exist some positive constants A, B such that

$$A^{-1}|v|^2 - B \le H(x,p) \le A|v|^2 + B \qquad \text{for } (x,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{B.0.5}$$

($\mathcal{L}_8$) $\mathrm{T}(x,v) \mapsto L(x,v)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$.

Assumptions on the regularity of the domain

($\mathcal{A}_1$) Ω a bounded star-shaped (with respect to the origin) open subset of $\mathbb{R}^n$ and there exists some $\kappa > 0$ such that $\operatorname{dist}(x, \overline{\Omega}) \ge \kappa r$ for all $x \in (1+r)\partial\Omega$ and $r > 0$.

($\mathcal{A}_2$) There exists a universal pair of positive numbers (r,h) and $\eta \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R}^n)$ such that

$$B(x + t\eta(x), rt) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0,h].$$

($\mathcal{A}_3$) There exists $\sigma \in (0,1)$, a universal pair of positive numbers (r,h), and $\eta \in \mathrm{BUC}(\overline{\Omega}; \mathbb{R}^n)$ such that

$$B(x + t\eta(x), rt^{\sigma}) \subset \Omega \qquad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0,h].$$

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