Solutions to some Applied Math Qual Questions From 2010 to 2017

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January 2010

Problem (Jan 2010 - 4/6). Solve $\frac{\partial u}{\partial t} + (1+x)\frac{\partial u}{\partial x} = 0$ for $x > 0$, $t > 0$, with $u(0,x) = f(x)$ and $u(t,0) = g(t)$.

Proof. Using method of characteristic, we consider $x'(t) = 1 + x(t)$ with $x(t_0) = x_0$, for $x_0 > 0$ and $t_0 > 0$. We have $x(t) = (x_0+1)e^{t-t_0} - 1$. Thus along $(t, x(t))$ we have $u(t,x(t)) = u(t_0,x_0)$ for $t > 0$. Thus

$$u(t_0,x_0) = u\left(t,(x_0+1)e^{t-t_0} - 1\right).$$

- First case, $(x_0+1)e^{-t_0} - 1 > 0$ i.e $x_0 > e^{t_0} - 1$ then

$$u(t_0,x_0) = u\left(0,(x_0+1)e^{-t_0} - 1\right) = f\left((x_0+1)e^{-t_0} - 1\right).$$

- Second case, $(x_0+1)e^{-t_0} - 1 < 0$, then observe that $\zeta(t) = (x_0+1)e^{t-t_0} - 1$ has $\zeta(0) < 0$ and $\zeta(t_0) = x_0$, also ζ is increasing, thus there exists a unique $t* > 0$ such that

$$(x_0+1)e^{t^*-t_0} - 1 = 0 \qquad \Longrightarrow \qquad t^* = \ln\left(\frac{e^{t_0}}{x_0+1}\right) > 0.$$

Then

$$u(t_0,x_0) = u\left(t^*,(x_0+1)e^{t^*-t_0} - 1\right) = u(t^*,0) = g(t^*) = g\left(\ln\left(\frac{e^{t_0}}{x_0+1}\right)\right).$$

On the curve $x_0 = e^{t_0} - 1$, if $\lim_{t\longrightarrow 0^+} g(0) = \lim_{x\longrightarrow 0^+} f(0)$ then $u(t_0,x_0)$ will be the common value. If $\lim_{t\longrightarrow 0^+} g(0) \neq \lim_{x\longrightarrow 0^+} f(0)$ then $u(t_0,x_0)$ then the solution $u(t_0,x_0)$ at these points is undefined. □

August 2011

Problem (Aug 2011 - 1/6). The radius $r(t)$ of a cloud droplet grows in time approximately as $dr/dt = s/r$. Here $s > 0$ is taken to be a positive constant that represents the "supersaturation," i.e., the extent to which the ambient air is above the saturation point (100% humidity). A population of cloud droplets can be represented by a density function $f(r,t)$, which gives the number of droplets with radius between r and $r + dr$, at time t. The density $f(r,t)$ then evolves according to

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial r}\left(\frac{s}{r}f\right) = 0, \qquad f(0,r) = f_0(r).$$

Find a formula for $f(t,r)$ in terms of the initial profile $f_0(r)$. Assume $f_0(r)$ vanishes rapidly as $r \longrightarrow 0$ and $r \longrightarrow \infty$.

Proof. We write the equation in the form

$$f_t + \frac{s}{r} f_r = \frac{s}{r^2} f.$$

The characteristic curve started at $(0, r_0)$ is

$$\begin{cases} r'(t) & = \frac{s}{r(t)} \\ r(0) & = r_0 \end{cases} \qquad \text{and} \qquad \begin{cases} z'(t) & = \frac{s}{r(t)^2} z(t) \\ z(0) & = f_0(r_0). \end{cases}$$

where $z(t) = f(t, r(t))$. Solve them we have $x(t) = \sqrt{r_0^2 + 2st}$, and

$$\Big(\ln z(t)\Big)' = \frac{s}{r_0^2 + 2st} = \left(\frac{\ln(r_0^2 + 2st)}{2}\right)' \qquad \Longrightarrow \qquad f(t, r(t)) = z(t) = \frac{f_0(r_0)}{|r_0|} \sqrt{r_0^2 + 2st}.$$

Using the fact that $r_0 = \sqrt{|r(t)^2 - 2st|}$ we obtain the general formula for $f(t, r)$ as

$$f(t, r) = \frac{r}{\left|\sqrt{|r(t)^2 - 2st|}\right|} f_0\left(\sqrt{|r^2 - 2st|}\right).$$

□

Problem (Aug 2011 - 2/6). Determine the exact entropy solution to Burger's equation $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ for all $t > 0$ with initial data:

$$u(0, x) = \begin{cases} 1 & \text{for} \quad x < -1 \\ 0 & \text{for} \quad -1 < x < 1 \\ -1 & \text{for} \quad 1 < x. \end{cases}$$

Proof. There will be two shocks moving as time goes (small time after $t = 0$). Let $\sigma_1(t)$ and $\sigma_2(t)$ denote the shocks positions started from $x = -1$ and $x = 1$ respectively. Using the Rakine-Hugoniot jump condition, we have the speed of the shocks are

$$\sigma_1'(t) = \frac{\frac{1}{2}.1^2 - \frac{1}{2}.0^2}{1 - 0} = \frac{1}{2} \qquad \text{and} \qquad \sigma_2'(t) = \frac{\frac{1}{2}.0^2 - \frac{1}{2}.(-1)^2}{0 - (-1)} = \frac{-1}{2}.$$

Since $\sigma_1(0) = -1$ and $\sigma_2(0) = 1$, we have $\sigma_1(t) = -1 + \frac{t}{2}$ and $\sigma_2(t) = 1 - \frac{t}{2}$. At time $t = 2$, the two shocks meet at $x = 0$ and the speed of the only one shock now becomes 0, thus after time $t = 2$, there will be one shock moving at speed 0. The exact entropy solution can be written as

$$u(t, x) = \left[\sigma\left(-x - 1 + \frac{t}{2}\right) - \sigma\left(x - 1 + \frac{t}{2}\right)\right]\sigma(2 - t) + \sigma(t - 2)\text{sign}(x).$$

□

Problem (Aug 2011 - 4/6). Solve

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + Q(t, x), \qquad 0 \le x \le L, t \ge 0$$

with the conditions

$$u(0, x) = f(x), \qquad u(t, 0) = A(t), \qquad u(t, L) = B(t).$$

Proof. Let

$$v(t, x) = u(t, x) + A(t)\left(\frac{x}{L} - 1\right) - B(t)\frac{x}{L}$$

then $v(t, x)$ solves

$$v_t - \gamma v_{xx} = \underbrace{Q(t, x) + \frac{A'(t) - B'(t)}{L} x - A'(t)}_{\tilde{Q}(t,x)}.$$

with the conditions

$$v(0,x) = \underbrace{f(x) + \left(\frac{A'(0)-B'(0)}{L}\right)x - A'(0)}_{F(x)}, \qquad v(t,0) = v(t,L) = 0. \tag{1}$$

We seek solution $v(t,x)$ in the form $v = U + V$ where

$$\begin{cases} U_t - \gamma U_{xx} = 0 \\ U(t,0) = U(t,L) = 0 \\ U(0,x) = F(x) \end{cases} \quad \text{and} \quad \begin{cases} V_t - \gamma V_{xx} = \tilde{Q}(t,x) \\ V(t,0) = V(t,L) = 0 \\ V(0,x) = 0. \end{cases} \tag{2}$$

For the homogeneous problem $U(t,x)$ we can use separation of variable, let's assume $U(t,x) = e^{-\lambda t}w(x)$ and then substitute to the equation we obtain

$$w''(x) + \frac{\lambda}{\gamma}w(x) = 0, \qquad w(0) = w(L) = 0.$$

The only possible case is $\frac{\lambda}{\gamma} = \omega^2 > 0$. We then obtain $\omega_n = \frac{n\pi}{L}$ and $\lambda_n = \gamma\left(\frac{n\pi}{L}\right)^2$ where $n = 1, 2, \ldots$. The corresponding eigensolution is $w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$. Thus solution has the form

$$U(t,x) = \sum_{n=1}^{\infty} b_n e^{\frac{-\gamma n^2 \pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \qquad \text{where} \qquad b_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

by using the initial condition $U(0,x) = F(x)$. For $V(t,x)$ we need to use the Duhamel's principle, let $V(t,x;s)$ solves

$$\begin{cases} V_t - \gamma V_{xx} = 0 & \text{in} \quad (t,x) \in (s,\infty) \times (0,L) \\ V(t,0) = V(t,L) = 0 \\ V(s,x;s) = \tilde{Q}(s,x). \end{cases}$$

By the same method using separation of variable we obtain

$$V(s,x;s) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L \tilde{Q}(s,x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)e^{-\frac{\gamma n^2\pi^2(t-s)}{L^2}}\sin\left(\frac{n\pi x}{L}\right).$$

Then the solution $V(t,x)$ of (2) can be found by

$$V(t,x) = \int_0^s V(t,x;s)\,ds = \int_0^s \left[\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L \tilde{Q}(s,x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)e^{-\frac{\gamma n^2\pi^2(t-s)}{L^2}}\sin\left(\frac{n\pi x}{L}\right)\right]ds.$$

Finally using the fact that $v = U + V$ is the solution of (1) can be written by, the solution $u(t,x)$ of the original problem can be recover by

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)e^{\frac{-\gamma n^2\pi^2 t}{L^2}}\sin\left(\frac{n\pi x}{L}\right) \\ &+ \int_0^s \left[\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L \tilde{Q}(s,x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)e^{-\frac{\gamma n^2\pi^2(t-s)}{L^2}}\sin\left(\frac{n\pi x}{L}\right)\right]ds - \left(\frac{A(t)-B(t)}{L}\right)x + A(t). \end{aligned}$$

□

January 2012

Problem (Jan 2012 - 2/6).

(a) Solve
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \geq 0, t \geq 0.$$
with the conditions
$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad \frac{\partial u(0,t)}{\partial x} = h(t).$$

(b) For the special case $h(t) = 0$, explain how you could use a symmetry argument to help construct the solution.

Proof.

(a) First let $v(t,x) = u(t,x) - h(t)x$ then
$$\begin{cases} v_{tt} - c^2 v_{xx} = -h''(t)x, & x \geq 0, t \geq 0 \\ v(0,x) = f(x) - h(0)x = F(x) \\ v_t(0,x) = g(x) - h'(0)x = G(x) \\ v_x(t,0) = 0. \end{cases}$$
We find solution $v = U + V$ where
$$\begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U(0,x) = F(x) \\ U_t(0,x) = G(x) \\ U_x(t,0) = 0. \end{cases} \quad \text{and} \quad \begin{cases} V_{tt} - c^2 V_{xx} = -h''(t)x \\ V(0,x) = 0 \\ V_t(0,x) = 0 \\ V_x(t,0) = 0. \end{cases} \tag{3}$$
U can be found by d'Alembert formula with the even extension $\tilde{F}, \tilde{G}$ of F and G as
$$U(t,x) = \frac{\tilde{F}(x-ct) + \tilde{F}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{G}(z)\, dz.$$
For V, we use the Duhamel's principle, let $V(t,x;s)$ solves
$$\begin{cases} V_{tt} - c^2 V_{xx} = 0 & \text{in} \quad (t,x) \in (s,\infty) \times (0,+\infty) \\ V(s,x;s) = 0 \\ V_t(s,x;s) = -h''(s)x \\ V_x(t,0) = 0. \end{cases}$$
Then by d'Alembert formula again, with the even extension of $-h''(t)x$ is $-h''(t)|x|$,
$$V(t,x;s) = \frac{1}{2c}\int_{x-ct+cs}^{x+ct-cs} \Big(-h''(s)|y|\Big)\, dy$$
and by Duhamel's principle
$$V(t,x) = \int_0^t V(t,x;s)\, ds = -\frac{1}{2c}\int_0^t h''(s) \int_{x-ct+cs}^{x+ct-cs} |y|\, dy\, ds.$$
Finally we have the solution $v(t,x)$ of (3) is
$$v(t,x) = \frac{\tilde{F}(x-ct) + \tilde{F}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{G}(z)\, dz - \frac{1}{2c}\int_0^t h''(s) \int_{x-ct+cs}^{x+ct-cs} |y|\, dy\, ds$$
where $\tilde{F}, \tilde{G}$ are the even extensions of F and G respectively. We recover the solution $u(t,x)$ by
$$u(t,x) = v(t,x) + h(t)x = \frac{\tilde{F}(x-ct) + \tilde{F}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{G}(z)\, dz - \frac{1}{2c}\int_0^t h''(s) \int_{x-ct+cs}^{x+ct-cs} |y|\, dy\, ds + h(t)x.$$

(b) If $h(t) \equiv 0$ then basically we can use the d'Alembert formula with the some appropriate extension of F and G so that

$$u(t,x) = \frac{\tilde{F}(x-ct)+\tilde{F}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{G}(z)\,dz.$$

We have

$$u_x(t,x) = \frac{\tilde{F}'(x-ct)+\tilde{F}'(x+ct)}{2} + \frac{1}{2c}\Big(\tilde{G}(x+ct)-\tilde{G}(x-ct)\Big)$$

and we observe that if we choose the even extensions of F and G then $u_x(t,0)=0$.

□

Problem (Jan 2012 - 4/6). Find the value(s) of the constant c for which the 2D boundary value problem

$$\Delta u = c, \qquad x^2+y^2=1,$$

with

$$\frac{\partial u}{\partial r} = 2 \qquad \text{on} \qquad r=1$$

has a solution. Then find the general regular solution.

Proof. It is easy to see that $\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial r}$ on $r=1$, where ν denotes the outward normal vector of the unit ball. Thus by Green formula we have

$$\pi c = \int_{B(0,1)} \Delta u(x,y)\,dxdy = \int_{\partial B(0,1)} \frac{\partial u}{\partial r}\,dS = 4\pi \qquad \Longrightarrow \qquad c=4.$$

Now we use separation of variable to seek for a solution, first in polar coordinates we have

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Let $u(r,\theta)=v(r)w(\theta)$, substitute into the equation we obtain

$$r^2\frac{v''}{v} + r\frac{v'}{v} = 4 - \frac{w''}{w} = \lambda \qquad \Longrightarrow \qquad \begin{cases} r^2v''+rv'-\lambda v = 0 \\ w''+(\lambda-4)w=0. \end{cases}$$

Let's solve for $w(\theta)$ first. We require $w(\theta+2\pi)=w(\theta)$, the only possible value of λ so that $w(\theta)$ is periodic is $\lambda-4=\omega^2>0$ or $\lambda=4$, thus $w(\theta)$ can be $1, \sin\omega\theta, \cos\omega\theta$. Since $w(\theta+2\pi)=w(\theta)$, we need $\omega\in\mathbb{N}$, which means the nonzero eigenfunctions are $1, \sin n\theta, \cos n\theta$ for $n=1,2,\ldots$. So $\lambda=n^2+4$ where $n=0,1,2,\ldots$. Then for the Euler equation of $v(r)$, we seek for solution of the form r^k, which implies

$$r^2v''+rv'-\lambda v = \Big(k(k-1)-k-n^2-4\Big)r^k = (k^2-n^2-4)r^k = 0.$$

Thus $k^2=n^2+4$ for $n=0,1,2,\ldots$. Thus the solutions is $v(r)=r^{-\sqrt{n^2+4}}$ or $v(r)=r^{\sqrt{n^2+4}}$ where $n=0,1,2,\ldots$. Since we want the solution to be regular, we want it to be bounded at $r=0$, which means a candidate series solution we should look at is

$$u(r,\theta) = C + a_0r^2 + \sum_{n=1}^{\infty}\Big(a_n r^{\sqrt{n^2+4}}\cos n\theta + b_n r^{\sqrt{n^2+4}}\sin n\theta\Big).$$

The Neumann boundary condition reads

$$\frac{\partial u}{\partial r}(1,\theta) = 2a_0 + \sum_{n=1}^{\infty}\Big(a_n\sqrt{n^2+4}\cos n\theta + a_n\sqrt{n^2+4}\sin n\theta\Big) = 2.$$

From that we obtain $a_n=b_n=0$ for all $n\geq 1$, and $a_0=1$, which mean the solution is $u(r,\theta)=C+r^2$, i.e $u(x,y)=C+x^2+y^2$. □

Problem (Jan 2012 - 5/6). Suppose that

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\delta\left(x-\frac{t^2}{2}\right), \qquad t>0, \qquad u(0,x)=0.$$

Reformulate the problem by changing variables from (x,t) to (ξ,t), where $\xi=x-\frac{t^2}{2}$. Then solve the problem using the method of characteristics.

Proof. Let $\xi=x-\frac{t^2}{2}$ and consider $v(t,\xi)=u(t,x)$, then

$$\begin{cases} u_t(t,x) &= v_t(t,\xi)-tv_\xi(t,\xi) \\ u_x(t,x) &= v_\xi(t,\xi) \end{cases} \qquad \Longrightarrow \qquad \frac{\partial v}{\partial t}(t,\xi)+(1-t)\frac{\partial v}{\partial \xi}(t,\xi)=\delta(\xi).$$

We will use the Duhamel's principle to solve this transport equation with the source term. For each $s>0$, let's define

$$V(t,\xi;s) \qquad \text{solves} \qquad \begin{cases} V_t+(1-t)V_\xi &= 0 \qquad \text{on} \qquad t>s \\ V(s,\xi;s) &= \delta(\xi). \end{cases}$$

The characteristic curve is $\xi(t)=t-\frac{t^2}{2}-s+\frac{s^2}{2}+\xi(s)$. And along the characteristic curve we have

$$V\Big(t,\xi(t);s\Big)=V(s,\xi(s);s)=\delta\Big(\delta(\xi(s))\Big) \qquad \Longrightarrow \qquad V\Big(t,\xi\Big)=\delta\left(\xi-t+\frac{t^2}{2}+s-\frac{s^2}{2}\right).$$

Then the solution is

$$u(t,x)=v(t,\xi)=\int_0^t V(t,\xi,s)\,ds=\int_0^t \delta\left(\xi-t+\frac{t^2}{2}+s-\frac{s^2}{2}\right)ds=\int_0^t \delta\left(x-t+s-\frac{s^2}{2}\right)ds.$$

□

Remark. A slightly different problem is $\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\delta\left(x-\frac{t}{2}\right)$ where $t>0$, $u(0,x)=0$.

Proof 1. Change of variable $u(t,x)=v(t,\xi)$ with $\xi=x-\frac{t}{2}$ again we obtain $v_t+\frac{1}{2}v_\xi=\delta(\xi)$. Using method of characteristic, we have $\xi(t)=\xi_0+\frac{t}{2}$ and $\frac{dv}{dt}(t,\xi(t))=\delta\left(\xi_0+\frac{t}{2}\right)$, which implies

$$v(t,\xi(t))=2\sigma\left(\xi_0+\frac{t}{2}\right)+C \qquad \Longrightarrow \qquad 0=v(0,\xi_0)=2\sigma(\xi_0)+C \qquad \Longrightarrow \qquad C=-2\sigma(\xi_0)$$

by the initial condition. Thus using $\xi_0=\xi(t)-\frac{t}{2}$ we conclude that

$$u(t,x)=v(t,\xi)=2\sigma(\xi)-2\sigma\left(\xi-\frac{t}{2}\right)=2\sigma\left(x-\frac{t}{2}\right)-2\sigma(x-t).$$

□

Proof 2. Seeking solution in the form $v(t,x)=v\left(x-\frac{t}{2}\right)=v(\xi)$ where $\xi=x-\frac{t}{2}$ we obtain $v'(\xi)-\frac{1}{2}v'(\xi)=\delta(\xi)$, i.e $v'(\xi)=2\delta(\xi)$. We can pick $v(\xi)=2\sigma(\xi)$, then $u_1(t,x)=2\sigma\left(x-\frac{t}{2}\right)$ satisfies the equation but not the initial condition $u_1(0,x)=2\sigma(x)$. Thus we can seek for u_2 solves $u_t+u_x=0$ with $u(0,x)=-2\sigma(x)$, we have $u_2(t,x)=-2\sigma(x-t)$ and then the final solution is $u=u_1+u_2$. □

Proof 3. Change of variable $u(t,x)=v(t,\xi)$ with $\xi=x-\frac{t}{2}$ again we obtain $v_t+\frac{1}{2}v_\xi=\delta(\xi)$. Using Duhamel's principle, for each $s>0$ we seek for $V(t,\xi;s)$ solves $V_t+\frac{1}{2}v_\xi=0$ on $t>s$ and $V(s,\xi;s)=\delta(\xi)$. We have $V(t,\xi;s)=\delta\left(\xi-\frac{t}{2}+\frac{s}{2}\right)$, thus by Duhamel's principle

$$u(t,x)=v(t,\xi)=\int_0^t V(t,\xi;s)\,ds=\int_0^t \delta\left(\xi-\frac{t}{2}+\frac{s}{2}\right)ds=\int_0^t \delta\left(x-t+\frac{s}{2}\right)ds.$$

□

August 2012

Problem (Aug 2012 - 3/6).
(a) Explain how to reduce the problem

$$u_t - u_{xx} = \delta(x)\delta(t) \qquad \text{for} \qquad x \in \mathbb{R}, t \geq 0$$

to an ordinary differential equation.
(b) Compute the fundamental solution of the diffusion equation found above for initial condition $u(0,x) = 0$ and boundary condition $u(t, \pm\infty) = 0$.

Proof. Using the convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\, dx$ we can take Fourier transform of both side (assuming u is a tempered distribution)

$$\widehat{u}_t - (2\pi i\xi)^2\widehat{u} = \delta(t) \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\widehat{u}(t,\xi)e^{4\pi^2\xi^2 t}\right) = \delta(t)e^{4\pi^2\xi^2 t}.$$

This is an ODE with the initial condition $\widehat{u}(0,\xi) = 0$. Now in order to find the fundamental solution for the diffusion equation, we have to put the point source $\delta(x-x_0)\delta(t-\tau)$ (Otherwise it makes no sense to integrate $\int_0^t \delta(s)f(x,s)\, ds$). Doing similarly we have

$$\widehat{u}_t - (2\pi i\xi)^2\widehat{u} = e^{-2\pi i\xi x_0}\delta(t-\tau) \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\widehat{u}(t,\xi)e^{4\pi^2\xi^2 t}\right) = e^{4\pi^2\xi^2 t - 2\pi i\xi x_0}\delta(t).$$

Integrate both sides from 0 to t and using the initial condition we obtain

$$\widehat{u}(t,\xi)e^{4\pi^2\xi^2 t} = e^{-2\pi i\xi x_0}\left(e^{4\pi^2\xi^2 t}\sigma(t-\tau)\right) \qquad \Longrightarrow \qquad \widehat{u}(t,\xi) = e^{-4\pi^2\xi^2(t-\tau)-2\pi i\xi x_0}\sigma(t-\tau)$$

where $\sigma(s)$ is the step function (aka Heavyside function). We deduce the solution $u(t,x)$ correspond to the point source $\delta(x-x_0)\delta(t-\tau)$ is

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{+\infty} e^{-4\pi^2\xi^2(t-\tau)-2\pi i\xi x_0}\sigma(t-\tau)e^{2\pi i\xi x}\, d\xi \\ &= e^{-\frac{(x-x_0)^2}{4(t-\tau)}}\sigma(t-\tau)\int_{-\infty}^{+\infty} e^{-4\pi^2(t-\tau)\left(\xi - \frac{i(x-x_0)}{4\pi(t-\tau)}\right)^2}\, d\xi = \frac{1}{\sqrt{4\pi(t-\tau)}}e^{-\frac{(x-x_0)^2}{4(t-\tau)}}\sigma(t-\tau). \end{aligned}$$

Now we can think about taking $x_0 = 0$, $\tau = 0$ to obtain the somewhat fundamental solution to the original point source

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\sigma(t).$$

Here we make use of the formula: (need some complex argument to justify)

$$\int_{\mathbb{R}} e^{-\pi a x^2}\, dx = \frac{1}{\sqrt{a}}.$$

□

Problem (Aug 2012 - 5/6). Consider the inviscid Burgers' equation with constant source term

$$u_t + uu_x = 1.$$

Use the method of characteristic curves to solve this equation for the initial condition $u(x,0) = x$. Explain any singularities that may occur.

Proof. The characteristic curves is

$$\begin{cases} x'(t) &= u(t,x(t)) \\ x(0) &= x_0 \end{cases} \quad \text{and} \quad \begin{cases} \frac{d}{dt}u(t,x(t)) &= 1 \\ u(0,x_0) &= u_0(x_0) = x_0. \end{cases}$$

It is easy to see that $x(t) = x_0 + x_0 t + \frac{t^2}{2}$, i.e $x_0 = \frac{x(t)-\frac{t^2}{2}}{t+1}$, thus

$$u(t,x(t)) = t + x_0 \qquad \Longrightarrow \qquad u(t,x) = t + \frac{x - \frac{t^2}{2}}{t+1} = \frac{x + \frac{t^2}{2} + t}{t+1}.$$

From that we have

$$u_x(t,x) = \frac{1}{t+1}$$

for all $t \geq 0$, thus no singularity can occur. □

January 2013

Problem (Jan 2013 - 5/6). Derive the solution to the heat equation on the semi-infinite domain,

$$u_t - u_{xx} = 0, \qquad 0 \leq x < \infty, \qquad 0 \leq t < \infty,$$

with zero initial condition

$$u(x,0) = 0,$$

and a prescribed boundary condition at $x = 0$:

$$u(0,t) = g(t), \qquad t > 0.$$

Also assume that $u(\infty, t) = 0$.

Proof. First of all from the zero initial condition, we can assume $g(0) = 0$. Let $v(x,t) = u(x,t) - g(t)$, we now solve the inhomogeneous heat equation

$$\begin{cases} v_t - v_{xx} = -g'(t), \qquad 0 \leq x < \infty, \qquad 0 \leq t < \infty \\ v(x,0) = 0 \\ v(0,t) = 0. \end{cases} \tag{4}$$

Using Duhamel's principle, we seek for each $s > 0$ the solution $v(x,t;s)$ of

$$\begin{cases} v_t - v_{xx} = 0, \qquad 0 \leq x < \infty, \qquad s \leq t < \infty \\ v(x,s) = -g'(s) \\ v(0,t) = 0. \end{cases} \tag{5}$$

We can use method of images to solve this problem, let's recall that the fundamental solution of the heat equation in 1D is

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \qquad \text{solves} \qquad \begin{cases} \Phi_t - \Phi_{xx} = 0, \qquad -\infty < x < +\infty, t > 0 \\ \Phi(x,0) = \delta(x), \qquad -\infty < x < +\infty, \{t = 0\}. \end{cases}$$

Then the heat kernel with respect to the point source $\delta(\xi)$ at location ξ is

$$\Phi(x-\xi,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-\xi)^2}{4t}}.$$

By method of images, if we put another point source $-\delta(-\xi)$ then

$$F(x,t;\xi) = \Phi(x-\xi,t) - \Phi(x+\xi,t) \qquad \text{solves} \qquad \begin{cases} F_t - F_{xx} = 0, & -\infty < x < +\infty, t > 0 \\ \Phi(x,0) = \delta(x-\xi) - \delta(x+\xi), & -\infty < x < +\infty, \{t=0\}. \end{cases}$$

Thus if we define

$$V(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) f(y)\, dy = \int_{-\infty}^{\infty} \Big(\Phi(x-y,t) - \Phi(x+y,t) \Big) f(y)\, dy$$

then we have $V_t - V_{xx} = 0$, $V(x,0) = f(x) - f(-x)$ and $u(0,t) = 0$. From this observation, we can define

$$f(x) = \operatorname{sign}(x)\Big(-g'(s) \Big) = \begin{cases} -g'(s) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Then the corresponding solution of (5) can be found by

$$v(x,t;s) = \int_{-\infty}^{\infty} \Big(\Phi(x-y,t-s) - \Phi(x+y,t-s) \Big) f(y)\, dy = g'(s) \int_0^{\infty} \Big(\Phi(x+y,t-s) - \Phi(x-y,t-s) \Big)\, dy.$$

We can easily verify that $v(x,s;s) = -g'(s)$. By Duhamel's principle for (4) we obtain the solution $u(x,t)$ to the original problem is:

$$u(x,t) = \int_0^t v(x,t;s)\, ds = \int_0^t \int_0^{\infty} \Big(\Phi(x+y,t-s) - \Phi(x-y,t-s) \Big)\, dy\, ds + g(t).$$

□

January 2014

Problem (Jan 2014 - 6/6). Consider the inviscid Burgers' equation $u_t + uu_x = 0$ with smooth initial condition $u(x;0) = g(x)$ for $-\infty < x < \infty$.

(a) Using the method of characteristics, find an implicit formula for the solution $u(x;t)$ involving the initial condition g.

(b) For $g(x) = e^{-x^2}$, find the time T_c when a shock forms.

Proof.

(a) Consider

$$\begin{cases} x'(t) & = u(x(t),t) \\ x(0) & = x_0 \end{cases} \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(u(t,x(t)) \Big) = (u_t + uu_x)|_{(x(t),t)} = 0.$$

Thus $u(x(t),t) = u(x_0,0) = g(x_0)$ for all t, which implies $x(t) = x_0 + tg(x_0)$. Thus

$$u(x(t),t) = g(x_0) = g\Big(x(t) - g(x_0)t \Big) = g\Big(x(t) - u(x(t),t)t \Big) \qquad \Longrightarrow \qquad u(x,t) = g\Big(x - tu(x,t) \Big).$$

(b) For $g(x) = e^{-x^2}$, we first derive the derivative $u_x(x,t)$ by implicit differentiation

$$u_x(x,t) = g'(x - tu(x,t))\Big(1 - tu_x(x,t) \Big) \qquad \Longrightarrow \qquad u_x(x,t) = \frac{g'(x - tu(x,t))}{1 + tg'(x - tu(x,t))}.$$

Let $\xi = x - tu(x,t)$, then the shocks occur at (T_c, x_c) when

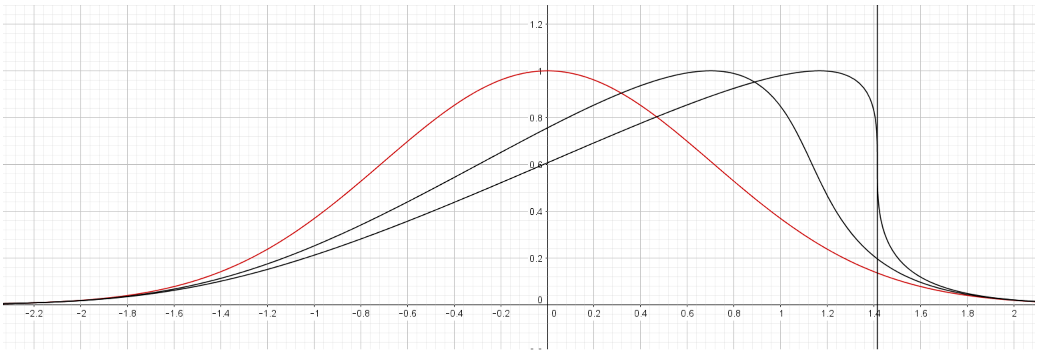
$$\lim_{t \longrightarrow T_c} |u_x(t,x_c)| = +\infty \qquad \Longrightarrow \qquad t \longrightarrow -\frac{1}{f'(\xi)}$$

when slope blows up, thus the earliest critical time is

$$T_c = \min\left\{-\frac{1}{f'(\xi)} : f'(\xi) < 0\right\}.$$

In our case with $g(x) = e^{-x^2}$, we have $-1/f'(\xi)$ is minimum at $\xi = \frac{1}{\sqrt{2}}$, which implies $T_c = \sqrt{\frac{e}{2}}$. The position of the shock is

$$x_c = T_c g(\xi) + \xi = \sqrt{\frac{e}{2}} e^{-\frac{1}{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$



□

August 2014

Problem (August 2014 - 5/6). Solve the following pair of initial-boundary value problems that together describe the flow of traffic just after a green light turns red. Provide equations and also diagrams to describe your solution.

(a) For the traffic approaching the red light:

$$\begin{cases} \rho_t + \big(\rho(1-\rho)\big)_x & = 0 \quad \text{for } x < 0 \\ \rho(x,0) & = \frac{1}{4} \quad \text{for } x < 0 \\ \rho(0^-,t) & = 1 \quad \text{for } t > 0. \end{cases}$$

(b) For the traffic beyond the red light:

$$\begin{cases} \rho_t + \big(\rho(1-\rho)\big)_x & = 0 \quad \text{for } x > 0 \\ \rho(x,0) & = \frac{1}{4} \quad \text{for } x > 0 \\ \rho(0^+,t) & = 0 \quad \text{for } t > 0. \end{cases}$$

Proof.

(a) Using method of characteristic, we consider

$$\begin{cases} x'(t) & = 1 - 2\rho(x(t),t) \\ x(0) & = x_0 < 0. \end{cases}$$

then along the characteristic curve, $\rho(x(t),t) = \frac{1}{4}$, thus $x(t) = x_0 + \frac{t}{2}$. But for $t > 0$, since the shock moves to the left as $1 - 2\rho^+ = -1$, we have the characteristic curve started closed to the origin when $t > 0$ is $x_0 - t$. These lines intersect when the shock form, which has the speed according to the Rankine-Hugoniot jump condition is

$$\sigma'(t) = \frac{\frac{1}{4}\left(1 - \frac{1}{4}\right) - 1(1-1)}{\frac{1}{4} - 1} = -\frac{1}{4}.$$

Since $\sigma(0) = 0$, we have $\sigma(t) = -\frac{t}{4}$, as described in the following (t, x)-plane.

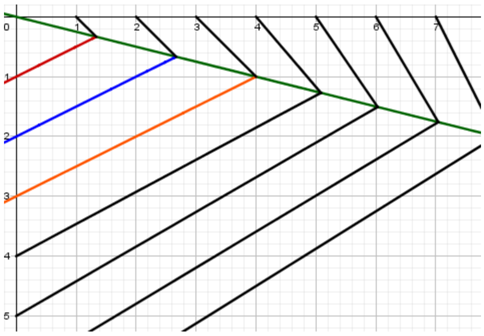


Figure 1: Characteristic lines in (t, x)-plane.

A formula for the solution is

$$\rho(x,t) = \begin{cases} 1 & \text{if} \quad \frac{x}{t} > \frac{-1}{4} \\ \frac{1}{4} & \text{if} \quad \frac{x}{t} < \frac{-1}{4}. \end{cases}$$

(b) Using method of characteristic, we consider

$$\begin{cases} x'(t) & = 1 - 2\rho(x(t), t) \\ x(0) & = x_0 > 0. \end{cases}$$

then along the characteristic curve, $\rho(x(t), t) = \frac{1}{4}$, thus $x(t) = x_0 + \frac{t}{2}$. But for $t > 0$, since the shock moves to the right as $1 - 2\rho^+ = 1$, we have the characteristic curve started closed to the origin when $t > 0$ is $x_0 + t$. These lines intersect when the shock form, which has the speed according to the Rankine-Hugoniot jump condition is

$$\sigma'(t) = \frac{\frac{1}{4}\left(1 - \frac{1}{4}\right) - 0(0 - 0)}{\frac{1}{4} - 0} = \frac{3}{4}.$$

Since $\sigma(0) = 0$, we have $\sigma(t) = \frac{3t}{4}$, as described in the following (t, x)-plane.

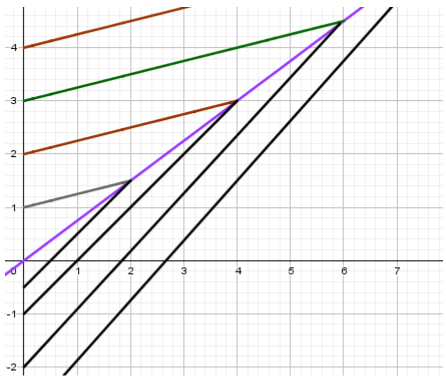


Figure 2: Characteristic lines in (t, x)-plane.

A formula for the solution is

$$\rho(x,t) = \begin{cases} \frac{1}{4} & \text{if} \quad \frac{x}{t} > \frac{3}{4} \\ 0 & \text{if} \quad \frac{x}{t} < \frac{3}{4}. \end{cases}$$

□

January 2015

Problem (Jan 2015 - 1/6). Are the claims below true or false? Explain your answer.

(a) There exist two positive definite matrices A and B such that $\det(AB) = 0$.

(b) If a, b, c are distinct positive numbers, then the matrix

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

is invertible.

(c) If the matrix M has the property that all its principal minors are positive, then all its eigenvalues are positive.

(d) There exist 4 by 4 matrices A and B, such that A has rank 2, B has rank 1, and $A + B$ is invertible.

Proof.

(a) **FALSE**. Recall that a positive definite matrix is invertible. Indeed if $x \neq 0$ but $Ax = 0$ then clearly $x^T Ax = 0$ will be a contradiction to the fact that A is positive definite. Now A, B are both positive definite, thus they are both invertible and thus $\det(AB) \neq 0$.

(b) **TRUE**. We have

$$\det A = a^3 + b^3 + c^3 - 3abc > 0$$

by Cauchy-Schwartz inequality (the equality happens if and only if $a = b = c$).

(c) **TRUE**. The determinant of a matrix equal to the product if its eigenvalues, also is the product of its pivots. And since the sub-matrix $A^{(k)}$ is k by k in the upper left corner has k-minor principals is $\det(A^{(k)})$, we deduce that the k-principal minor is the product of the first k pivots. Thus $\lambda_1, \lambda_1\lambda_2, \ldots, \lambda_1\lambda_2 \ldots \lambda_n$ are positive (λ_i may repeats itself) we obtain $\lambda_i > 0$ for all $i = 1, \ldots, n$.

(d) **FALSE**. We have the rank inequality for $A, B \in \mathbb{M}_{n\times n}(\mathbb{R})$ then $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$. Recall that $\operatorname{rank}(A)$ is the dimension of the column space. I.e if we define the linear operator $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $f_A(x) = Ax$ then clearly $\operatorname{Im}(f_A) = \operatorname{span}(\text{columns of } A)$, which means $\operatorname{rank}(A) = \dim(\operatorname{Im} f_A)$. Now clearly $\operatorname{Im} f_{A+B} \subset \operatorname{Im} f_A + \operatorname{Im} f_B$ which implies $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$. Using this we have $\operatorname{rank}(A+B) \leq 3$, thus it cannot be invertible.

□

Problem (Jan 2015 - 2/6). Solve the differential equation $\frac{du}{dx} + au = \delta(x)$ by using Fourier transform.

Proof. Using the convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi\xi x}\, dx$ we have

$$(2\pi i\xi)\widehat{u}(\xi) + a\widehat{u}(\xi) = 1 \qquad \Longrightarrow \qquad \widehat{u}(\xi) = \frac{1}{a + 2\pi i\xi}.$$

Thus the inversion formula reads

$$u(x) = \int_{-\infty}^{\infty} \widehat{u}(\xi)e^{2\pi i\xi x}\, d\xi = \int_{-\infty}^{\infty} \frac{e^{2\pi i\xi x}}{a + 2\pi i\xi}\, d\xi = e^{-ax}\int_{-\infty}^{+\infty} \frac{e^{(2\pi i\xi + a)x}}{2\pi i\xi + a}\, d\xi.$$

Given $a > 0$, we can write down the integral above in a closed form. Indeed, let's observe that

$$\mathcal{F}\Big(e^{-ax}\sigma(x)\Big)(\xi) = \int_0^{\infty} e^{-(2\pi i\xi + a)x}\, dx = \frac{1}{2\pi i\xi + a}. \tag{6}$$

And thus the inversion formula reads

$$u(x) = e^{-ax}\sigma(x) \qquad \text{when} \qquad a > 0$$

where σ is the Heavyside function. Here in order to justify (6) we use the fact that for $a > 0$ then

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \qquad \text{and} \qquad \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}.$$

These facts can be proven using integration by parts or complex analysis. □

August 2015

Problem (August 2015 - 1/6). Here A, A_1, A_2 denote $n \times n$ matrices.

(a) If A is positive definite, are A^{-1} and A^2 also positive definite? Same question if A is negative definite.

(b) If $n = 2$ and $A_1A_2 = 0$ and $\mathrm{rank}(A_1) > 0$ and $\mathrm{rank}(A_2) > 0$, is it true that $\mathrm{rank}(A_1) = \mathrm{rank}(A_2)$? How would you generalize this for $n > 2$?

(c) Give a geometric explanation for the method of Lagrange multipliers. (You may consider a positive definite quadratic form $Q(x_1, x_2)$ in $\mathbb{R}^2$, subject to a linear constraint $L(x_1, x_2) = 0$. Explain geometrically why the method of Lagrange multipliers allows us to find the minimum of Q subject to the constraint L. For example, you may want to start by drawing some level curves of Q).

Proof.

(a) First we prove that a positive definite matrix is invertible. It is obvious but worth noticing. We show it by prove that $Ax = 0$ implies $x = 0$. Indeed if $x \neq 0$, then clearly $x^TAx = 0$ will be a contradiction to the fact that A is positive definite. Now the rest of proof consists of two parts.

- For A is positive definite implies A^2 is positive definite. **FALSE**. A counter example can be constructed by

$$B = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \qquad \text{has} \qquad \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - xy + y^2 = \left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4}.$$

But B^2 is not positive definite, as

$$B^2 = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 2 & -1 \end{pmatrix} \quad \text{has} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 < 0.$$

Remark. It will be true if A is symmetric. The proof is using the special characterization of symmetric matrix, that is "positive definite" if and only if all eigenvalue is positive.

- For A is positive definite implies A^{-1} is positive definite. **TRUE**. For every $x \in \mathbb{R}^n \backslash \{0\}$ there exists a unique $y \in \mathbb{R}^n \backslash \{0\}$ such that $x = Ay$, since A is invertible, and

$$x^TA^{-1}x = y^TA^TA^{-1}Ay = y^TA^Ty = y^TAy > 0.$$

The same conclusions hold for A is negative definite, since we just simply consider $-B$ as a counter example in the first case and in the second case the argument is similar.

(b) As $AB = 0$, we see that each column of B belongs to the null space of A, thus in terms of dimension we have $\mathrm{rank}(B) \leq \dim(\mathrm{null}\, A) = n - \mathrm{rank}(A)$. Thus $\mathrm{rank}(A) + \mathrm{rank}(B) \leq n$. In case $n = 2$, it is obvious to see that $\mathrm{rank}(A) = \mathrm{rank}(B) = 1$. In case $n > 2$ we can only say $\mathrm{rank}(A) + \mathrm{rank}(B) \leq n$.

(c) See Wikipedia.

□

Problem (August 2015 - 2/6). Consider a predator-prey system given by the differential equations

$$\frac{du_1}{dt} = u_1 - u_1^2 + bu_1u_2$$
$$\frac{du_2}{dt} = u_2 - u_2^2 + cu_1u_2$$

Assume that $b < 0$ and $c > 0$. (This means that u_1 is the prey population and u_2 is the predator population.)

(a) Are there any values of b and c for which the origin $(0,0)$ is a stable fixed point?

(b) Note that $(1,0)$ and $(0,1)$ are also fixed points. Determine their stability in terms of the parameters b and c.

(c) Depending on the values of the parameters b and c, there can be a 4th biologically relevant (i.e., nonnegative) fixed point, different from the three fixed points discussed above. Find this fixed point and describe the conditions that b and c must satisfy.

Proof.

(a) Let's compute the Jacobian at a point (u_1, u_2) as

$$J(u_1,u_2) = \begin{pmatrix} 1-2u_1+bu_2 & bu_1 \\ cu_2 & 1-2u_2+cu_1 \end{pmatrix}.$$

At $(u_1,u_2) = (0,0)$ we have

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad \det(J-\lambda I) = (\lambda-1)^2.$$

Thus we have a repeated eigenvalue $\lambda_1 = \lambda_2 = 1$, which means the origin is an unstable fixed point. Indeed it is an unstable node, there are no values of the parameters b, c such that $(0,0)$ is stable.

(b) The Jacobian at (1,0) and (0,1) are

$$J(1,0) = \begin{pmatrix} -1 & b \\ 0 & 1+c \end{pmatrix} \qquad \text{and} \qquad J(0,1) = \begin{pmatrix} 1+b & 0 \\ c & -1 \end{pmatrix}.$$

- For $(1,0)$ we have $\det(J-\lambda I) = (\lambda-c-1)(\lambda+1)$, thus there are two different eigenvalues $\lambda_1 = -1$ and $\lambda_2 = c+1 > 0$. Thus (1,0) is a saddle point.
- For $(0,1)$ we have $\det(J-\lambda I) = (\lambda-b-1)(\lambda+1)$, thus there are two different eigenvalues $\lambda_1 = -1$ and $\lambda_2 = b+1$. There are some cases:
  - $-1 < b < 0$ then $\lambda_1 = -1 < 0$, $\lambda_2 = b+1 > 0$, we have $(0,1)$ is a saddle node, unstable.
  - $b = -1$ then $\lambda_1 = -1$, $\lambda_2 = 0$, then $(1,0)$ is not an isolated fixed point.
  - $b < -1$ then $\lambda_1 = -1$, $\lambda_2 = b+1 < 0$, then $(0,1)$ is an stable fixed point (a degenerate node).

(c) We have another fixed point

$$(u_1^*, u_2^*) = \left(\frac{1+b}{1-bc}, \frac{1+c}{1-bc}\right).$$

This is a non-negative fixed point if $1+b > 0$, i.e $-1 < b < 0$. The Jacobian matrix at this point is

$$J\left(u_1^*, u_2^*\right) = \begin{pmatrix} \dfrac{-1-b}{1-bc} & \dfrac{b+b^2}{1-bc} \\ \dfrac{c+c^2}{1-bc} & \dfrac{-1-c}{1-bc} \end{pmatrix}.$$

We have

$$\tau = \text{Trace}(J) = \lambda_1 + \lambda_2 = -\frac{2+b+c}{1-bc} = -\frac{(1+b)+(1+c)}{1-bc} < 0.$$

And

$$\Delta = \det(J) = \lambda_1\lambda_2 = (1+b)(1+c) > 0.$$

That means in this case $\lambda_1 < 0, \lambda_2 < 0$, i.e this is a stable fixed point.

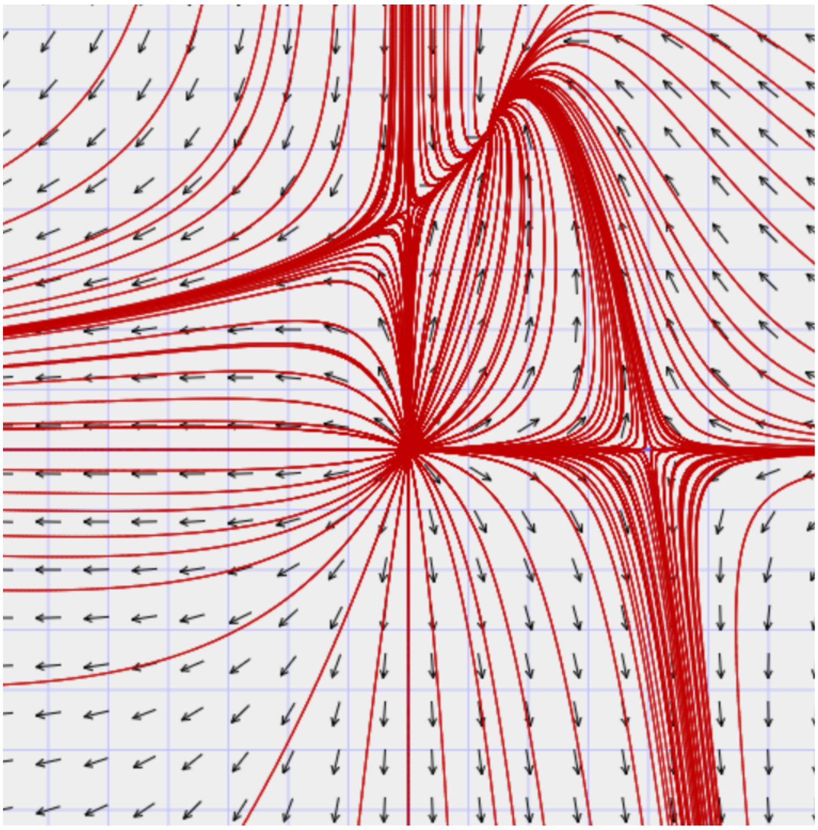


Figure 3: Phase portrait when $b = -1/2, c = 1$.

$\square$

Problem (August 2015 - 3/6). Consider a square wave f that has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = 1$ on the right side $0 < x < \pi$ (and is extended periodically on $\mathbb{R}$).

(a) Calculate the Fourier expansion of f.

(b) Find the same Fourier series expansion of f in a different way: first show that $\frac{df}{dx} = 2\delta(x) - 2\delta(x+\pi)$ (where $\delta(x)$ is the Dirac delta function extended periodically on $\mathbb{R}$), then calculate the Fourier series expansion of $\delta(x)$, and then integrate to obtain the Fourier series of f.

Proof.

(a) The Fourier series expansion of f has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx.$$

Since f is odd, this is a sine series, i.e $a_n = 0$ for all n and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\, dx = \frac{2\left(1-(-1)^n\right)}{n\pi}.$$

Finally we obtain

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases} \quad \sim \quad \sum_{n=1}^{\infty} \frac{2\left(1-(-1)^n\right)}{n\pi} \sin(nx).$$

(b) It is easy to see that f is discontinuous at $x=0$ with magnitude 2, and is discontinuous at $x=-\pi$ with magnitude -2, thus

$$\frac{df}{dx} = 2\delta(x) - 2\delta(x+\pi).$$

We don't include discontinuities at other points $k\pi$ where $k \in \mathbb{Z}$ since we only focus on $[-\pi,\pi)$. Recall that the Fourier series of $\delta(x)$ is

$$\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx, \qquad \delta(x+\pi) \sim \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(nx+n\pi).$$

Thus using $\cos(a+b) = \cos a \cos b - \sin a \sin b$ we obtain

$$2\delta(x) - 2\delta(x+\pi) = \frac{2}{\pi}\sum_{n=1}^{\infty}\Big(1-(-1)^n\Big)\cos nx \qquad \Longrightarrow \qquad f(x) = C + \sum_{n=1}^{\infty}\frac{2\Big(1-(-1)^n\Big)}{n\pi}\sin(nx).$$

Finally using the fact that $\int_{-\pi}^{\pi} f(x)\,dx = 0$ we obtain $C=0$ and thus we have the same Fourier series expansion.

□

Problem (August 2015 - 6/6).

(a) Rankine-Hugoniot jump conditions following a singular source or sink. Consider the following general conservation law with singular sink term:

$$\rho_t + \Big(f(\rho)\Big)_x = -D\delta(x - X(t))$$

where the sink term is at location $X(t)$ and moves at velocity $X'(t) = \sigma(t)$.

i. Derive a modified Rankine-Hugoniot jump condition that is valid for this problem, for the jump at the location $X(t)$ of the singular sink.

ii. What is the form of the Rankine-Hugoniot jump condition from part (i) in the special case of $X(t) = x_0$, i.e., the case of a singular sink at a fixed location?

(b) Traffic flow with an off-ramp. Consider the following initial value problem for traffic flow with an off-ramp:

$$\begin{cases} \rho_t + \Big[\rho(1-\rho)\Big]_x &= -D\delta(x) \\ \rho(x,0) &= \rho_i. \end{cases}$$

Derive the solution to the initial value problem for the case of $\rho_i = \frac{1}{3}$, $D = \frac{5}{144}$. Present your solution in all of the following ways:

i. an analytic formula for $\rho(x,t)$ for general values of x and t,

ii. a plot of the characteristic curves in the $x-t$ plane, and

iii. a plot of $\rho(x,t)$ at time $t=12$.

Proof.

(a) Pick $[x_1,x_2] \sim [x, x+\Delta x]$ small so that $\sigma(t) \in (x_1,x_2)$, then since $X'(t) = \sigma(t)$ we have $X(t) \in (x_1,x_2)$, thus integrating both side of the equation we get

$$\frac{d}{dt}\left(\int_{x_1}^{\sigma(t)} \rho(x,s)\,dx + \int_{\sigma(t)}^{x_2}\rho(x,s)\,dx\right) = -\int_{x_1}^{x_2}\Big(f(\rho)\Big)_x\,dx - D\int_{x_1}^{x_2}\delta(x-X(t))\,dx.$$

From that if we let $x_1, x_2 \longrightarrow \sigma(t)$ we obtain the approximation

$$\sigma'(t)\Big(\rho^- - \rho^+\Big) = f(\rho^-) - f(\rho^+) - D.$$

Thus the modified Rankine-Hugoniot jump condition is

$$\sigma'(t) = \frac{f(\rho^-) - f(\rho^+) - D}{\rho^- - \rho^+}.$$

If $X(t) \equiv x_0$, then when integrating the integral involving δ will be zero, since the sink now is located at x_0 which is away from the the shock $\sigma(t)$ when t varies. Thus the there will be no change in the Rankine-Hugoniot jump condition

$$\sigma'(t) = \frac{f(\rho^-) - f(\rho^+)}{\rho^- - \rho^+}.$$

(b) Using method of characteristic, we consider

$$\begin{cases} x'(t) &= 1 - 2\rho(x(t), t) \\ x(0) &= x_0 \end{cases} \quad \text{and} \quad \begin{cases} z'(t) &= -D\delta(x(t)) \\ z(0) &= \rho_i. \end{cases}$$

Where $z(t) = \rho(x(t), t)$. If $x(t)$ remains its sign from x_0, then $\rho(x(t), t)$ remains the same value as ρ_i. Things only change at the time $x(t)$ switches its sign.

- If we start from $x_0 > 0$, then since $\rho_i = \frac{1}{3}$, we have $1 - 2\rho_i = \frac{1}{3} > 0$, which means $x(t) = x_0 + \frac{t}{3}$ and remains positive all time. Thus $\rho(x(t), t) = \rho_i = \frac{1}{3}$ for all time.
- If we start from $x_0 < 0$, then $x(t) = x_0 + \frac{t}{3} < 0$ when $t < -3x_0$, which means $\rho(x(t), t) = \rho_i = \frac{1}{3}$ when $t < -3x_0$. After $t > -3x_0$, the value of $\rho(x(t), t)$ must be constant $\rho_i - D = \frac{1}{3} - \frac{5}{144} = \frac{43}{144}$. And the characteristic line from this time will be $x(t) = \Big(1 - 2\rho(x(t), t)\Big)(t + 3x_0) = \frac{29}{72}(t + 3x_0)$ for $t > -3x_0$.

We can see that the shocks will form immediately after $t > 0$. One shock will stay at 0 for all time, and another shock is moving with the speed according to the Rankine-Hugoniot jump condition

$$\sigma'(t) = \frac{f(\rho^-) - f(\rho^+)}{\rho^- - \rho^+}.$$

Here $f(\rho) = \rho(1 - \rho)$, $\rho^- = \frac{43}{144}$, while $\rho^+ = \frac{1}{3}$, thus

$$\sigma'(t) = \frac{f(\rho^-) - f(\rho^+)}{\rho^- - \rho^+} = \frac{53}{144} \qquad \Longrightarrow \qquad \sigma(t) = \frac{53}{144}t.$$

A formula for $\rho(x, t)$ can be given by

$$\rho(x, t) = \begin{cases} \frac{1}{3} & x \le 0 \\ \frac{43}{144} & 0 \le x \le \sigma(t) \\ \frac{1}{3} & \sigma(t) \le x. \end{cases}$$

□

January 2016

Problem (January 2016 - 1/6).

(a) Find a formula that expresses all the solutions of the second order differential equation $u'' = u$. Explain why there cannot exist any other solutions, different from the ones given by your formula.

(b) Now consider the initial value problem with initial conditions $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$, where g and h have compact support.

(c) Give an example of a dynamical system with parameter $\lambda \in \mathbb{R}$ such that for some value of λ you obtain a transcritical bifurcation, and for another value of λ you obtain a pitchfork bifurcation. Explain your answer.

Proof.

(a) A general formula is $u(x) = ae^x + be^{-x}$. Given any solution $\tilde{u}$ of $\tilde{u}'' = \tilde{u}$, we can compute

$$a = \frac{\tilde{u}(0) + \tilde{u}'(0)}{2}, \qquad \text{and} \qquad b = \frac{\tilde{u}(0) - \tilde{u}'(0)}{2}.$$

Then $v(x) = ae^x + be^{-x}$ solved $v'' = v$ with $v(0) = \tilde{u}(0), v'(0) = \tilde{v}'(0)$, which implies $\tilde{u} \equiv v$ by the uniqueness of the second order ODE with 2 initial conditions $u(0), u'(0)$.

(b) The answer is **YES**. We can choose a system in $\mathbb{R}^2$ with pure imaginary eigenvalues, then the orbits are circles, there is some smaller circle contained inside a bigger one.

(c) Let's recall a helpful criterion for bifurcation behaviors in 1D system. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ is smooth of (x, μ) satisfies the necessary bifurcation conditions

$$f(x_0, \mu_0) = 0, \qquad \text{and} \qquad \frac{\partial f}{\partial x}(x_0, \mu_0) = 0.$$

Then

- If $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$, $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$ then a saddle-node bifurcation occurs at (x_0, μ_0).
- If $\frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0$, $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$ and $\frac{\partial^2 f}{\partial x \partial \mu}(x_0, \mu_0) \neq 0$ then a transcritical bifurcation occurs at (x_0, μ_0).
- If $\frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0$, $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) = 0$, $\frac{\partial^2 f}{\partial x \partial \mu}(x_0, \mu_0) \neq 0$ and $\frac{\partial^3 f}{\partial x^3}(x_0, \mu_0) \neq 0$ then a pitchfork bifurcation occurs at (x_0, μ_0).

Now let $\eta_1, \eta_2 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\eta_1) \subset (1, 3)$ and $\eta_1 \equiv 1$ for $\frac{3}{2} \leq x \leq \frac{5}{2}$, and η_2 be the reflection of η_1 around the y-axis. Let's consider

$$f(x, \mu) = \Big((\mu - 1)(x - 2) - (x - 2)^3\Big)\eta_1(x) + \Big((\mu + 1)(x + 2) - (x + 2)^2\Big)\eta_2(x).$$

It is easy to see that f is smooth, and a transcritical bifurcation occurs at $(-2, -1)$, while a (super-critical) pitchfork bifurcation occurs at $(2, 1)$.

□

Problem (January 2016 - 2/6).

(a) Calculate the Fourier transform of the constant function $f(x) = 2\pi$.

(b) Is the identity

$$\sin x + \frac{\cos 2x}{2} + \frac{\sin 3x}{3} + \ldots = \cos x + \frac{\sin 2x}{2} + \frac{\cos 3x}{3} + \ldots$$

true or false? Explain your answer.

(c) Calculate the Fourier transform of the function $h(x) = e^{-|x|}$ and verify that your solution satisfies Plancherel's formula.

(d) Consider the differential equation $-\frac{d^2 u}{dx^2} + a^2 u = h(x)$. Use the Fourier transform to calculate its Green's function, i.e., the solution of $-\frac{d^2 G}{dx^2} + a^2 G = \delta(x)$. How would you use the function $G(x)$ for solving the original equation for any given right-hand side function $h(x)$?

Proof. We use the convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi \cdot x}\, dx$.

(a) In terms of tempered distribution, we have the Fourier transform of $2\pi\delta$ is

$$\left\langle \widehat{2\pi\delta}, \phi \right\rangle = \left\langle 2\pi\delta, \widehat{\phi} \right\rangle = 2\pi\widehat{\phi}(0) = 2\pi \int_{\mathbb{R}} \phi(x)\, dx = \langle 2\pi, \phi \rangle$$

where $\phi \in \mathcal{S}(\mathbb{R})$ is the Schwartz test function. Now let's replace ϕ by $\widehat{\phi}$, recalling that the Fourier transform is an isormorphism from $\mathcal{S} \longrightarrow \mathcal{S}$, we have

$$\left\langle \widehat{2\pi\delta}, \widehat{\phi} \right\rangle = \langle 2\pi, \widehat{\phi} \rangle \qquad \Longrightarrow \qquad \langle 2\pi\delta, \mathcal{F}\mathcal{F}(\phi) \rangle = \langle \widehat{2\pi}, \phi \rangle = \langle 2\pi\delta, \mathcal{R}\phi \rangle = 2\pi\phi(0) = \langle 2\pi\delta, \phi \rangle$$

where we use the Fourier inversion formula $\mathcal{F}\mathcal{F}\phi(x) = \phi(-x)$. This implies $\widehat{2\pi} = 2\pi\delta$.

(b) It is **FALSE**, since if it is true then we can multiply both sides to $\sin x$, and integrate from $-\pi$ to π, then the left hand side we obtain π, while the right hand side equals to 0, which is a contradiction.

(c) First of all, by integration by parts we have the damped cosine integral for $\lambda > 0$:

$$\boxed{\int_0^\infty e^{-\lambda x}\cos(ax)\,dx = \frac{\lambda}{\lambda^2+a^2}.}$$

By definition and the above formula we obtain

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-|x|}e^{2\pi i\xi\cdot x}\,dx = \int_0^\infty e^{-x}e^{-2\pi i\xi\cdot x}\,dx + \int_0^\infty e^{-x}e^{+2\pi i\xi\cdot x}\,dx = \frac{2}{1+4\pi^2\xi^2}.$$

We need to verify that

$$\int_{-\infty}^{+\infty}\left(e^{-|x|}\right)^2\,dx = \int_{-\infty}^{+\infty}\left(\frac{2}{1+4\pi^2\xi^2}\right)^2\,d\xi.$$

It is easy to see that the left hand side is 1, while the right hand side equals to

$$\int_{-\infty}^{+\infty}\left(\frac{2}{1+4\pi^2\xi^2}\right)^2\,d\xi = \frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{d\zeta}{(1+\zeta^2)^2} = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta = 1$$

by setting $\zeta = \tan\theta$.

(d) Recall that under our convention then $\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i\xi)^\alpha\widehat{f}(\xi)$ for any multi-index α. We consider the Green function subject to a unit impulse concentrated at $x = \eta$,

$$-G_\eta''(x) + a^2G_\eta(x) = \delta_\eta(x) = \delta(x-\eta).$$

Thus taking the Fourier transform both side we obtain

$$-(2\pi i\xi)^2\widehat{G_\eta}(\xi) + a^2\widehat{G_\eta}(\xi) = e^{-2\pi i\eta\cdot\xi} \qquad\Longrightarrow\qquad \widehat{G_\eta}(\xi) = \frac{e^{-2\pi i\eta\cdot\xi}}{a^2+4\pi^2\xi^2}.$$

At $\eta = 0$, we have

$$\mathcal{F}\Big(G_0\Big)(\xi) = \frac{1}{a^2+4\pi^2\xi^2}. \tag{7}$$

From part (c), by re-scaling we obtain

$$\mathcal{F}\Big(e^{-a|x|}\Big)(\xi) = \frac{2a}{a^2+4\pi^2\xi^2} \qquad\Longrightarrow\qquad \mathcal{F}\left(\frac{e^{-a|x|}}{2a}\right)(\xi) = \frac{1}{a^2+4\pi^2\xi^2}. \tag{8}$$

From (11) and (12) we deduce that

$$G_0(x) = \frac{e^{-a|x|}}{2a} \qquad\Longrightarrow\qquad G_\eta(x) = \frac{e^{-a|x-\eta|}}{2a}.$$

Finally the superposition principle based on the Green's function tells us the solution of the original ODE with source $h(x)$ is

$$u(x) = \int_{-\infty}^{+\infty} G_\eta(x)h(\eta)\,d\eta = \frac{1}{2a}\int_{-\infty}^{+\infty} e^{-a|x-\eta|}h(\eta)\,d\eta.$$

□

Problem (January 2016 - 3/6).

(a) In an n-dimensional space, what is the distance from the origin (0, 0, ..., 0) to the hyperplane given by the equation $x_1 + \sqrt{2}x_2 + \ldots + \sqrt{n}x_n = 1$?

(b) True or false: If the matrix B is symmetric and positive definite then any power B^n is also positive definite. Explain your answer.

(c) True or false: If the matrix B is positive definite then any power B^n is also positive definite. Explain your answer.

(d) Consider the matrix equation $A^2 = I$, where A is an $n \times n$ matrix and I is the identity matrix. How many solutions does this equation have? Same question for the equation $A^3 = I$. [Hint: Try to think about these matrix equation geometrically, in terms of linear transformations. Also, note that, depending on how you solve this problem, the equation $A^3 = I$ in the case $n = 2$ may require a different approach that in the case $n \geq 3$.]

Proof.

(a) The hyperplane $(P) := x_1 + \sqrt{2}x_2 + \ldots + \sqrt{n}x_n = 1$ above passing through $(1, 0, \ldots, 0)$ and has a normal vector is $(1, \sqrt{2}, \ldots, \sqrt{n})$. Since $O \notin (P)$, we only need to find the projection $O' = (x_1, \ldots, x_n)$ of O to (P), it can be done by solving the system

$$\begin{cases} x_1 + \sqrt{2}x_2 + \ldots + \sqrt{n}x_n = 1 \\ \overrightarrow{OO'} = (x_1, \ldots, x_n) = \lambda(1, \sqrt{2}, \ldots, \sqrt{n}). \end{cases}$$

for some $\lambda \in \mathbb{R}$. Since $\overline{OO'}$ must be parallel to $\overrightarrow{n} = (1, \sqrt{2}, \ldots, \sqrt{n})$. Plug this into the equation of (P) we obtain

$$\lambda\Big(1{+}2{+}3{+}\ldots{+}n\Big) = 1 \quad \Longrightarrow \quad \lambda = \frac{2}{n(n+1)} \quad \Longrightarrow \quad |OO'| = \frac{2}{n(n+1)}\sqrt{1+2+\ldots+n} = \left(\frac{2}{n(n+1)}\right)^{\frac{3}{2}}.$$

(b) **TRUE**. If B is (real) symmetric then its eigenvalues are real and its eigenvectors can be chosen to be orthonormal. I.e there exists a orthogonal matrix Q such that $QQ^T = I$ and $B = Q\Lambda Q^{-1}$. Where $\Lambda = \text{diag}\Big[\lambda_1, \ldots, \lambda_n\Big]$ where $\lambda_i > 0$ are eigenvalues of B, since B is positive definite. Then clearly $B^n = Q\Lambda^n Q^{-1}$ is also positive definite since $\Lambda^n = \text{diag}\Big[\lambda_1^n, \ldots, \lambda_n^n\Big]$ has $\lambda_i^n > 0$.

(c) **FALSE**. A counter example can be constructed by

$$B = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \qquad \text{has} \qquad \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - xy + y^2 = \Big(x - \frac{y}{2}\Big)^2 + \frac{3y^2}{4}.$$

But B^2 is not positive definite, as

$$B^2 = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 2 & -1 \end{pmatrix} \quad \text{has} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 < 0.$$

(d) For example in 2D, we can think about rotation and reflection are only two example of these kind of matrices $A^2 = I$. But even in 2D then the solution must be infinite, since they maybe differ by an equivalent relation $A \sim S^{-1}AS$. Nevertheless, we can characterize all the solution of $A^2 = I$ upto equivalent classes by using Jordan form. Recall that every matrices admit equivalent relation based on the their unique Jordan forms. So the question is reduced to count how many Jordan forms satisfy $J^2 = I$. It is easy to see that the Jordan form satisfies $J^2 = I$ (even in n-dimension) must be a diagonal matrix. So we only need to count how many ways we can arrange 1 and -1 on the diagonal, which is $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ equivalent classes.

In case $A^3 = I$, we must have $\det(A) = 1$, i.e the number of -1 on the diagonal should be an even number, thus the number of equivalent classes is

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}$$

where $[a]$ means the integer part of a, i.e the biggest integer number which is not bigger than a.

□

Problem (January 2016 - 4/6). Consider the heat equation with forcing: $u_t = u_{xx} + f(x,t)$, for $x \in [0, 2\pi)$ with periodic boundary conditions. Let the forcing take the form $f(x,t) = F\cos(2x)\cos(t)$, where F is a constant.

(a) Find a particular solution to this problem that is temporally periodic. Call it $u_\infty(x,t)$.

(b) Consider now the initial value problem. Let $u(x,t)$ be the solution corresponding to the initial condition $u(x,0) = g(x)$, where $g(x)$ is an L^2 function. Show that $u(x,t)$ converges to $u_\infty(x,t)$ in the L^2 norm as $t \longrightarrow \infty$.

Proof. The statement of this problem is a little bit incorrect, since u_∞ maybe found upto some constants, and the constant depends on $g \in L^2$.

(a) We look for an solution with is periodic in time in the form

$$u_\infty(x,t) = a(x)\cos t + b(x)\sin t \quad \Longrightarrow \quad \begin{cases} a(x) &= -b''(x) \\ b(x) &= a''(x) + F\cos 2x \end{cases} \quad \Longrightarrow \quad b(x) = -b^{(4)}(x) + F\cos(2x).$$

Then if we seek for $b(x)$ of the form $C\cos(2x)$, plug it in we obtain $C = \frac{F}{17}$, thus

$$u_\infty(x,t) = \left(\frac{4F}{17}\cos t + \frac{F}{17}\sin t\right)\cos 2x.$$

(b) Let $u(x,t)$ be a solution to the heat source problem with periodic boundary condition, we claim that there is a constant c such that

$$w(x,t) = u(x,t) - (u_\infty(x,t) + c)$$

converges to 0 in L^2. It is clear that $w(x,t)$ satisfies the classical heat equation. We know that $E(t) = \frac{1}{2}\int_0^{2\pi} |w(x,t)|^2\, dx$ satisfies $E'(t) \le 0$ and $N(t) = \int_0^{2\pi} w(x,t)\, dx$ is constant $N(t) = N(0)$. On the other hand we have

$$E(t) \le E(0) = \frac{1}{2}\int_0^{2\pi} |g(x) - u_\infty(x,0) - c|^2\, dx$$

We seek for solution of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n(t)\cos nx + b_n(t)\sin nx\Big).$$

We have

$$u_t - u_{xx} = \sum_{n=1}^{\infty}\Big[\Big(a_n'(t) + n^2 a_n(t)\Big)\cos nx + \Big(b_n'(t) + n^2 b_n(t)\Big)\sin nx\Big] \equiv F\cos 2x\cos t.$$

That means

$$\begin{cases} b_n'(t) &= -n^2 b_n(t) \quad \text{for all } n \in \mathbb{N} \\ a_n'(t) &= -n^2 a_n(t) \quad \text{for all } n \in \mathbb{N}\backslash\{2\} \end{cases} \quad \Longrightarrow \quad \begin{cases} b_n(t) &= b_n(0)e^{-n^2 t} \quad \text{for all } n \in \mathbb{N} \\ a_n(t) &= a_n(0)e^{-n^2 t} \quad \text{for all } n \in \mathbb{N}\backslash\{2\} \end{cases}.$$

For $n = 2$, we want

$$a_2'(t) + 2^2 a_2(t) = F\cos t \quad \Longrightarrow \quad a_2(t) = \frac{4F\cos t + F\sin t}{17} + \Big(a_2(0) - \frac{4F}{17}\Big)e^{-4t}.$$

Thus the solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n(0)e^{-n^2 t}\cos nx + b_n(0)e^{-n^2 t}\sin nx\Big) + u_\infty(x,t) - \frac{4F}{17}e^{-4t}.$$

With the initial data $g \in L^2([0,2\pi))$ we require

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n(0)\cos nx + b_n(0)\sin nx\Big) + \frac{4F}{17}\cos 2x - \frac{4F}{17} \equiv g(x).$$

Comparing the coefficient we obtain

$$\frac{a_0}{2} - \frac{4F}{17} = \frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx \qquad \Longrightarrow \qquad \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx + \frac{4F}{17}.$$

Then we obtain

$$u(x,t) - \left(u_\infty + \frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx + \frac{4F}{17}\right) = \sum_{n=1}^{\infty}\left(a_n(0)e^{-n^2 t}\cos nx + b_n(0)e^{-n^2 t}\sin nx\right) - \frac{4F}{17}e^{-4t}.$$

The right hand side converges to zero exponentially as $t \longrightarrow \infty$, thus we conclude that

$$u(x,t) \longrightarrow u_\infty + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx + \frac{4F}{17}}_{C_\infty}$$

as $t \longrightarrow \infty$, while the right hand side is another periodic in time solution of the forced heat equation (with a appropriate constant).

□

Remark. The convergence of the series above can be made rigorous. Indeed let g_N to be the truncation of the first $N+1$ terms of the Fourier series of g, then we know that $g_N \longrightarrow g$ in L^2. Let u_N be the solution of the forced heat equation, then we obtain

$$u_N(x,t) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n(0)e^{-n^2 t}\cos nx + b_n(0)e^{-n^2 t}\sin nx\right) + u_\infty(x,t) - \frac{4F}{17}e^{-4t}.$$

Then clearly with C_∞ as above we have

$$\|u_N - (u_\infty + C_\infty)\|_{L^2}^2 = \sum_{n=3}^{N}\left(|a_n(0)|^2 + |b_n(0)|^2\right)e^{-2n^2 t} + \left(|a_n(1)|^2 + |b_n(1)|^2\right)e^{-2t} + \left(\left|a_2(0) - \frac{4F}{17}\right|^2 + |b_2(0)|^2\right)e^{-8t}$$

which is clearly vanish as $t \longrightarrow \infty$ since it is a finite sum. Now $u - u_N$ is the solution to the classical heat equation, we have

$$\begin{aligned}\|u - (u_\infty + C_\infty)\|_{L^2} &\le \|u - u_N\|_{L^2}^2 + \|u_N - (u_\infty + C_\infty)\|_{L^2} \\ &\le \|g - g_N\|_{L^2} + \|u_N - (u_\infty + C_\infty)\|_{L^2} \longrightarrow 0\end{aligned}$$

as $t \longrightarrow \infty$.

Problem (January 2016 - 5/6). Consider the wave equation $u_{tt} - u_{xx} = 0$ on the real line.

(a) Show that if a function $u(x,t)$ satisfies $u_{tt} - u_{xx} = 0$, then it must be of the form $u(x,t) = F(x-t) + G(x+t)$.

(b) Now consider the initial value problem with initial conditions $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$, where g and h have compact support.

  (i) Show that the energy $E(t) = \frac{1}{2}\int_\infty^\infty \left[u_t^2 + u_x^2\right] dx$ is constant in t.

  (ii) Show that $K(t) = V(t)$ for all sufficiently large times t, where $K(t) = \frac{1}{2}\int_{-\infty}^\infty u_t^2\,dx$ is the kinetic energy and $V(t) = \frac{1}{2}\int_{-\infty}^\infty u_x^2\,dx$ is the potential energy.

Proof.

(a) From d'Alembert's formula.

(b) Recall the d'Alembert's formula

$$u(x,t) = \frac{g(x-t) + g(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz.$$

Thus using the Lebnitz's formula for derivative we have

$$u_t(x,t) = \frac{-g'(x-t)+g'(x+t)}{2} + \frac{1}{2}\left[h(x+t)+h(x-t)+\int_{x-t}^{x+t} h'(z)\,dz\right]$$
$$u_x(x,t) = \frac{g'(x-t)+g'(x+t)}{2} + \frac{1}{2}\left[h(x+t)-h(x-t)+\int_{x-t}^{x+t} h'(z)\,dz\right].$$

Since g, h has compact support, we deduce that for each fixed $t > 0$, $x \longmapsto u(x,t), u_t(x,t), u_x(x,t)$ have compact support in $\mathbb{R}$.

(i) Define the total energy as

$$E(t) = k(t) + p(t) = \frac{1}{2}\int_{\mathbb{R}} \left(u_t(x,t)^2 + u_x(x,t)^2\right) dx.$$

For a fixed t, the function

$$x \longmapsto \frac{\partial}{\partial t}\left(\frac{u_t(x,t)^2 + u_x(x,t)^2}{2}\right) = u_t u_{tt} + u_x u_{xt} = \frac{\partial}{\partial x}\left(u_x u_t\right)$$

has compact support in $\mathbb{R}$ since u_t, u_x has compact support in $\mathbb{R}$, thus we can use the Lebesgue dominated convergence theorem to deduce that

$$E'(t) = \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{\mathbb{R}} \left(u_t(x,t)^2 + u_x(x,t)^2\right) dx\right)$$
$$= \int_{\mathbb{R}} \frac{\partial}{\partial t}\left(\frac{u_t(x,t)^2 + u_x(x,t)^2}{2}\right) dx = \int_{\mathbb{R}} \frac{\partial}{\partial x}\left(u_t(x,t)u_x(x,t)\right) dx = u_t(x,t)u_x(x,t)\Big|_{x=-\infty}^{x=+\infty} = 0$$

since u_t, u_x has compact support for each fixed t. Thus $E(t) = E(0)$ is a constant for all time $t > 0$.

(ii) Also from d'Alembert's formula, we have

$$u(x,t) = F(x-t) + G(x+t)$$

where

$$F(s) = \frac{1}{2}g(s) - \frac{1}{2}\int_0^s h(\xi)\,d\xi + C \qquad \text{and} \qquad G(s) = \frac{1}{2}g(s) + \frac{1}{2}\int_0^s h(\xi)\,d\xi - C.$$

where C is a constant, i.e

$$F'(s) = \frac{g'(s)-h(s)}{2} \qquad \text{and} \qquad G'(s) = \frac{g'(s)+h(s)}{2}$$

which are compactly supported in $\mathbb{R}$. From that we have

$$\begin{cases} u_t(x,t) &= -F'(x-t) + G'(x+t) \\ u_x(x,t) &= \ \ F'(x-t) + G'(x+t) \end{cases} \implies u_t^2 - u_x^2 = -4F'(x-t)G'(x+t).$$

Thus

$$k(t) - p(t) = -2\int_{-\infty}^{\infty} F'(x-t)G'(x+t)\,dx.$$

Assume the support of F, G is $[-R, R]$, then the domain of the above integral for a fixed t is

$$-R + t \le x \le R - t.$$

It is easy to see that if we have t large enough, then $-R+t \ge R-t$, then clearly $k(t) = p(t)$ follows.

□

Problem (January 2016 - 6/6). Traffic flow with an off-ramp. Consider the following initial value problem for traffic flow with an off-ramp:

$$\begin{cases} \rho_t + \Big[\rho(1-\rho)\Big]_x &= -D\delta(x) \\ \rho(x,0) &= \rho_i. \end{cases}$$

Derive the solution to the initial value problem for the case of $\rho_i = \frac{3}{4}$, $D = \frac{5}{144}$. Present your solution in all of the following ways:

(a) an analytic formula for $\rho(x,t)$ for general values of x and t,

(b) a plot of the characteristic curves in the $x-t$ plane, and

(c) a plot of $\rho(x,t)$ at time $t = 12$.

Proof. Using method of characteristic, we consider

$$\begin{cases} x'(t) &= 1-2\rho(x(t),t) \\ x(0) &= x_0 \end{cases} \quad \text{and} \quad \begin{cases} z'(t) &= -D\delta(x(t)) \\ z(0) &= \rho_i. \end{cases}$$

Where $z(t) = \rho(x(t),t)$. If $x(t)$ remains its sign from x_0, then $\rho(x(t),t)$ remains the same value as ρ_i. Things only change at the time $x(t)$ switches its sign.

- If we start from $x_0 < 0$, then since $\rho_i = \frac{3}{4}$, we have $1-2\rho_i = -\frac{1}{2} < 0$, which means $x(t) = x_0 - \frac{t}{2}$ and remains negative all time. Thus $\rho(x(t),t) = \rho_i = \frac{3}{4}$ for all time.
- If we start from $x_0 > 0$, then $x(t) = x_0 - \frac{t}{2}$ when $t < 2x_0$, which means $\rho(x(t),t) = \rho_i = \frac{3}{4}$ when $t < 2x_0$. After $t > 2x_0$, the value of $\rho(x(t),t)$ must be constant $\rho_i - D = \frac{3}{4} - \frac{5}{144} = \frac{103}{144}$. And the characteristic line from this time will be $x(t) = \Big(1-2\rho(x(t),t)\Big)(t-2x_0) = -\frac{31}{72}(t-2x_0)$ for $t > 2x_0$.

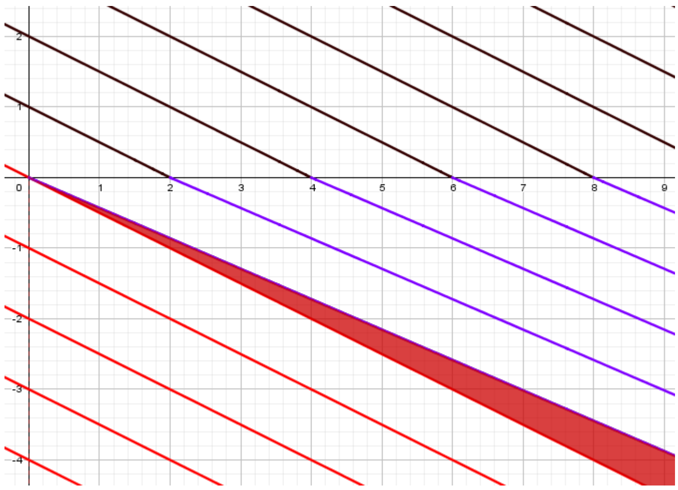


Figure 4: Characteristic curves in the $t-x$ plane.

There is a region on which the value of $\rho(x,t)$ is not determined by the characteristic curves. We can fill in the value of ρ in this region in a linear way as

$$u(x,t) = \begin{cases} \frac{3}{4} & x \geq 0 \\ \frac{103}{144} & -\frac{31}{72}t \leq x \leq 0 \\ \frac{3}{4} - \frac{1}{2}\left(\frac{x}{t} + \frac{1}{2}\right) & -\frac{t}{2} \leq x \leq -\frac{31}{72}t \\ \frac{3}{4} & x \leq -\frac{t}{2}. \end{cases}$$

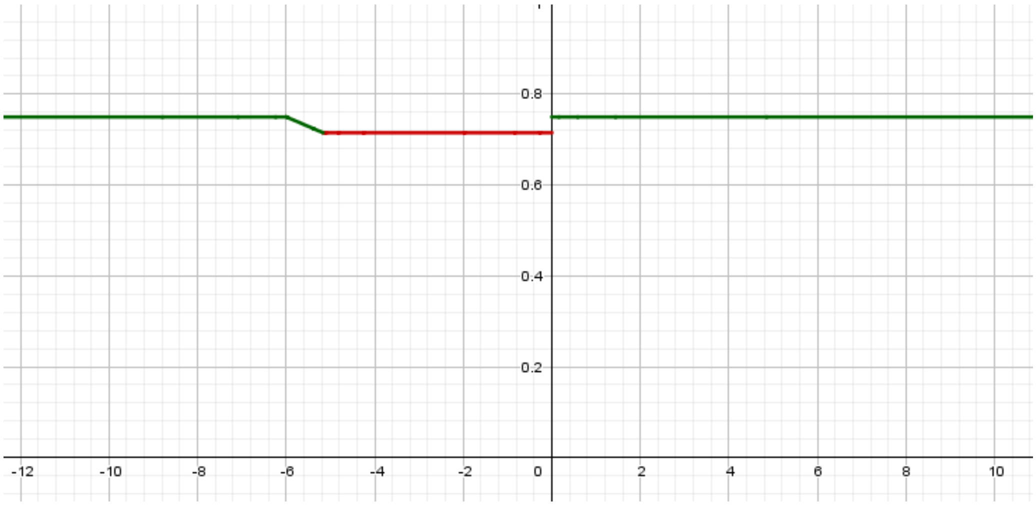


Figure 5: A plot of $\rho(t,x)$ at time $t=12$.

$\square$

August 2016

Problem (Aug 2016 - 1/6).

(a) Consider two 5×5 matrices A and B. If $\text{rank}(A)=4$ and the matrix $A+B$ is invertible, what is the minimal rank of B? Explain your answer.

(b) True or false: if the square matrix A is symmetric and positive definite, then there exists a matrix B such that $A=B^2$. Explain your answer.

(c) True or false: if the square matrices A and B satisfy $A=B^2$ and B is invertible, then A is positive definite. Explain your answer.

(d) Consider the line in $\mathbb{R}^4$ that contains the origin and the point (1,1,1,1). Also, consider the line in $\mathbb{R}^4$ that contains the points (2; 0; 0; 0) and (0; 0; 0; 2). What is the distance between these lines? Explain your answer.

Proof.

(a) We have the rank inequality for $A,B\in\mathbb{M}_{n\times n}(\mathbb{R})$:

$$\text{rank}(A+B)\le\text{rank}(A)+\text{rank}(B).$$

Proof. Recall that rank(A) is the dimension of the column space. I.e if we define the linear operator $f_A:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ by $f_A(x)=Ax$ then clearly $\text{Im}(f_A)=\text{span}(\text{columns of }A)$, which means $\text{rank}(A)=\dim(\text{Im } f_A)$. Now clearly $\text{Im } f_{A+B}\subset\text{Im } f_A+\text{Im } f_B$ which implies $\text{rank}(A+B)\le\text{rank}(A)+\text{rank}(B)$. $\square$

Now using this fact we have $\text{rank}(B)\ge 1$. One example is as following

$$\begin{pmatrix}0 & \\ & I_4\end{pmatrix}+\begin{pmatrix}1 & \\ & 0_4\end{pmatrix}=\begin{pmatrix}1 & \\ & I_4\end{pmatrix}=I_5.$$

(b) **TRUE**. If A is (real) symmetric then its eigenvalues are real and its eigenvectors can be chosen to be orthonormal. I.e there exists a orthogonal matrix Q such that $QQ^T = I$ and $A = Q\Lambda Q^{-1}$. Where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

where $\lambda_i > 0$ are eigenvalues of A, since A is positive definite. Thus we can define

$$\tilde{\Lambda} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix} \implies A = Q\Lambda Q^{-1} = Q\tilde{\Lambda}\tilde{\Lambda}Q^{-1} = \underbrace{\left(Q\tilde{\Lambda}Q^{-1}\right)}_{B}\underbrace{\left(Q\tilde{\Lambda}Q^{-1}\right)}_{B} = B^2.$$

(c) **FALSE**. A counter-example is

$$B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \implies B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ has } (1,0)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1.$$

While $\det(B) = 1$.

Remark. It will be true if B is symmetric, since in this case for $x \in \mathbb{R}^n \backslash \{0\}$ as a column vector then $x^T A x = x^T B^2 x = x^T B^T B x = (Bx)^T (Bx) = \|Bx\|^2 > 0$. Note that B is invertible, thus the only x that makes $Bx = 0$ is $x = 0$.

(d) Let $(d_1) : \{x(1,1,1,1) : x \in \mathbb{R}\}$ and $(d_2) : \{(2,0,0,0) + y(2,0,0,-2) = (2+2y, 0, 0, -2y) : y \in \mathbb{R}\}$.

- The first proof. Pick $A \in (d_1)$ and $B \in (d_2)$ then the distant between them is

$$|AB|^2 = (x - 2y - 2)^2 + 2x^2 + (x + 2y)^2 := f(x, y).$$

We want to minimize this distance, by taking derivative and set it equal to zero, we have $\nabla f(x, y) = 0$ if and only if $(x, y) = \left(\frac{x}{2}, -\frac{1}{2}\right)$. From that we have the distance between these lines should be 1. We can easily check that the line passing A, B with in this case is both perpendicular to these lines.

- The second proof. The idea is to find a space (P) with is both perpendicular to (d_1) and (d_2), and for any $A \in (d_1), B \in (d_2)$ we have the distance between (d_1) and (d_2) will be the projection of AB onto (P). Let

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & -2 \end{pmatrix}$$

then $(P) = \text{span(null space of } \mathbb{A})$. The null space is generated by two linearly independent vectors $(1,0,-2,1)^T$ and $(0,1,-1,0)^T$. Given a vector $\overrightarrow{AB}$ with $A \in (d_1)$ and $B \in (d_2)$, the projection of $\overrightarrow{AB}$ into (P) will be $a(1,0,-2,1)^T + b(0,1,-1,0)^T$ for some (a, b) satisfies

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -1 \\ 1 & 0 \end{pmatrix}}_{\mathbb{P}}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{x} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \overrightarrow{AB} \quad \text{and} \quad \overline{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \perp (P).$$

Choosing $A = (0,0,0,0)$ and $B = (2,0,0,0)$ then $\overrightarrow{AB} = (2,0,0,0)$, we have

$$\mathbb{P}^T\mathbb{P}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbb{P}^T\overrightarrow{AB} \implies \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \implies \begin{cases} a & = \frac{1}{2} \\ b & = -\frac{1}{2} \end{cases}.$$

Thus the distance will be $|\mathbb{P}x|$, which is

$$\left|\mathrm{P}\begin{pmatrix} a \\ b \end{pmatrix}\right|^2 = \left|\left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right|^2 = 1.$$

□

Problem (Aug 2016 - 2/6). Consider the predator-prey model given by

$$\dot{x} = x\left(b - x - \frac{y}{1+x}\right), \qquad \dot{y} = y\left(\frac{x}{1+x} - ay\right)$$

where $x, y \geq 0$ are the populations and $a, b > 0$ are parameters.

(a) Sketch the nullclines and discuss the bifurcations that occur as b varies.

(b) Show that a positive fixed point $x^* > 0, y^* > 0$ exists for all $a, b > 0$.

(c) Show that a Hopf bifurcation occurs at the positive fixed point if

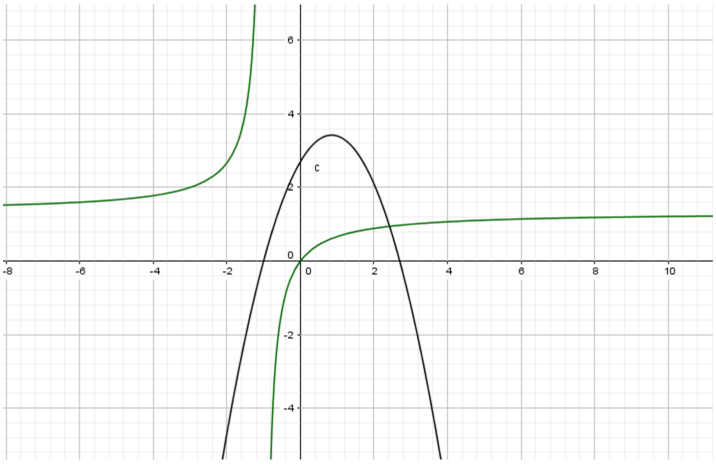
$$a = a_c = \frac{4(b-2)}{b^2(b+2)}$$

and $b > 2$.

(d) Using a computer, check the validity of the expression in (c) and determine whether the bifurcation is subcritical or supercritical. Plot typical phase por- traits above and below the Hopf bifurcation.

Proof.

(a-b) The nullclines include $x = 0$, $y = 0$, $y = (b-x)(1+x)$ and $y = \frac{1}{a}\frac{x}{1+x}$. The fixed points are $(0,0), (b,0)$ and (x^*, y^*) which is a positive fixed point where $0 < x^* < b$.



The Jacobian at a point (x, y) is given by

$$J_{(x,y)} = \begin{pmatrix} b - 2x - \frac{y}{(1+x)^2} & -\frac{x}{1+x} \\ \frac{y}{(1+x)^2} & \frac{x}{1+x} - 2ay \end{pmatrix}$$

- At (0,0) the system is stable with no movement at all, thus no bifurcation happens at b varies such that $b > 0$.
- At $(b, 0)$ we have

$$J_{(0,b)} = \begin{pmatrix} -b & -\frac{b}{1+b} \\ 0 & \frac{b}{1+b} \end{pmatrix}.$$

Then it has characteristic polynomial is $\lambda^2 + \frac{-b^2}{1+b} - \frac{b^2}{1+b}$ which has two eigenvalues, $\lambda < 0$ and $\lambda_2 > 0$ for all value of $b > 0$. Thus $(0, b)$ is an saddle node for all $b > 0$, and no bifurcation occurs.

- A simple calculus argument shows that for any $b > 0$ there exists a unique fixed point (x^*, y^*). Bifurcation may occur at this point. The Jacobian at this point is

$$J_{(x^*,y^*)} = \begin{pmatrix} b - 2x - \frac{b-x}{1+x} & -\frac{x}{1+x} \\ \frac{b-x}{1+x} & -\frac{x}{1+x} \end{pmatrix}.$$

Thus at b varies such that $b > 0$, the number of fixed point remains the same.

(c) At (x^*, y^*) we have the trace and the determinant are [1]

$$\operatorname{Tr} J = \frac{x^*}{1+x^*}(b-2-x^*) \qquad \text{and} \qquad \det J = \frac{x^*}{(1+x^*)^2}\Big(2(x^*)^2 - bx^* + b\Big).$$

A necessary condition for Hopf bifurcation happen at (x^*, y^*) is $\operatorname{Tr}(J_{(x^*,y^*)}) = 0$. It is equivalent to

$$\begin{aligned} b - 2x^* - \frac{y^*}{(1+x^*)^2} - ay = 0 \qquad &\Longleftrightarrow \qquad b - 2x^* - \frac{b-x^*}{1+x^*} - \frac{x^*}{1+x^*} = 0 \\ &\Longleftrightarrow \qquad (1+x^*)(b-2x^*) - b = 0 \\ &\Longleftrightarrow \qquad x^* = \frac{b-2}{2}. \end{aligned}$$

Thus

$$y^* = (b-x^*)(1+x^*) = \frac{b(b+2)}{4} \qquad \Longrightarrow \qquad \frac{b(b+2)}{4} = \frac{1}{a}\frac{x^*}{1+x^*} \qquad \Longrightarrow \qquad a_c = \frac{4(b-2)}{b^2(b+2)}$$

and clearly we need $b > 2$ to have $x^* > 0$. Also at (x^*, y^*) then the trace is zero we have

$$\det J_{(x^*,y^*)} = \frac{4(b-2)}{b^2} > 0$$

when $b > 2$. Thus at (x^*, y^*) when the Hopf bifurcation occurs we have two egienvalues are both pure imaginary. Now we actually prove that Hopf bifurcation actually occurs, by showing that:

- When $a \neq a_c$ then two eigenvalue of $J(x^*, y^*)$ are both complex numbers. It is proven using the fact that

$$\operatorname{Tr}(J)^2 - 4\det J = \frac{x^*}{(1+x^*)^2}\Big(x(x-b+2)^2 - 4(2(x^*)^2) - bx^* + b\Big) < 0 \qquad \text{for} \qquad x \in (0, b).$$

- When $a < a_c$ then $x^* < \frac{b-2}{2}$, which means $\operatorname{Tr}(J_{(x^*,y^*)}) > 0$, thus real parts of both eigenvalues are positive, hence (x^*, y^*) is a unstable fixed point (spiral). This fact is proven using the fact that when $a < a_c$ then

$$\frac{x^*}{1+x^*}\frac{1}{(b-x^*)(1+x^*)} < \frac{4(b-2)}{b^2(b-2)}.$$

  Note that when $x^* = \frac{b-2}{2}$ we have the left hand side and the right hand side are the same, and the function

$$x \longmapsto \frac{x^*}{1+x^*}\frac{1}{(b-x^*)(1+x^*)} - \frac{4(b-2)}{b^2(b-2)}$$

  is increasing on $(0, b)$, thus it is negative if and only if $x^* < \frac{b-2}{2}$.

- When $a > a_c$ then $x^* > \frac{b-2}{2}$, which means $\operatorname{Tr}(J_{(x^*,y^*)}) < 0$, thus real parts of both eigenvalues are negative, hence (x^*, y^*) is a stable fixed point (spiral).

These facts suggest that the Hopf bifurcation occurs when a varies passing a_c, as (x^*, y^*) changes from an unstable spiral to a stable spiral.

(d) It suggests that we have super-critical Hopf bifurcation at (x^*, y^*). Look at the pictures, we can see that before $a < a_c$ then (x^*, y^*) is an unstable spiral with a limit stable circle. After $a > a_c$ we only have a stable spiral, and also at $a = a_c$ we have limit circles, thus we have super-critical Hopf bifurcation.

□

Problem (Aug 2016 - 3/6).

(a) Find the Fourier series of $f(x) = \sin^3 x$ on $[-\pi, \pi]$.

(b) What multiple of $\sin x$ is closest in the least square sense to $\sin^3 x$? Explain your answer.

(c) Explain why functions of the form $u(r, \theta) = a_k r^k \sin k\theta$ are formal solutions of Laplace's equation.

(d) If a solution of Laplace's equation satisfies $u(1, \theta) = 1$ for $0 < \theta < \pi$ abd $u(1, \theta) = -1$ for $-\pi < \theta < 0$, find a formula for $u(r, \theta)$ inside the unit circle.

[1] Using $y^* = (b-x^*)(1+x^*) = \frac{1}{a}\frac{x^*}{1+x^*}$.

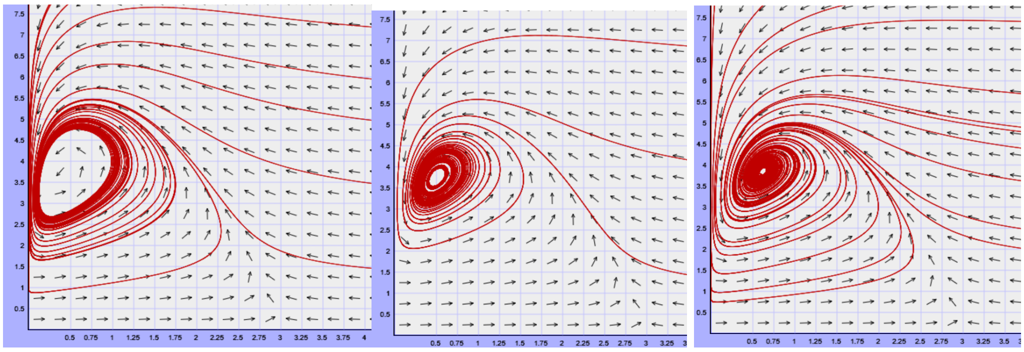


Figure 6: Phase portrait when $a < a_c, a = a_c$ and $a > a_c$ with $b = 3$.

Proof.

(a) We have $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

(b) Let's look at the first term $b_1 \sin x = \frac{3}{4}\sin x$. This is the closed approximation to $\sin^3 x$ by any multiple of $\sin x$ in the lest square sense. Indeed, let's look at the error

$$E(\zeta) = \int_{-\pi}^{\pi} \left|\sin^3 x - \zeta \sin x\right|^2 \, dx$$

and try to minimize this error among all $\zeta \in \mathbb{R}$, then taking derivative in ζ we see that

$$\begin{aligned} E'(\zeta) &= -2\int_{-\pi}^{\pi} \left(\sin^3 x - \zeta \sin x\right)\sin x \, dx \\ &= -2\int_{-\pi}^{\pi} \sin^3 x \sin x \, dx + 2\zeta \int_{-\pi}^{\pi} \sin^2 x \, dx = 2(-\pi b_1 + \zeta\pi). \end{aligned}$$

Thus $E'(\zeta) = 0$ if and only if $\zeta = b_1 = \frac{3}{4}$. It is clear that $\zeta \longmapsto E(\zeta)$ is decreasing on $(-\infty, b_1)$ and increasing on $(b_1, +\infty)$, which proves the claim.

(c) We have $r^k \sin k\theta = \mathrm{Im} z^k$ where $z = x + iy$, and recall the fact that the real and imaginary parts of a complex analytic function both satisfy the Laplace equation. I.e if $z = x + iy$ and

$$f(z) = u(x, y) + iv(x, y)$$

then the necessary condition that $f(z)$ be analytic is that the Cauchy-Riemann equations be satisfied:

$$u_x = v_y, \qquad u_y = -v_x \qquad \Longrightarrow u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0 = u_{xy} - u_{xy} = v_{xx} + v_{yy}.$$

Therefore u, v satisfies the Laplace equation. Now since $(x + iy)^k$ is an holomorphic function, which is also analytic, thus the imaginary part satisfies the Laplace equation.

(d) For the Dirichlet boundary problem we can use separation of variable to obtain the series solution

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right)$$

and then match the boundary condition, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} u(1, \theta)\, d\theta = 0.$$

and for $n \geq 1$ then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(1, \theta)\cos\theta \, d\theta = 0 \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(1, \theta)\sin\theta \, d\theta = \frac{2(1 - (-1)^n)}{n\pi}.$$

Thus inside the unit ball we have:

$$u(r,\theta)=\sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}r^n\sin n\theta.$$

□

Problem (Aug 2016 - 4/6). Consider Burger's equation with damping on the real line $-\infty < x < +\infty$:

$$\begin{cases} u_t + uu_x &= -Du \\ u(x,0) &= g(x) \end{cases}$$

where $D > 0$ is a constant damping coefficient.

(a) Using the method of characteristics, find explicit formulas
  (i) for characteristic curves $x(t)$ in terms of the initial location x_0 and the initial data g, and
  (ii) for $z(t) = u(x(t),t)$, the value of u along the characteristic curve, in terms of the initial location x_0 and the initial data g.

(b) Based on your method-of-characteristics solution, find an implicit formula for $u(x,t)$.

(c) What is the limiting form of your implicit formula in the limit $D \longrightarrow 0$?

(d) Find the time T_* for the first formation of a shock, as a function of D, for initial condition $g(x) = \sin(x)$. Also find the value D_* for which a shock never forms if $D \geq D_*$.

Proof.

(a) Let's consider $(0, x_0, u(x_0))$ be an initial point of the solution, we seek for

$$\begin{cases} x'(t) &= u(x(t),t) \\ x(0) &= x_0 \end{cases} \quad \text{and} \quad \begin{cases} z'(t) &= -Du(x(t),t) \\ z(0) &= u(x_0,0) = g(x_0). \end{cases} \tag{9}$$

Then if we consider $h(t) = u(x(t),t) - z(t)$ then

$$\alpha'(t) = u_t(x(t),t) + x'(t)u_x(x(t),t) - z'(t) = 0 \quad \Longrightarrow \quad \alpha(t) = \alpha(t_0) = 0.$$

Thus $u(x(t),t) = z(t)$. Substitute back to the equations (9) we obtain

$$\begin{cases} z'(t) &= -Dz(t) \\ z(0) &= g(x_0) \end{cases} \quad \Longrightarrow \quad \boxed{z(t) = g(x_0)e^{-Dt} = u(x(t),t).}$$

Thus we have

$$\begin{cases} x'(t) &= g(x_0)e^{-Dt)} \\ x(0) &= x_0 \end{cases} \quad \Longrightarrow \quad x(t) = \frac{g(x_0)}{D}\left(1-e^{-Dt}\right) + x_0 = \boxed{g(x_0)e^{-Dt}\left(\frac{e^{Dt}-1}{D}\right) + x_0.}$$

(b) In other words, we have

$$x(t) = x_0 + u(x(t),t)\left(\frac{e^{Dt}-1}{D}\right) \quad \Longrightarrow \quad \underbrace{x(t) - u(x(t),t)\left(\frac{e^{Dt}-1}{D}\right)}_{\xi} = x_0.$$

Thus we obtain the explicit formula for $u(x(t),t)$ in terms of the characteristic variable ξ is

$$u(x(t),t) = g\left(x(t) - u(x(t),t)\left(\frac{e^{Dt}-1}{D}\right)\right)e^{-Dt}.$$

Namely,

$$u(x,t) = g\left(x - u(x,t)\left(\frac{e^{Dt}-1}{D}\right)\right)e^{-Dt}. \tag{10}$$

Although it contains $u(x,t)$ inside the formula, (10) does provide an explicit formula to compute the value of $u(x,t)$. In fact, given a point (x^*, t^*), we seek for the initial point x_0 provided by

$$x^* = x_0 + g(x_0)e^{-Dt^*}\left(\frac{e^{Dt^*}-1}{D}\right).$$

And then

$$u(x^*, t^*) = g(x_0)e^{-Dt^*}.$$

(c) Assuming everything is continuous, then since for each $t \geq 0$ we have

$$\lim_{D\longrightarrow 0} \frac{e^{Dt}-1}{D} = t.$$

We conclude that as $D \longrightarrow 0$, the characteristic curve $x(t)$ becomes $x(t) = x_0 + tg(x_0) = x_0 + tu(x_0, 0)$, which is a line in (x,t)-plane. Also

$$\boxed{u(x_0 + tf(x_0), t) = f(x_0)}$$

for all t, which means $u(x,t)$ is constant on each characteristic line. Thus we recover the solution of the inviscid Burgers' equation.

(d) From (10) if we differentiate with respect to x then we obtain

$$u_x = g'(\xi)\left(1 - u_x\left(\frac{e^{Dt-1}}{D}\right)\right)e^{-Dt} \qquad \Longrightarrow \qquad u_x(x,t) = \frac{g'(\xi)e^{-Dt}}{1 + g'(\xi)\left(\frac{1-e^{-Dt}}{D}\right)}.$$

where

$$\xi = x(t) - f(x_0)e^{-Dt}\left(\frac{e^{Dt}-1}{D}\right) = x_0.$$

Shocks form at (x^*, t^*) if $|u_x(x^*, t)| \longrightarrow +\infty$ at $t \longrightarrow t^*$, i.e

$$1 + f'(\xi)\left(\frac{1-e^{-Dt}}{D}\right) = 0 \qquad \Longrightarrow \qquad t = \frac{-1}{D}\log\left(1 + \frac{D}{g'(\xi)}\right).$$

The condition for shock can be form is $t > 0$, which means

$$0 < 1 + \frac{D}{g'(\xi)} < 1 \qquad \Longrightarrow \qquad \begin{cases} g'(\xi) < 0 \\ g'(\xi) < -D. \end{cases}$$

Thus we conclude that the time T_* for the first formation of shock is

$$T_* = \min\left\{\frac{-1}{D}\log\left(1 + \frac{D}{g'(\xi)}\right) : g'(\xi) < -D\right\}.$$

For $g(\xi) = \sin(\xi)$ then $g'(\xi) = \cos(\xi)$. Thus if $D_* = 1$ then for any $D \geq 1$ we have $g'(\xi) = \cos\xi \geq -1$, thus shock cannot be formed. In this case we have

$$T_* = \min\left\{\frac{-1}{D}\log\left(1 + \frac{D}{\cos\xi}\right) : \cos\xi < -D\right\}.$$

□

Problem (Aug 2016 - 5/6). Consider the following PDE that describes the falling of rain through air:

$$\frac{\partial q}{\partial t} - \frac{\partial}{\partial z}\Big(V \cdot (q - q_s) \cdot H(q - q_s)\Big) = 0$$

where $V > 0$ is a constant advection velocity, and $H(q - q_s)$ is the Heaviside function, which equals 0 if $q - q_s < 0$ and equals 1 if $q - q_s > 0$. Assume the constant value $q_s = 0$ for simplicity.

(a) Based on the Rankine-Hugoniot jump conditions, at what velocity does a shock prop- agate in the case that $q > 0$ on each side of the jump?

(b) Based on the Rankine-Hugoniot jump conditions, at what velocity does a shock propagate in the case that $q_+ > 0$ on one side and $q_- < 0$ on the other side of the jump?

(c) Formulate a solution to this PDE for all $t > 0$ for the initial conditions

$$q(z, 0) = \begin{cases} q_- & \text{if} \quad z < 0 \\ q_+ & \text{if} \quad z > 0 \end{cases}$$

with $q_- < 0$ and $q_+ > 0$. Describe your solution using analytical formulas, using plots of $q(z, t)$ for some sample values of t, and using plots of characteristic curves.

Proof. For $q_s = 0$, we have the following conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial z}\underbrace{\Big(- VH(q)q \Big)}_{C(q)} = 0$$

Let's denote $M_{a,b}(t) = \int_a^b q(z, t)\, dz$ to be the total amount of water at time t in the interval $[a, b]$, then we have

$$\frac{dM_{a,b}}{dt} = \int_a^b q_t(z, t)\, dz = -\int_a^b \frac{d}{dz} C(q(z, t))\, dz = C(q(a, t)) - C(q(b, t)).$$

Assume the conservation of the total amount of water at time t remains constant, which is the area under the curve $q(x, t)$ from a to b and this continues to hold even if shocks occur. Let $\Delta t \ll 1$, $a = \sigma(t)$ to be the position of the shock at time t, $b = \sigma(t + \Delta t)$, then the total amount of water in $[a, b]$ at time t before the shock has passed through is

$$M(t) = \int_a^b q(z, t)\, dz \approx q^+(\sigma(t))\Big[\sigma(t + \Delta t) - \sigma(t)\Big].$$

On the other hand, the total amount of water in $[a, b]$ at time $t + \Delta t$ after the shock has passed is

$$M(t + \Delta t) = \int_a^b q(z, t + \Delta t)\, dz \approx q^-(\sigma(t + \Delta t))\Big[\sigma(t + \Delta) - \sigma(t)\Big].$$

Thus by the conservation of the total amount of water we have

$$\frac{M(t + \Delta t) - M(t)}{\Delta t} = \Big[q^-(\sigma(t + \Delta t)) - q^+(\sigma(t))\Big]\frac{\sigma(t + \Delta t) - \sigma(t)}{\Delta t}.$$

Let $\Delta t \longrightarrow 0$ we have

$$\frac{dM}{dt} = \Big[q^-(t) - q^+(t)\Big]\frac{d\sigma}{dt} = C\Big(q^-(t)\Big) - C\Big(q^+(t)\Big)$$

thus we conclude the speed of the shock is

$$\frac{d\sigma}{dt} = \frac{C\Big(q^-(t)\Big) - C\Big(q^+(t)\Big)}{q^-(t) - q^+(t)} = \frac{-VH(q^-)q^- + VH(q^+)q^+}{q^- - q^+}.$$

(a) When $q > 0$ on both side then from the above formula we obtain $\frac{d\sigma}{dt} = -V$.

(b) When $q_+ > 0$ and $q_- < 0$ then from the above formula we obtain $\frac{d\sigma}{dt} = \frac{Vq^+}{q^- - q^+}$.

(c) The equation can be rewrite to

$$\frac{\partial q}{\partial t} + (-VH(q))\frac{\partial q}{\partial z} = V\delta(q)q = 0.$$

The last identity came from the fact that $x\delta(x) \equiv 0$. Using method of characteristic, for a initial point $(0, z_0, q(z_0, 0))$ we seek for the curve $z(t)$ satisfy

$$\begin{cases} x'(t) & = -VH(q(x(t), t)) \\ x(0) & = z_0. \end{cases}$$

From that, if $q(x(t), t) < 0$ then $x(t)$ remains a constant, while in case $q(x(t), t) > 0$ then $x(t) = z_0 - Vt$. We can conclude that

$$x(t) = z_0 - VtH(q(x(t), t)) \qquad \Longrightarrow \qquad \underbrace{x(t) + VtH(q(x(t), t))}_{\xi} = z_0.$$

Then

$$q(x(t), t) = q(z_0, 0) = q_0(\xi) = q_0\Big(x(t) + VtH\big(q(x(t), t\big)\Big)$$
$$= q_0\Big(x(t) + VtH\big(q_0(z_0)\big)\Big).$$

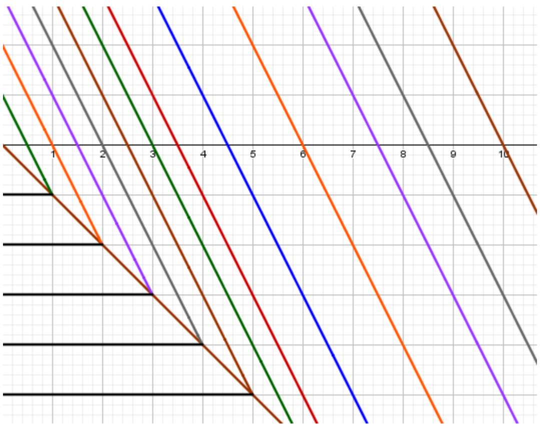
And thus

$$q(x, t) = q_0\Big(x + VtH\big(q(x, t)\big)\Big).$$

This is not a good formula, since the shock will form shortly after $t > 0$. Now consider the characteristic lines.

- Start from $z_0 < 0$, then since $q(z_0) = q^- < 0$, we have the characteristic line started from z_0 is $z(t) = z_0$, thus $q(t, z(t)) = q^-$ for all time $t > 0$.
- Start from $z_0 > 0$, then $q(z_0) > 0$ implies the characteristic line started from z_0 is $z(t) = z_0 - Vt$.

The characteristic lines will look like (in the (t, z)-plane)



where the characteristic meet at $\sigma(t) = \frac{Vq^+}{q^- - q^+}t$. A formula in short time can be given by

$$q(t, x) = \begin{cases} q^- & \text{if} \quad x < \frac{Vq^+}{q^- - q^+}t \\ q^+ & \text{if} \quad x > \frac{Vq^+}{q^- - q^+}t. \end{cases}$$

□

Problem (Aug 2016 - 6/6). Consider the energy functional

$$I[w] = \int_U |\nabla w|^2 - k_0^2 w^2 \, dV$$

defined by integrating over the domain U, where $k_0 > 0$ is a constant. Suppose that u minimizes $I[w]$ over the class of smooth functions that are equal to g on the boundary ∂U. Derive the PDE that is satisfied by u.

Proof. Let $\phi \in C_c^\infty(U)$, then for any $\varepsilon > 0$ we have $u + \varepsilon\phi$ is a smooth function that is equal to g on ∂U. Let's define

$$f(\varepsilon) = I[u + \varepsilon\phi] = \int_U |\nabla u + \varepsilon \nabla \phi|^2 - k_0^2 (u + \varepsilon\phi)^2 \, dV.$$

Since u is a minimizer of $I[\cdot]$, we have $\varepsilon = 0$ is a minimizer of $f(\varepsilon)$, thus

$$f'(0) = 0 \qquad \Longrightarrow \qquad \int_U \nabla u \cdot \nabla \phi - k_0^2 u\phi \, dV = 0.$$

Green formula yields

$$\int_U \Big(-\Delta u\phi - k_0^2 u\phi \Big) \, dV + \int_{\partial U} \frac{\partial u}{\partial \nu} \phi \, dS = 0.$$

Since $\phi = 0$ on ∂U, the latter term vanishes, and since it is true for all $\phi \in C_c^\infty(U)$ we deduce that u solves the PDE

$$\begin{cases} -\Delta u &= k_0^2 u \qquad \text{in } U \\ \quad u &= g \qquad \text{on } \partial U. \end{cases}$$

□

January 2017

Problem (Jan 2017 - 1/6).

(a) Consider a two-dimensional linear dynamical system $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2)^T$, and A is a 2×2 positive definite symmetric matrix. What can we say about the solutions of this dynamical system as $t \longrightarrow \infty$? (For example, what kind of fixed point is the origin? Are the solutions bounded or unbounded? Can any of the solutions be spirals? Can you draw a phase plane diagram for this system?) Explain your answers.

(b) Consider a matrix

$$A = \begin{pmatrix} a & b & 1 \\ b & a & b \\ 1 & b & a \end{pmatrix}$$

such that $a > b > 1$. Is the matrix A positive definite? Explain your answer.

(c) True or false: if the square matrices A and B satisfy $A = B^2$ and B is symmetric and invertible, then A is positive definite. Explain your answer.

(d) Consider two invertible 2×2 matrices A_0 and A_1 with positive entries such that $\det(A_0)$ and $\det(A_1)$ have the same sign. Does there exist a continuous path of invertible 2×2 matrices with positive entries

$$A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

such that $A(0) = A_0$ and $A(1) = A_1$? Explain your answer.

Proof.

(a) Since A is a positive definite symmetric matrix, it has real positive eigenvalues (is can be diagonalizable by orthonomal matrices) with orthonormal eigenvectors. There are two cases

- Case 1. A has two different eigenvalue $\lambda_1 > \lambda_2$. Then since A is positive, we have $\lambda_1 > \lambda_2 > 0$ and also we have two linearly independent corresponding eigenvectors $\mathbf{v}_1, \mathbf{v_2}$. The general solution in this case is given by
$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$
From that we obtain
  - The solution blows up at $t \longrightarrow +\infty$, it can not be unbounded.
  - The solution converges to the origin at $t \longrightarrow -\infty$.
  - The origin is the only fixed point, since A is invertible (positive matrix is inversible), and it is a unstable node (a source).
  - The solution can not be spiral, as one can show that it tends to parallel to the fast direction ($\mathbf{v}_1$) as $t \longrightarrow +\infty$, while it tends to parallel to the slow direction ($\mathbf{v}_2$) as $t \longrightarrow -\infty$.
- Case 2. A has repeated eigenvalue λ. Then since A is positive definite, $\lambda > 0$, and since A is symmetric, we have the eigenspace of A corresponding to λ is 2-dimensional, thus we have $\mathbf{v}_1$ and $\mathbf{v}_2$ are two eigenvectors. The general solution now is given by
$$\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2).$$
From that we obtain
  - The solution blows up at $t \longrightarrow +\infty$, it can not be unbounded.
  - The solution converges to the origin at $t \longrightarrow -\infty$.
  - The origin is the only fixed point, since A is invertible (positive matrix is inversible), and it is a unstable star node (a source).
  - The solution can not be spiral, all trajectories are straight lines through the origin.

(b) A matrix is positive definite if all of its pivots are positive. Doing the Gauss elimination for this matrix we get the pivots are
$$d_1 = a \qquad d_2 = a - \frac{b^2}{a} \qquad d_3 = \left(a - \frac{1}{a}\right) - \frac{b - \frac{b}{a}}{a - \frac{b^2}{a}}\left(b - \frac{b}{a}\right) = \frac{(a-1)(a^2 + a - 2b^2)}{a^2 - b^2}.$$
The third pivot d_3 may be negative, for example with $a = 5, b = 4$ then it is not positive definite.

(c) **TRUE**. Since B is symmetric, for $x \in \mathbb{R}^n \backslash \{0\}$ as a column vector then $x^T A x = x^T B^2 x = x^T B^T B x = (Bx)^T(Bx) = ||Bx||^2 > 0$. Note that B is invertible, thus the only x that makes $Bx = 0$ is $x = 0$.

(d) **TRUE**.

- The case $\det(A_0) > 0, \det(A_1) > 0$. First we observe that given any matrix A with positive entries and $\det(A) > 0$, we can connect A to a matrix A' also with positive entries with $\det(A') = 1$ by a continuous path of invertible matrices $\gamma(t) : [0,1] \longrightarrow \mathbb{M}_{2\times 2}(\mathbb{R}_+)$ with $\gamma(0) = A$ and
$$\gamma(t) = \left(\sqrt[n]{1 - t + \frac{t}{\det A}}\right) A \qquad \Longrightarrow \qquad \det(\gamma(1)) = 1.$$
Note that if $\det A = 1$ then $\gamma(t) \equiv A$ for all $t \in [0,1]$, while if $\det A \neq 1$ then
  - $\det A > 1$ then
$$0 < 1 - \frac{1}{\det A} < 1 \qquad \Longrightarrow \qquad 1 - t\left(1 - \frac{1}{\det A}\right) > 0.$$
  - $\det A < 1$ then
$$1 - \frac{1}{\det A} < 0 \qquad \Longrightarrow \qquad 1 - t\left(1 - \frac{1}{\det A}\right) > 0.$$

Thus equivalently, we can solve our problem with the assumption that $\det A_0 = \det A_1 = 1$, where

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad a_0d_0 - b_0c_0 = a_1d_1 - b_1c_1 = 1.$$

Without loss of generality we can assume that $a_1d_1 \geq a_0d_0$, then it implies $b_1c_1 \geq b_0c_0$. Let

$$\begin{cases} \mathcal{A}(t) & = (1-t)a_0 + ta_1 \\ \zeta(t) & = (1-t)a_0d_0 + ta_1d_1 \end{cases} \quad \text{and} \quad \mathcal{D}(t) = \frac{\zeta(t)}{\mathcal{A}(t)}.$$

Then $\zeta(t)$ is increasing with $a_0d_0 \leq \zeta(t) \leq a_1d_1$, also $\mathcal{D}(0) = d_0, \mathcal{D}(1) = d_1$. Let's define

$$A_1(t) = \begin{pmatrix} \mathcal{A}(t) & b_0 \\ c_0 & \mathcal{D}(t) \end{pmatrix} \quad \Longrightarrow \quad \det(A_1(t)) = \zeta(t) - b_0c_0 \geq a_0d_0 - b_0c_0 = 1 > 0.$$

We have $A_1(0) = A_0$, also it is obvious that all entries of $A_1(t)$ are positive. Now observe that

$$A_1(1) = \begin{pmatrix} a_1 & b_0 \\ c_0 & d_1 \end{pmatrix}$$

we now connect this matrix with A_1 and the proof will be done. Let

$$\begin{cases} \mathcal{B}(t) & = (1-t)b_0 + tb_1 \\ \xi(t) & = (1-t)b_0c_0 + tb_1c_1 \end{cases} \quad \text{and} \quad \mathcal{C}(t) = \frac{\xi(t)}{\mathcal{B}(t)}.$$

Then $\xi(t)$ is increasing with $b_0c_0 \leq \xi(t) \leq b_1c_1$, also $\mathcal{C}(0) = c_0, \mathcal{C}(1) = c_1$. Let's define

$$A_2(t) = \begin{pmatrix} a_1 & \mathcal{B}(t) \\ \mathcal{C}(t) & d_1 \end{pmatrix} \quad \Longrightarrow \quad \det(A_2(t)) = a_1d_1 - \xi(t) \geq a_1d_1 - b_1c_1 = 1 > 0.$$

We have $A_2(0) = A_1(1)$, while $A_2(1) = A_1$, also it is obvious that all entries of $A_2(t)$ are positive. Thus connecting these two paths we obtain a path from A_0 to A_1.

- The case $\det(A_0) < 0, \det(A_1) < 0$, we can start with $\det(A_0) = \det(A_1) = -1$. Indeed for given A with $\det A < 0$, we can choose a continuous path of invertible matrices $\gamma(t) : [0,1] \longrightarrow \mathbb{M}_{2\times 2}(\mathbb{R}_+)$ with $\gamma(0) = A$ and

$$\gamma(t) = \left(\sqrt[n]{1 - t - \frac{t}{\det A}}\right) A \quad \Longrightarrow \quad \det(\gamma(1)) = -1.$$

  Note that if $\det A = -1$ then $\gamma(t) \equiv A$ for all $t \in [0,1]$, while if $\det A \neq -1$ then the things inside the n-root is always non-zero for $t \in [0,1]$.

  - $\det A > -1$ then
$$1 + \frac{1}{\det A} < 0 \quad \Longrightarrow \quad 1 - t\left(1 + \frac{1}{\det A}\right) > 0.$$
  - $\det A < -1$ then
$$0 < 1 + \frac{1}{\det A} < 1 \quad \Longrightarrow \quad 1 - t\left(1 + \frac{1}{\det A}\right) > 0.$$

  Thus equivalently, we can solve our problem with the assumption that $\det A_0 = \det A_1 = -1$, where

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad a_0d_0 - b_0c_0 = a_1d_1 - b_1c_1 = -1.$$

  Without loss of generality we can assume that $a_1d_1 \geq a_0d_0$, then it implies $b_1c_1 \geq b_0c_0$. Let

$$\begin{cases} \mathcal{A}(t) & = (1-t)a_0 + ta_1 \\ \zeta(t) & = (1-t)a_0d_0 + ta_1d_1 \end{cases} \quad \text{and} \quad \mathcal{D}(t) = \frac{\zeta(t)}{\mathcal{A}(t)}.$$

Then $\zeta(t)$ is increasing with $a_0 d_0 \le \zeta(t) \le a_1 d_1$, also $\mathcal{D}(0) = d_0, \mathcal{D}(1) = d_1$. Let's define

$$A_1(t) = \begin{pmatrix} \mathcal{A}(t) & b_1 \\ c_1 & \mathcal{D}(t) \end{pmatrix} \quad \Longrightarrow \quad \det(A_1(t)) = \zeta(t) - b_1 c_1 \le a_1 d_1 - b_1 c_1 = -1 < 0.$$

We have $A_1(1) = A_1$, also it is obvious that all entries of $A_1(t)$ are positive. Now observe that

$$A_1(0) = \begin{pmatrix} a_0 & b_1 \\ c_1 & d_0 \end{pmatrix}$$

we now connect this matrix with A_0 and the proof will be done. Let

$$\begin{cases} \mathcal{B}(t) & = (1-t)b_0 + t b_1 \\ \xi(t) & = (1-t)b_0 c_0 + t b_1 c_1 \end{cases} \quad \text{and} \quad \mathcal{C}(t) = \frac{\xi(t)}{\mathcal{B}(t)}.$$

Then $\xi(t)$ is increasing with $b_0 c_0 \le \xi(t) \le b_1 c_1$, also $\mathcal{C}(0) = c_0, \mathcal{C}(1) = c_1$. Let's define

$$A_2(t) = \begin{pmatrix} a_0 & \mathcal{B}(t) \\ \mathcal{C}(t) & d_0 \end{pmatrix} \quad \Longrightarrow \quad \det(A_2(t)) = a_0 d_0 - \xi(t) \le a_0 d_0 - b_0 c_0 = -1 < 0.$$

We have $A_2(1) = A_1(0)$, while $A_2(0) = A_0$, also it is obvious that all entries of $A_2(t)$ are positive. Thus connecting these two paths we obtain a path from A_0 to A_1.

□

Problem (Jan 2017 - 2/6).
(a) Calculate the Fourier series of the function $f(x) = \cos^3 x$.
(b) Calculate the Fourier transform of the constant function $f(x) = 2\pi$.
(c) Consider the differential equation $-\frac{d^2u}{dx^2} + a^2 u = h(x)$. Use the Fourier transform to calculate its Green's function, i.e., $-\frac{d^2u}{dx^2} + a^2 u = \delta(x)$, and then explain how you can use G to find a solution u for a given right side function h.
(d) Find the relationship between the Fourier transform of the product of two functions and the convolution of the Fourier transforms of the two functions.

Proof.

(a) We have $\cos^3 x = \frac{1}{4}\cos 3x - \frac{3}{4}\cos x$ is exactly its Fourier series.

(b) We use the convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi \cdot x}\, dx$. In terms of tempered distribution, we have the Fourier transform of $2\pi\delta$ is

$$\left\langle \widehat{2\pi\delta}, \phi \right\rangle = \left\langle 2\pi\delta, \widehat{\phi} \right\rangle = 2\pi\widehat{\phi}(0) = 2\pi \int_{\mathbb{R}} \phi(x)\, dx = \langle 2\pi, \phi \rangle$$

where $\phi \in \mathcal{S}(\mathbb{R})$ is the Schwartz test function. Now let's replace ϕ by $\widehat{\phi}$, recalling that the Fourier transform is an isormorphism from $\mathcal{S} \longrightarrow \mathcal{S}$, we have

$$\left\langle \widehat{2\pi\delta}, \widehat{\phi} \right\rangle = \langle 2\pi, \widehat{\phi} \rangle \quad \Longrightarrow \quad \langle 2\pi\delta, \mathcal{F}\mathcal{F}(\phi) \rangle = \langle \widehat{2\pi}, \phi \rangle = \langle 2\pi\delta, \mathcal{R}\phi \rangle = 2\pi\phi(0) = \langle 2\pi\delta, \phi \rangle$$

where we use the Fourier inversion formula $\mathcal{F}\mathcal{F}\phi(x) = \phi(-x)$. This implies $\widehat{2\pi} = 2\pi\delta$.

(c) Recall that under our convention then $\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi)$ for any multi-index α. We consider the Green function subject to a unit impulse concentrated at $x = \eta$,

$$-G_\eta''(x) + a^2 G_\eta(x) = \delta_\eta(x) = \delta(x - \eta).$$

Thus taking the Fourier transform both side we obtain

$$-(2\pi i \xi)^2 \widehat{G_\eta}(\xi) + a^2 \widehat{G_\eta}(\xi) = e^{-2\pi i \eta \cdot \xi} \quad \Longrightarrow \quad \widehat{G_\eta}(\xi) = \frac{e^{-2\pi i \eta \cdot \xi}}{a^2 + 4\pi^2 \xi^2}.$$

At $\eta = 0$, we have

$$\mathcal{F}\Big(G_0\Big)(\xi) = \frac{1}{a^2 + 4\pi^2\xi^2}. \tag{11}$$

From part (b), by re-scaling we obtain

$$\mathcal{F}\Big(e^{-a|x|}\Big)(\xi) = \frac{2a}{a^2 + 4\pi^2\xi^2} \quad \Longrightarrow \quad \mathcal{F}\left(\frac{e^{-a|x|}}{2a}\right)(\xi) = \frac{1}{a^2 + 4\pi^2\xi^2}. \tag{12}$$

From (11) and (12) we deduce that

$$G_0(x) = \frac{e^{-a|x|}}{2a} \quad \Longrightarrow \quad G_\eta(x) = \frac{e^{-a|x-\eta|}}{2a}.$$

Finally the superposition principle based on the Green's function tells us the solution of the original ODE with source $h(x)$ is

$$u(x) = \int_{-\infty}^{+\infty} G_\eta(x) h(\eta)\, d\eta = \frac{1}{2a} \int_{-\infty}^{+\infty} e^{-a|x-\eta|} h(\eta)\, d\eta.$$

(d) Within our convention, we have

$$\mathcal{F}(f * g) = \widehat{f}\,\widehat{g}.$$

First of all this is true for $f, g \in L^1(\mathbb{R}^n)$, then by Young's inequality $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$, and for a.e $x \in \mathbb{R}^n$ then

$$\begin{aligned}
\mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) g(y)\, dy \right) e^{-2\pi i \xi \cdot x}\, dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) e^{-2\pi i \xi (x-y)} g(y) e^{-2\pi i \xi \cdot y}\, dy \right) dx
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) e^{-2\pi i \xi (x-y)} g(y) e^{-2\pi i \xi \cdot y}\, dx \right) dy &= \int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot y} \left(\int_{\mathbb{R}^n} f(x-y) e^{-2\pi i \xi (x-y)}\, dx \right) dy \\
&\le \|\widehat{f}\|_{L^\infty} \|\widehat{g}\|_{L^\infty} \le \|\widehat{f}\|_{L^1} \|\widehat{g}\|_{L^1} < \infty.
\end{aligned}$$

Thus we can use Fubini's theorem to deduce that

$$\mathcal{F}(f * g)(\xi) = \int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot y} \left(\int_{\mathbb{R}^n} f(x-y) e^{-2\pi i \xi (x-y)}\, dx \right) dy = \widehat{f}(\xi)\widehat{g}(\xi).$$

We can extend this to $L^2(\mathbb{R}^n)$.

□

Problem (Jan 2017 - 3/6).

(a) Give a geometric explanation of the constrained optimization method of Lagrange multipliers.

(b) Find the rectangle with corners at points $(\pm x_1, \pm x_2)$ on the ellipse $9x_1^2 + x_2^2 = 1$, such that the perimeter of the rectangle is as large as possible.

(c) Draw a figure that shows your solution for part (b) and use it to explain how this solution fits the geometric explanation you gave in part (a).

(d) Consider the line in $\mathbb{R}^4$ that contains the points (-1, -1, -1, -1) and (1,1,1,1). Also, consider the line in $\mathbb{R}^4$ that contains the points (2, 0, 0, 0) and (0, 0, 0, 2). What is the distance between these lines? Explain your answer.

Proof.

(a) (From Wikipedia) Consider the two-dimensional problem introduced above maximize $f(x,y)$ subject to $g(x,y)=0$. Observe maximum, $f(x,y)$ cannot be increasing in the direction of any neighboring point where $g=0$. If it were, we could walk along $g=0$ to get higher, meaning that the starting point wasn't actually the maximum.

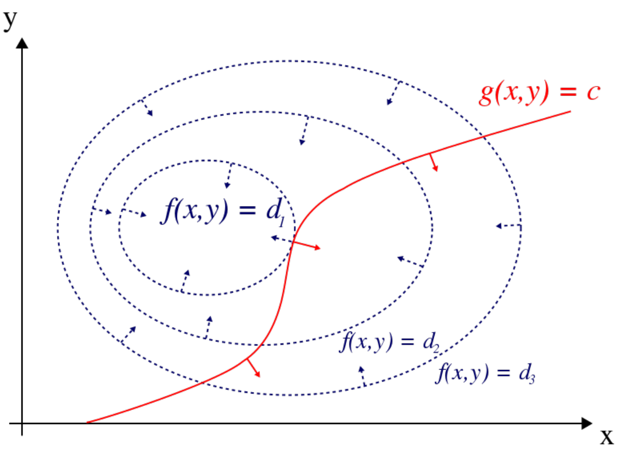
We can visualize contours of f given by $f(x,y)=d$ for various values of d, and the contour of g given by $g(x,y)=0$. Suppose we walk along the contour line with $g=0$. We are interested in finding points where f does not change as we walk, since these points might be maxima. There are two ways this could happen:

- First, we could be following a contour line of f, since by definition f does not increase as we walk along its contour lines. This would mean that the contour lines of f and g are parallel here.
- The second possibility is that we have reached a "level" part of f, meaning that f does not change in any direction.

To check the first possibility, notice that since the gradient of a function is perpendicular to the contour lines, the contour lines of f and g are parallel if and only if the gradients of f and g are parallel. Thus we want points (x,y) where $g(x,y)=0$ and

$$\nabla_{x,y} f=\lambda \nabla_{x,y} g \nabla_{x,y} f=\lambda \nabla_{x,y} g.$$

Notice that this method also solves the second possibility: if f is level, then its gradient is zero, and setting $\lambda=0$ is a solution regardless of g.



(b) Let $f(x,y)=x+y$ and $g(x,y)=9x^2+y^2-1$, we want to maximize $f(x,y)$ subjected to the constraint $g(x,y)=0$. Let $\lambda>0$, we consider

$$L(x,y)=f(x,y)-\lambda g(x,y).$$

Using the method of Lagrange multipliers, we compute

$$\nabla L(x,y)=\nabla f(x,y)-\lambda \nabla g(x,y)=(1-18\lambda x, 1-2\lambda y).$$

Thus

$$\nabla L(x,y)=0 \quad\Longleftrightarrow\quad \begin{cases}\lambda x &= \frac{1}{18}\\ \lambda y &= \frac{1}{2}.\end{cases}$$

It is easy to see that $\lambda=0$ is not possible, thus $(x,y)=\left(\frac{1}{18\lambda},\frac{1}{2\lambda}\right)$. Substitute to the constant we obtain

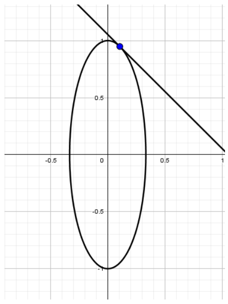
$$9\left(\frac{1}{18\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=1 \quad\Longrightarrow\quad \lambda^2=\frac{5}{18} \quad\Longrightarrow\quad \lambda=\pm\frac{\sqrt{10}}{6}.$$

Since we want to maximize $x+y$, we choose $\lambda>0$ and the corresponding result is

$$f(x,y)=x+y=\frac{1}{18\lambda}+\frac{1}{2\lambda}=\frac{\sqrt{10}}{3}.$$

Thus the maximum parameter is $\frac{2\sqrt{10}}{3}$, which is archived at the rectangle located at $\left(\pm\frac{\sqrt{10}}{30},\pm\frac{3\sqrt{10}}{10}\right)$.

(c) The level sets of $f(x, y) = x + y = d$ are lines with slopes -1, the largest value achieved when this line is tangent to the ellipse $9x^2 + y^2 = 1$, as being showed in the picture.



(d) It has the same answer with Problem 1d August 2016. The answer is 1, which can be done by minimizing the distance function or calculating the projection of any vector between 2 points on these lines to the perpendicular plane.

□

August 2017

Problem (Aug 2017 - 1/6).

(a) True or false: if A and B are symmetric and positive definite $n \times n$ matrices, then their sum $A + B$ is also symmetric and positive definite. Explain your answer.

(b) True or false: if A and B are symmetric and positive definite $n \times n$ matrices, then their product AB is also symmetric and positive definite. Explain your answer.

(c) Consider the matrix

$$A = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}.$$

Find the values of α and β for which this matrix is positive definite. Explain your answer.

Proof.

(a) **TRUE**. It is obvious from the definition.

(b) **TRUE**. It is easy to find a counter example.

(c) The condition are all the principal, i.e the determinant of $k \times k$ matrices in the left corner must be positive.

$$\alpha > 0, \qquad \alpha^2 - \beta^2 > 0, \qquad \det A > 0.$$

□

Problem (Aug 2017 - 6/6). Consider

$$u_{xx} + u_{yy} + u_{zz} = 0, \qquad z > 0,$$

with

$$u(x, y, 0) = f(x, y), \qquad |u(x, y, z)| \text{ bounded.}$$

(a) Let $\widehat{u}(\xi_x, \xi_y, z)$ be the Fourier transform of $u(x, y, z)$ in x and y. Find the ODE and boundary conditions for $\widehat{u}$.
(b) Solve the ODE for $\widehat{u}$, then write down the solution as an inverse Fourier transform.
(c) By interchanging order of integration if needed, rewrite the solution in (b) in the form

$$u(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x}, \tilde{y}) K\Big(x - \tilde{x}, y - \tilde{y}\Big)\, d\tilde{x} d\tilde{y}$$

where K is a kernel defined by integrals over ξ_x and ξ_y.
(d) To evaluate the integrals in K first define L such that $K(x, y, z) = -\frac{\partial}{\partial z} L(x, y, z)$. Change to polar coordinates in the (ξ_x, ξ_y) space to evaluate L. You may use

$$\int_0^{2\pi} \frac{d\theta}{z - ix\cos\theta - iy\sin\theta} = \frac{2\pi}{\sqrt{x^2 + y^2 + z^2}}.$$

Proof. We use the convention $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2-\pi\xi x}\, dx$.

(a) The ODE is

$$\frac{\partial^2}{\partial z^2}\Big(\widehat{u}(\xi_x, \xi_y, z)\Big) - 4\pi^2 \|(\xi_x, \xi_y)\|^2 \widehat{u}(\xi_x, \xi_y, z) = 0.$$

for each $\xi = (\xi_x, \xi_y)$, with boundary condition

$$\widehat{u}(\xi_x, \xi_y, 0) = \widehat{f}(\xi_x, \xi_y).$$

Also we need to impose the condition $|u(x, y, z)|$ is bounded.

(b) The characteristic equation of the second order ODE in z is $\lambda^2 - 4\pi^2\|\xi\|^2 = 0$, it has two eigenvalues $\lambda_{1,2} = \pm 2\pi\|\xi\|$, thus the general solution will be

$$\widehat{u}(\xi_x, \xi_y, z) = C_1 e^{-2\pi\|\xi\| z} + C_2 e^{2\pi\|\xi\| z}$$

for some suitable constant C_1, C_2. Using the fact that $u(x, y, z)$ is bounded, we expect $\widehat{u}$ is bounded as $z \longrightarrow +\infty$, thus we should expect $C_2 = 0$, then C_1 will be determined using the initial condition

$$\widehat{u}(\xi_x, \xi_y, 0) = C_1 = \widehat{f}(\xi_x, \xi_y).$$

We conclude that the solution to the ODE is

$$\widehat{u}(\xi_x, \xi_y, z) = \widehat{f}(\xi_x, \xi_y) e^{-2\pi\|\xi\| z}.$$

By the Fourier inversion formula we have the solution $u(x, y, z)$ is

$$u(x, y, z) = \int_{\mathbb{R}^2} \widehat{f}(\xi_x, \xi_y) e^{-2\pi\|\xi\| z} e^{2\pi i\left(\xi_x \cdot x + \xi_y \cdot y\right)}\, d\xi_x d\xi_y.$$

(c) Using the definition of $\widehat{f}(\xi_x, \xi_y)$, then we have for

$$K(x, y, z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi\|\xi\| z} e^{2\pi i\left(\xi_x \cdot x + \xi_y \cdot y\right)}\, d\xi_x d\xi_y.$$

Then

$$u(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x}, \tilde{y}) K\big(x - \tilde{x}, y - \tilde{y}\big)\, d\tilde{x} d\tilde{y}.$$

(d) Define

$$L(x,y,z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{-2\pi\|\xi\|z}}{2\pi\|\xi\|}e^{2\pi i\left(\xi_x\cdot x+\xi_y\cdot y\right)}\,d\xi_x d\xi_y$$

then since $e^{-2\pi\|\xi\|z}$ is integrable in (ξ_x,ξ_y), by Lebesgue dominated convergence theorem we have

$$\frac{\partial L}{\partial z}(x,y,z)=-K(x,y,z).$$

We can compute L using polar coordinates in (ξ_x,ξ_y) space, we have

$$\begin{aligned}L(x,y,z)&=\int_0^{2\pi}\int_0^r\frac{e^{-2\pi rz}}{2\pi r}e^{-2\pi i\left(r\cos\theta\cdot x+r\sin\theta\cdot y\right)}r\,dr\,d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}\int_0^r e^{-2\pi r\left(z-ix\cos\theta-iy\sin\theta\right)}\,dr\,d\theta\\&=-\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{z-ix\cos\theta-iy\sin\theta}=\frac{-1}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

by the formula. From that we obtain

$$K(x,y,z)=\frac{\partial}{\partial z}L(x,y,z)=\frac{-z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}.$$

Thus the solution $u(x,y,z)$ is given by

$$u(x,y,z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{zf(\tilde{x},\tilde{y})}{\left(\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+z^2}\right)^3}\,d\tilde{x}d\tilde{y}.$$

□

Appendix

Theorem 1. *Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into*

$$\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda) \qquad (*)$$

and making a clever choice of λ.

Proof. First of all we know that $\det(A-\lambda I)$ is the characteristic polynomial of degree n of the matrix A, and if we assume its zeros can be complex value, then the representation (*) is always true according to the fundamental theorem of algebra. Now let $\lambda=0$ in (*) we obtain

$$\det(A)=\lambda_1\lambda_2\dots\lambda_n.$$

□

Theorem 2. *Show that the trace equals the sum of the eigenvalues.*

Proof. Assume

$$\det(A-\lambda I)=c_n(-\lambda)^n+c_{n-1}(-\lambda)^{n-1}+\dots+c_1(-\lambda)+c_0$$

and by Viete's theorem, we have ($c_n=1$)

$$\lambda_1+\dots+\lambda_n=\frac{(-1)^{n-1}c_{n-1}}{(-1)^n}=-c_{n-1}.$$

On the other hand, we have

$$\det(A-\lambda I)=\det\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}. \tag{**}$$

Now look at this expression and count for the coefficient of $(-\lambda)^{n-1}$. We can do it easily by mathematical induction. We claim that the term $(-\lambda)^{n-1}$ is only included in the product $(a_{11}-\lambda)\dots(a_{nn}-\lambda)$.

1. For $n=2$, it is clear that

$$\det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}=(a_{11}-\lambda)(a_{12}-\lambda)-a_{12}a_{21}$$

and thus the term involving $(-\lambda)$ is only included in the product of the entries on the main diagonal.

2. Assume it is true for n.

   - Denoted A_{ij} to be the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A.
   - We have the expansion of $\det(A)$ along the 1-th row is

$$\det A=\sum_{k=1}^{n} a_{1k}(-1)^{1+k}\det A_{1k}.$$

   We obtain in (**) then

$$\det(A-\lambda I)=(a_{11}-\lambda)(-1)^{1+1}A_{11}+\sum_{k=2}^{n} a_{1k}(-1)^{1+k}A_{1k}.$$

   For $k\geq 2$, then after deleting 1-row and k-th column the matrix we obtained only have at most $(n-2)$-terms involving λ on the main diagonal, thus by the induction hypothesis there is no way a term like $a_{1k})(-1)^{1+k}A_{1k}$ produce $(-\lambda)^{n-1}$. That means all the terms have $(-\lambda)^{n-1}$ is included in the first term

$$(a_{11}-\lambda)A_{11}.$$

   Look at the $(n-1)\times(n-1)$ matrix after deleting 1-row and 1-column, by the induction hypothesis again, all the terms have $(-\lambda)^{n-2}$ is only included in the product of the entries on the main diagonal, which implies our result.

Using this result, we only need to count for the coefficient of $(-\lambda)^{n-1}$ in

$$(a_{11}-\lambda)\dots(a_{nn}-\lambda).$$

Again by Viete's theorem, we have it equals to

$$-a_{11}+\dots-a_{nn}.$$

Comparing these results we have

$$-a_{11}+\dots-a_{nn}=-\lambda_1+\dots+-\lambda_n$$

which implies $\text{trace}(A)=\lambda_1+\dots+\lambda_n$. □